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Properties of generalized derangement graphs

Hannah Jackson, Kathryn Nyman and Les Reid

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A permutation on n elements is called a *k-derangement* ($k \le n$) if no k-element subset is mapped to itself. One can form the *k-derangement graph* on the set of all permutations on n elements by connecting two permutations σ and τ if $\sigma\tau^{-1}$ is a k-derangement. We characterize when such a graph is connected or Eulerian. For n an odd prime power, we determine the independence, clique and chromatic numbers of the 2-derangement graph.

1. Introduction

Permutations which leave no element fixed, known as derangements, were first considered in [de Montmort 1708] and have been extensively studied since. A derangement graph is a graph whose vertices are the elements of the symmetric group S_n and whose edges connect two permutations that differ by a derangement. Derangement graphs have been shown to be connected (for $n > 3$) and Hamiltonian, and their independence number, clique number, and chromatic number have been calculated [Renteln 2007].

Here we consider the generalization of derangements known as k-derangements, which are those permutations in S_n that do not fix any k-element subset of the set being permuted. A k-derangement graph is defined in an analogous manner to a derangement graph. We examine some of the graph-theoretical properties of k-derangement graphs.

2. Preliminaries

Let S_n be the group of permutations on the set $\{1, 2, \ldots, n\}$. A permutation $\sigma \in S_n$ maps any k-element subsetof $\{1, \ldots, n\}$ to a k-element subset of $\{1, \ldots, n\}$; in the usual notation,

$$\sigma(\{a_1, \ldots, a_k\}) = \{\sigma(a_1), \ldots, \sigma(a_k)\}.$$

If $\{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$ (as sets, that is, without regard to order), we

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say that σ fixes the unordered k-tuple $\{a_1, \dots, a_k\}$. ("Unordered k-tuple" is another name for a k-element set.)

If σ does not map *any* of the $\binom{n}{k}$ possible unordered k-tuples to itself, we say that σ is a *k-derangement*. For example, with $n = 4$, the cyclic permutation $\sigma = (1234)$ is a 2-derangement, because (taking $k = 2$) we have

$$
\begin{aligned}
(1234)(\{1,2\}) &= \{(1234)(1), (1234)(2)\} = \{2,3\},\\
(1234)(\{1,3\}) &= \{(1234)(1), (1234)(3)\} = \{2,4\},\\
(1234)(\{1,4\}) &= \{(1234)(1), (1234)(4)\} = \{2,1\} = \{1,2\},\\
(1234)(\{2,3\}) &= \{(1234)(2), (1234)(3)\} = \{3,4\},\\
(1234)(\{2,4\}) &= \{(1234)(2), (1234)(4)\} = \{3,1\} = \{1,3\},\\
(1234)(\{3,4\}) &= \{(1234)(3), (1234)(4)\} = \{4,1\} = \{1,4\}.
\end{aligned}
$$

This extends the ordinary notion of a derangement, defined as a permutation $\sigma \in S_n$ such that $\sigma(x) \neq x$ for all $x \in \{1, \dots, n\}$.

The set of k-derangements in S_n is denoted by $\mathscr{D}_{k,n}$, and its cardinality $|\mathscr{D}_{k,n}|$ — the number of k-derangements in S_n — is denoted by $D_k(n)$. As we have seen, (1234) is in $\mathscr{D}_{2,4}$. Specifically,

$$
\begin{aligned}
\mathscr{D}_{2,4} = \big\{&(1234), (1243), (1324), (1342), (1423), (1432), (123)(4), (124)(3),\\
&(132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)\big\},
\end{aligned}
$$

and thus $D_2(4) = 14$. The sequence $D_2(n)$ appears as A137482 in the *On-Line Encyclopedia of Integer Sequences*; see [Henshaw 2008]. The number $D_1(n)$ is also simply called the derangement number.

The cycle structure of a permutation σ, denoted by C_σ, is the multiset of the lengths of the cycles in its cycle decomposition (e.g., $C_{(12)(3)(45)} = \{2, 2, 1\}$). Note that the cycle structure of $\sigma \in S_n$ is a partition of n. Given a partition r of n, let P_r be the set of all permutations in S_n whose cycle structure is r. For example (as usual, excluding singletons in our notation) $P_{\{2,1,1\}} = \{(12), (13), (14), (23), (24), (34)\}$.

We first note that if the cycle structure of a permutation σ contains a multiset which partitions k, then σ is not a k-derangement. For example, $(12)(34)$ is a 3-derangement in S_4, but $(12)(3)(4)$ is not, because it fixes the set $\{1, 2, 3\}$, for example. And we see that $\{2, 1\} \subseteq C_{(12)(3)(4)} = \{2, 1, 1\}$ is a partition of 3. Thus we observe that the cycle structure of a permutation determines whether or not it is a k-derangement, and we have the following.

Proposition 1. *A permutation $\sigma \in S_n$ is a k-derangement if and only if the cycle decomposition of σ does not contain a set of cycles whose lengths partition k.*

Proof. If $\{q, r, \dots, s\}$ is a partition of k, and $(a_1 \cdots a_q)(b_1 \cdots b_r) \cdots (c_1 \cdots c_s)$ are cycles of σ, then, for $x = \{a_1, \dots, a_q, b_1, \dots, b_r, c_1, \dots, c_s\}$, $\sigma(x) = x$. Conversely,

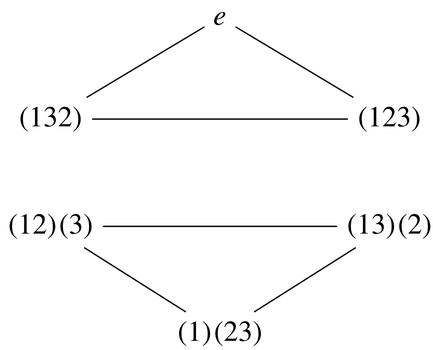

Figure 1. The 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

if σ has no set of cycles whose lengths partition k, then, given any k-element subset x of $\{1, \ldots, n\}$, there is a cycle in σ which contains at least one element in x and contains some element not in x. Hence σ sends an element in x to an element not in x and so $\sigma(x) \neq x$. $\square$

Let $CD_{k,n}$ be the set of cycle structures corresponding to k-derangements in S_n; for example, $CD_{2,4} = \{\{4\}, \{3, 1\}\}$. Since a cycle structure C_σ is in $CD_{k,n}$ if and only if it is in $CD_{n-k,n}$, we have $\mathfrak{D}_{k,n} = \mathfrak{D}_{n-k,n}$.

Let G be a group, and let S be a subset of G that is closed under taking inverses. The *Cayley graph* $\Gamma(G, S)$ is the graph whose vertices are the elements of G such that an edge connects two vertices $u, v \in G$ if $su = v$ for some $s \in S$. A *k-derangement graph* is a Cayley graph defined by $\Gamma_{k,n} := \Gamma(S_n, \mathfrak{D}_{k,n})$. (Note that $\mathfrak{D}_{k,n}$ is symmetric, as the inverse of a k-derangement is a k-derangement, and thus satisfies the requirements for a Cayley graph.) It is worth noting that $\Gamma_{k,n}$ is, by construction, $D_k(n)$-regular, and that, since $\mathfrak{D}_{k,n} = \mathfrak{D}_{(n-k),n}$, $\Gamma_{k,n} = \Gamma_{(n-k),n}$. Figure 1 illustrates the 2-derangement graph on 6 vertices, $\Gamma_{2,3}$.

It is possible to consider k-derangements in S_n for any positive k and n. However, if $k = n$, there will be no k-derangements in S_n, since every partition in S_n will have a cycle structure such that the cycle lengths partition k. As such, $\Gamma_{k,n}$ will be the empty (edgeless) graph on n vertices. If $k > n$, then every permutation in S_n is a k-derangement vacuously, and thus $\Gamma_{k,n}$ will be the complete graph on $|S_n|$ vertices. As neither of these cases is particularly interesting, henceforth we will only consider k-derangements where $k < n$.

3. Properties of derangement graphs

Figure 1 shows that $\Gamma_{2,3}$ is not a connected graph, and, since $\Gamma_{2,3} = \Gamma_{1,3}$, we see that $\Gamma_{k,3}$ is disconnected for all $k < n$. But this is an exception rather than the rule, as the following theorem demonstrates.

Theorem 2. *For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is connected.*

Proof. Every permutation in S_n can be written as the product of adjacent transpositions $(h\ (h{+}1))$. These, in turn, can be expressed as products of two k-derangements, so long as $n > 3$, as we will demonstrate. As a result, for $n > 3$, the elements of $\mathcal{D}_{k,n}$ generate S_n, which means that every vertex of $\Gamma_{k,n}$ can be reached by a path from the identity.

We show that the permutation $(1\ 2)$ can be written as the product of two k-derangements and then note that, since it is the form and not the individual labels that are important, any adjacent transposition can be written as the product of two k-derangements. We consider two cases: $k = 1$ and $k \geq 2$.

<u>Case 1</u>: If $k = 1$, then $(1\ 2) = (1\ 2\ \cdots\ n)^2 \cdot (n\ (n{-}1)\ \cdots\ 1)^2(1\ 2)$. We claim that $(1\ 2\ \cdots\ n)^2$ and $(n\ (n{-}1)\ \cdots\ 1)^2(1\ 2)$ are each 1-derangements in S_n for all $n > 3$. If n is even, then $(1\ 2\ \cdots\ n)^2 = (1\ 3\ \cdots\ (n{-}3)\ (n{-}1))(2\ 4\ \cdots\ (n{-}2)\ n)$, which is a 1-derangement in S_n for all n. Additionally,

$$(n\ (n{-}1)\ \cdots\ 1)^2(1\ 2) = (1\ n\ (n{-}2)\ (n{-}4)\ \cdots\ 2\ (n{-}1)\ (n{-}3)\ \cdots\ 3),$$

which is also a 1-derangement in S_n for any n.

On the other hand, if n is odd, then

$$(1\ 2\ \cdots\ n)^2 = (1\ 3\ \cdots\ (n{-}2)\ n\ 2\ 4\ \cdots\ (n{-}3)\ (n{-}1)),$$

which is a 1-derangement in S_n for all n. And

$$\begin{aligned}(n\ (n{-}1)\ \cdots\ 1)^2(1\ 2) &= (n\ (n{-}2)\ (n{-}4)\ \cdots\ 3\ 1\ (n{-}1)\ (n{-}3)\ \cdots\ 4\ 2)(1\ 2)\\ &= (1\ n\ (n{-}2)\ (n{-}4)\ \cdots\ 3)(2\ (n{-}1)\ (n{-}3)\ \cdots\ 4),\end{aligned}$$

which is a 1-derangement in S_n so long as $n > 3$. (If $n = 3$, $(312)(12) = (13)(2)$, which is not a 1-derangement.)

Thus, for $n > 3$, we have shown that $(1\ 2)$ can be written as the product of two 1-derangements, and, by extension, every adjacent transposition can be written as the product of two 1-derangements.

<u>Case 2</u>: For $k \geq 2$, $(1\ 2) = (1\ 2\ \cdots\ n)^{-1}(1\ 3\ 4\ \cdots\ n)$. We know $(1\ 2\ \cdots\ n)^{-1}$ is a k-derangement for all k since the inverse of a k-derangement is a k-derangement. And, by the cycle structure, we see that $(1\ 3\ 4\ \cdots\ n) = (1\ 3\ 4\ \cdots\ n)(2)$ is a k-derangement for all k, except $k = 1$ and $k = (n{-}1)$ (however, since $\Gamma_{1,n} = \Gamma_{(n-1),n}$, Case 1 addresses $(n{-}1)$-derangements as well as 1-derangements).

So we have shown that, for $k \geq 2$, $(1\ 2)$ can be written as the product of two k-derangements, and again, by extension, we can write any adjacent transposition as the product of two k-derangements. Thus every vertex is connected by a path to the identity, and $\Gamma_{k,n}$ is connected. □

It is worth noting that Theorem 2 holds for $n = 2$ as well. Since we are only interested in k-derangements in S_n such that $k < n$, when $n = 2$, k must equal 1, and so $\Gamma_{1,2}$ is the connected graph on two vertices.

Next, we give a characterization in terms of n and k for when a derangement graph is Eulerian. We will require the following result.

Lemma 3. *If a cycle structure includes a cycle of length greater than* 2, *then there are an even number of permutations with that cycle structure.*

Proof. Consider P_r, the set of permutations with a given cycle structure, r. We can pair each $\sigma \in P_r$ with its inverse $\sigma^{-1} \in P_r$, and, so long as $\sigma \neq \sigma^{-1}$ for any $\sigma \in P_r$, $|P_r|$ will be even. Suppose there exists a $\sigma \in P_r$ such that $\sigma = \sigma^{-1}$. Then $\sigma^2 = e$, and so the order of σ is at most 2. The order of a permutation is the least common multiple of the orders of the elements of its cycle structure, so σ must not include a cycle of length greater than 2. This is a contradiction; thus $|P_r|$ is even. □

Theorem 4. *For $n > 3$ and $k < n$, $\Gamma_{k,n}$ is Eulerian if and only if k is even or k and n are both odd.*

Proof. A graph is Eulerian if and only if it is connected and each vertex has an even degree. In light of Theorem 2 and the previously noted fact that $\Gamma_{k,n}$ is $D_k(n)$-regular, in order to ascertain if $\Gamma_{k,n}$ is Eulerian, we must determine whether $D_k(n)$ is even or odd.

If k is even, we claim that $D_k(n)$ is the sum of even numbers. Any cycle structure composed entirely of 2- or 1-cycles will partition an even k, and thus any permutation which is in $\mathcal{D}_{k,n}$ for an even k will contain a cycle of length 3 or greater in its cycle decomposition. Now, $\mathcal{D}_{k,n} = P_{r_1} \dot{\cup} P_{r_2} \dot{\cup} \cdots \dot{\cup} P_{r_m}$ (disjoint union) such that no r_i partitions k, and, by Lemma 3, $|P_{r_i}|$ is even for all $i \in \{1, \ldots, m\}$. Thus, when k is even, $D_k(n)$ is even.

If k and n are both odd, again we see that every permutation in $\mathcal{D}_{k,n}$ will contain a cycle of length 3 or greater in its cycle decomposition, since an odd k can be partitioned by a set of cycles of lengths 1 or 2 if there is at least one 1-cycle. Furthermore, since n is odd, there are no permutations whose cycle structure is composed only of length-2 cycles. Thus, $D_k(n)$ is even.

Finally, we show that, if k is odd and n is even, then $\Gamma_{k,n}$ is not Eulerian. In this case, $P_{\{2,2,\ldots,2\}}$ is in $CD_{k,n}$. By choosing pairs of elements for the cycles and dividing by the number of ways to order the cycles, we see that the number of permutations in $P_{\{2,2,\ldots,2\}}$ is given by

$$\begin{aligned}\frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{2}{2}}{\left(\frac{n}{2}\right)!} &= \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{\left(2\cdot\frac{n}{2}\right)\left(2\cdot\left(\frac{n}{2}-1\right)\right)\cdots(6)(4)(2)} \\ &= \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{n(n-2)\cdots(6)(4)(2)} = (n-1)(n-3)\cdots(5)(3)(1).\end{aligned}$$

Since n is even, the product $(n-1)(n-3)\cdots(5)(3)(1)$ is odd. Every other k-derangement in S_n will contain a cycle with length greater than 2, since any combination of 1-cycles or 1- and 2-cycles will partition k. So $D_k(n)$ is the sum of one odd number and even numbers, and so is odd. □

4. Chromatic, independence and clique numbers for $k = 2$ and n an odd prime power

For the majority of this section, we will think of permutations in terms of the result of their application to the ordering $\{1, 2, 3, \ldots, n\}$. Thus, $\{2, 3, 1, 4, 5\}$ represents the permutation which has moved 2 to the first position, 3 to the second, 1 to the third, and left 4 and 5 fixed; that is, the permutation $(132)(4)(5)$ in cycle notation, or the inverse of the permutation $\binom{12345}{23145}$ in two line notation.

We note that in order for vu^{-1} (or, equivalently, $v^{-1}u$) to be a k-derangement, it is necessary and sufficient that no unordered k-tuple of elements be sent to the same unordered k-tuple of positions by both u and v. For example, the permutations $u = \{2, 3, 1, 4, 5\}$ and $v = \{4, 1, 3, 5, 2\}$ both send the pair $\{1, 3\}$ to the second and third positions. Thus $(vu^{-1})(\{2, 3\}) = \{2, 3\}$, and so vu^{-1} is not a 2-derangement and there is no edge between u and v in the 2-derangement graph. More formally, suppose u and v both send the k-tuple $M' = \{a'_1, a'_2, \ldots, a'_k\}$ to positions $M = \{a_1, a_2, \ldots, a_k\}$. Then, $(vu^{-1})(M) = v(M') = M$. Thus, vu^{-1} is not a k-derangement.

On the other hand, if u and v send no k-tuple to the same positions we claim vu^{-1} is a k-derangement. Consider an arbitrary k-tuple, $M = \{a_1, a_2, \ldots, a_k\}$, and suppose u maps the k-tuple $M' = \{a'_1, a'_2, \ldots, a'_k\}$ to the positions given in M. Then $(vu^{-1})(M) = v(M') \neq M$ since v cannot send the k-tuple M' to the same positions as u does. Thus, vu^{-1} is a k-derangement.

In Theorem 6, we find the clique number of the 2-derangement graph, $\omega(\Gamma_{2,n})$, for n an odd prime power, by constructing a clique of maximal size. Before establishing this clique number, we note an upper bound on the clique number of a general k-derangement graph.

Lemma 5. *For $k < n$, $\omega(\Gamma_{k,n}) \leq \binom{n}{k}$.*

Proof. The clique number of the k-derangement graph, $\omega(\Gamma_{k,n})$, cannot be greater than $\binom{n}{k}$, since there are only $\binom{n}{k}$ subsets of size k and hence at most $\binom{n}{k}$ different unordered k-tuples of positions for an arbitrary k-tuple of elements to be sent under a permutation. □

Theorem 6. *If n is an odd prime power, then $\omega(\Gamma_{2,n}) = \binom{n}{2}$.*

Proof. We will explicitly construct a clique with $\binom{n}{2}$ elements. Let $n = p^r$, with p a prime greater than 2, and let $\mathbb{F}_{p^r}$ denote the field with p^r elements. Rather than letting S_n act on $\{1, \dots, n\}$, we will let it act on $\mathbb{F}_{p^r}$ and construct $\Gamma_{2,n}$ accordingly. Let $v = (x_1, \dots, x_n)$ be an ordered n-tuple whose entries are the elements of $\mathbb{F}_{p^r}$ in some order. Given any function $\phi : \mathbb{F}_{p^r} \to \mathbb{F}_{p^r}$, we define $\phi(v) = (\phi(x_1), \dots, \phi(x_n))$. Partition the nonzero elements of $\mathbb{F}_{p^r}$ by pairing each element with its (additive) inverse, and let T be a set obtained by choosing exactly one element from each pair, giving $|T| = (p^r - 1)/2$.

Define $f_{s,\alpha}(x) = sx + \alpha$, and consider the set $X = \{f_{s,\alpha}(v) \mid s \in T \text{ and } \alpha \in \mathbb{F}_{p^r}\}$. Since $s \neq 0$, $f_{s,\alpha}$ is a bijection and $f_{s,\alpha}(v)$ is a permutation of the elements of $\mathbb{F}_{p^r}$. We claim that X is a clique in $\Gamma_{2,n}$. Suppose not; that is, suppose there are $s, t \in T$ and $\alpha, \beta \in \mathbb{F}_{p^r}$, $(s, \alpha) \neq (t, \beta)$, such that $f_{s,\alpha}(v)$ is not a 2-derangement of $f_{s,\beta}(v)$. In that case there exist $x, y \in \mathbb{F}_{p^r}$, $x \neq y$, such that either $f_{s,\alpha}(x) = f_{t,\beta}(x)$ and $f_{s,\alpha}(y) = f_{t,\beta}(y)$ or $f_{s,\alpha}(x) = f_{t,\beta}(y)$ and $f_{s,\alpha}(y) = f_{t,\beta}(x)$. In the first case, subtracting the two equations and rewriting yields $(s - t)(x - y) = 0$. If $s = t$, then $\alpha = \beta$, giving a contradiction. If $s \neq t$, then $x = y$ and again we have a contradiction. In the second case, subtracting and rewriting yields $(s + t)(x - y) = 0$ and, since $s + t \neq 0$ for $s, t \in T$, $x = y$ and this also give a contradiction. Thus, X is a clique of size $p^r(p^r - 1)/2 = \binom{n}{2}$. □

The next example illustrates the construction when $n = 7$.

Example 7. We build a clique of size $\binom{7}{2}$ in the derangement graph $\Gamma_{2,7}$ consisting of $\frac{7-1}{2}$ blocks, each of which contains 7 permutations. We let $v = (1, 2, 3, 4, 5, 6, 7)$ (writing 7 instead of 0) and take $T = \{1, 4, 5\}$. Then

$$f_{1,0}(v) = (1, 2, 3, 4, 5, 6, 7), \quad f_{4,0}(v) = (4, 1, 5, 2, 6, 3, 7),$$
$$f_{5,0}(v) = (5, 3, 1, 6, 4, 2, 7).$$

Increasing α from 0 cyclically permutes the 7-tuples. Block 1 consists of the arrangements $\{f_{1,\alpha}(v) \mid \alpha \in \mathbb{F}_7\}$, that is, the arrangement $(1, 2, 3, 4, 5, 6, 7)$ and the remaining 6 rotations of this arrangement (e.g., $(2, 3, 4, 5, 6, 7, 1)$, $(3, 4, 5, 6, 7, 1, 2)$, etc.). Block 2 consists of the arrangement $f_{4,0}(v)$ along with all of its rotations. Finally, block 3 consists of $f_{5,0}(v)$ and its rotations. To see that these permutations form a clique, consider, for example, the pair $\{1, 2\}$. These elements are one position apart in block 1, two positions apart in block 2 and three positions apart in block 3 (counting the shortest distance between them either forwards or backwards). So the pair $\{1, 2\}$ cannot occupy the same positions in two permutations which appear in different blocks. Furthermore, within a block, the rotations insure that the pair never occupies the same positions.

Remark 8. Cliques achieving the upper bound of Lemma 5 are known as *sharply k-homogeneous sets* of permutations. A corollary in [Nomura 1985] shows that, for $2k \leq n$, the existence of such a k-homogeneous set implies $n + 1 \equiv 0 \mod k$. Thus Theorem 6 cannot be extended to even n, and we have the following.

Corollary 9. *For n even and $n \geq 4$, $\omega(\Gamma_{2,n}) < \binom{n}{2}$.*

A computer search confirms that $\omega(\Gamma_{2,4}) = 5 < \binom{4}{2}$.

Next we turn to the independence number $\alpha(\Gamma_{k,n})$ and the chromatic number $\chi(\Gamma_{k,n})$ of the k-derangement graph. We will require the following lemma which has been adapted from Frankl and Deza's lemma [1977] and applied to k-tuples of elements.

Lemma 10. *For $k < n$, $\alpha(\Gamma_{k,n})\omega(\Gamma_{k,n}) \leq n!$.*

Proof. Let $\mathcal{P}$ be a set of permutations in S_n, every pair of which has at least one unordered k-tuple of elements in the same unordered k-tuple of positions. That is, for any $u, v \in \mathcal{P}$, there exists a set $M = \{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\}$ such that $(v^{-1}u)(M) = M$. Note that $\mathcal{P}$ is an independent set in the k-derangement graph. Let $\mathcal{Q}$ be a set of permutations in S_n such that each pair of permutations has no k-tuple of elements in the same positions; that is, $\mathcal{Q}$ is a clique in the k-derangement graph. We claim that products of the form PQ with $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ give distinct permutations of n. Suppose, for the sake of contradiction, that $P_1 Q_1 = P_2 Q_2$ for $P_1, P_2 \in \mathcal{P}$ and $Q_1, Q_2 \in \mathcal{Q}$ with $P_1 \neq P_2$ and $Q_1 \neq Q_2$. This implies that $P_1^{-1} P_2 = Q_1 Q_2^{-1}$. Now, since P_1 and P_2 are in $\mathcal{P}$, there is a k-tuple of elements $M = \{a_1, \ldots, a_k\}$ such that $(P_1^{-1} P_2)(M) = M$. However, this implies $(Q_1 Q_2^{-1})(M) = M$. But we know that the permutations in $\mathcal{Q}$ agree on no k-tuples, and so we must have $Q_1 = Q_2$ and, hence, $P_1 = P_2$. Finally, since each product gives a unique permutation of n, there can be no more than $n!$ such products. □

Theorem 11. *For $k < n$, $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$ and $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$.*

Proof. Consider H, the set of all permutations in S_n that send $\{1, 2, \ldots, k\}$ to itself (and hence $\{k+1, \ldots, n\}$ to itself). It is clear that H is a subgroup of S_n isomorphic to $S_k \times S_{n-k}$ and that $|H| = k!(n-k)!$. Since the unordered k-tuple $\{1, 2, \ldots, k\}$ is fixed, none of these are k-derangements of each other, so H is an independent set and $\alpha(\Gamma_{k,n}) \geq k!(n-k)!$.

The cosets of H partition S_n, and each forms an independent set, since $\tau_1, \tau_2 \in \sigma H$ implies that $\tau_1^{-1}\tau_2 \in H$ is not a k-derangement and hence the vertices associated to τ_1 and τ_2 are not connected by an edge. Giving each of the $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ cosets a different color results in a valid coloring of $\Gamma_{k,n}$, so $\chi(\Gamma_{k,n}) \leq \binom{n}{k}$. □

Corollary 12. *For n an odd prime power, $\alpha(\Gamma_{2,n}) = 2(n-2)!$ and $\chi(\Gamma_{2,n}) = \binom{n}{2}$.*

Proof. By Lemma 10 and Theorem 6, we have $\binom{n}{2} \cdot \alpha(\Gamma_{2,n}) \leq n!$. Thus

$$\alpha(\Gamma_{2,n}) \leq n! \cdot \frac{2(n-2)!}{n!} = 2(n-2)!$$

and Theorem 11 gives the reverse inequality. For any graph G, $\chi(G) \geq \omega(G)$, so, by Theorem 6, $\chi(\Gamma_{2,n}) \geq \binom{n}{2}$ and again Theorem 11 gives the reverse inequality. □

5. Further questions

In the last section, we showed that the clique number of the 2-derangement graph is equal to $\binom{n}{2}$ when n is an odd prime power and strictly less than that if n is even (and at least 4). The clique construction of Theorem 6 fails to work when n is odd and not a prime power since there is no field of that cardinality. We believe that in this case the clique number is strictly smaller than $\binom{n}{2}$. For arbitrary k, we have some faint hope that the bounds given in Theorem 11 for $\alpha(\Gamma_{k,n})$ and $\chi(\Gamma_{k,n})$ are actually equalities, but the situation for $\omega(\Gamma_{k,n})$ remains unclear.

In another direction, the numerical evidence is overwhelming that the derangement graphs are Hamiltonian. We hope to explore these and other questions in future work.

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