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Convergence and the Lebesgue Integral

Ryan Vail Thomas

CONVERGENCE AND THE LEBESGUE INTEGRAL

A Masters Thesis

Presented to

the Graduate College of

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science, Mathematics

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ABSTRACT

In this paper, we examine the theory of integration of functions of real variables. Background information in measure theory and convergence is provided and several examples are considered. We compare Riemann and Lebesgue integration and develop several important theorems. In particular, the Monotone Convergence Theorem and Dominated Convergence Theorem are considered under both pointwise convergence and convergence in measure.

KEYWORDS: Lebesgue, Riemann, convergence, integration, measure

This abstract is approved as to form and content

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CHAPTER 1

INTRODUCTION

Integration has been an essential part of the theory of calculus since its beginning. First seriously pursued by both Newton and Leibniz in the late 17th century, the concept has always been recognized for its importance, both theoretically and in applications. However, the rigorous background of integration was slow to develop. In the late 18th and early 19th centuries, mathematicians, in particular Cauchy and Riemann, began constructing and refining the theory of integration. Their work established a firm theoretical background for the sound, but often somewhat loose ideas proposed by Newton and Leibniz. Riemann's well-known theory of integration became the seminal work in the field for half a century, and is still useful in many practical circumstances. In fact, throughout the second half of the 19th century, many believed that the theory of integration had been developed as completely as possible. However, a young French mathematician would soon change that.

At the turn of the 20th century, Henri Lebesgue was completing his doctoral studies at the University of Nancy. His dissertation, titled *Integral, Length, Area*, was published in 1902 in the Annali di Matematica. This work was quickly recognized as a great advance in the field of real analysis [H]. Rather than partitioning the x-axis to find the area under a curve, Lebesgue proposed partitioning the y-axis, then applying measure theory to compute the integral. As we will see, Lebesgue's revolutionary idea greatly expanded the scope of integrable functions, giving rise to several important results concerning integration.

In this paper, we will examine several of these theorems, providing examples where appropriate. Many of the results studied are fairly standard, though we have endeavored to supply our own methods and style in the proofs. Interested readers may refer to Royden ([R]) or Wheeden and Zygmund ([WZ]) for more detail.

CHAPTER 2

BACKGROUND INFORMATION

In order to better appreciate the rich theory we will be investigating, it is necessary to be familiar with some definitions and basic properties. These will be presented in this chapter, with the intention of allowing the remainder of this paper to flow uninterrupted. In particular, the concepts of measure and the convergence of sequences of functions are essential. Accordingly, this chapter will be divided into three sections. In Section 1, we discuss Lebesgue measure. In Section 2, we address measurable functions. In Section 3, we study convergence of various types.

2.1 Lebesgue Measure

The theory of Lebesgue measure was developed by several mathematicians, most notably Lebesgue and his doctoral advisor, Émile Borel, at the turn of the 20th century. Since then, measure has become a key component and an important tool in the study of real analysis. The work of Borel and Lebesgue essentially expands the familiar notions of length and area to allow a real number "size" to be assigned to any suitable subset of a Euclidean space. Here we will primarily consider important results and examples in $\mathbb{R}$.

DEFINITION 2.1.1 *We begin by considering closed intervals* $I_i = \{x : a_i \leq x \leq b_i\}, i = 1, 2, \ldots, n$ *and define* $l(I_i) = b_i - a_i$. *Then for a given set* E, *let* $\mathcal{X}$ *be a countable collection of such intervals which covers* E *and define* $\sigma(\mathcal{X}) = \sum_{I_i \in \mathcal{X}} l(I_i)$. *Then the* Lebesgue outer measure *of* E, *denoted by* m^*E, *is defined as*

$$m^*E = \inf \sigma(\mathcal{X}),$$

where the infimum is taken over all such covers of E.

This definition can easily be extended to general n-dimensional space by simply taking higher-dimensional "intervals" and considering their volume, rather than length, in covering E.

Some properties of the outer measure are readily apparent, such as the fact that the measure of an interval is simply its length, as the smallest possible covering is in fact the interval itself. Another simple property is that if $E_1 \subset E_2$, then $m^*E_1 \leq m^*E_2$. Also, if $\{E_1, E_2, \ldots\}$ is a countable collection of sets, then

$$m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*E_n;$$

that is, countable sub-additivity holds. The following proposition shows another useful property of outer measure concerning countable sets.

PROPOSITION 2.1.1 *If E is countable, then $m^*E = 0$.*

Proof. It suffices to show that $m^*E < \varepsilon$ for any given $\varepsilon > 0$. Since E is countable, we can write $E = \{a_1, a_2, \ldots\}$. Consider

$$\bigcup_{n=1}^{\infty}(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}).$$

Since each member of this union is an open interval, we have

$$\begin{aligned}\sum_{n=1}^{\infty} l(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}) &= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\ &= \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon.\end{aligned}$$

Now $E \subset \bigcup_{n=1}^{\infty}(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}})$. By definition, $m^*E = \inf \sum_{n=1}^{\infty} l(I_n)$. Thus we have that $m^*E \leq \varepsilon$. Now letting $\varepsilon \to 0$, we have $m^*E = 0$. □

The converse to this result is not true in general; that is, there are sets with outer measure zero which are not countable. As an example, we consider the Cantor ternary set on the interval $[0, 1]$ of the real line.

EXAMPLE 2.1.1 To construct the Cantor ternary set, we first divide the interval

$[0,1]$ into thirds, then remove the interior of the middle third; i.e., the open interval $(\frac{1}{3},\frac{2}{3})$. We then divide the two remaining subintervals into thirds, then remove the interior of each of these, leaving four closed subintervals. This process is repeated until, after an infinite number of steps, we are left with the Cantor set, denoted by $\mathcal{C}$; that is, if $\mathcal{C}_i$ denotes the union of the intervals remaining after the i^{th} iteration, then

$$\mathcal{C} = \bigcap_{i=1}^{\infty} \mathcal{C}_i.$$

Now clearly each $\mathcal{C}_i$ is closed, since we started with a closed interval and repeatedly removed open subintervals. Then $\mathcal{C}$ is the intersection of countably many closed sets, and thus is itself closed.

To show $\mathcal{C}$ is uncountable, we will define a function $f : \mathcal{C} \to [0,1]$. First we consider the above construction again in terms of its ternary, or base-3, representation. In base-3, the first step still removes the middle third of the interval, but we now write it as $(0.1, 0.122\ldots)$. However, we could also express this same interval with an infinitely repeating numeral; that is, $(0.1, 0.122\ldots)$ is equivalent to $(0.0222\ldots, 0.2)$. When written in this form, it is clear that the first step in our construction removes all numbers with a 1 as the first digit. Continuing in this manner, the second step will remove any numbers with a 1 as the second digit, and so on until finally any remaining point in $\mathcal{C}$ is composed entirely of 0's and 2's. Now to define f, we will consider the binary representation of an arbitrary element in $[0,1]$. Clearly this numeral will be a string composed of 0's and 1's. Now if we replace each instance of a 1 with a 2, then we will have a similar string composed entirely of 0's and 2's. In this way, for each $b \in [0,1]$, we may produce a ternary numeral, say a, which contains only 0's and 2's. Then if we let $f(a) = b$, we see that clearly f is a surjective function. Then $|\mathcal{C}| \geq |[0,1]|$, and applying Cantor's diagonalisation argument shows that $\mathcal{C}$ is uncountable.

However, $\mathcal{C}$ has outer measure zero. Since each step in the construction effectively doubles the number of subintervals contained in the set, $\mathcal{C}_i$ contains 2^i closed intervals and contains the endpoints of each of these. Furthermore, each of these intervals has length

3^{-i}. Observe that $m^*\mathcal{C}_i \leq (2^i) \times (3^{-i})$. Then since $\mathcal{C}_i$ covers $\mathcal{C}$ for every i, as $i \to \infty$, $m^*\mathcal{C} \to 0$. Hence we have $m^*\mathcal{C} = 0$.

Outer measure can be a powerful tool, as it is defined for all sets, but is not without flaws. In defining a generalized measuring function φ, we might hope to satisfy the following four properties:

1. φ is well-defined for every subset of $\mathbb{R}$,
2. $\varphi(E + x) = \varphi(E)$; that is, φ is translation invariant,
3. if $E = (a, b)$, then $\varphi(E) = b - a$, and
4. if $E_i \cup E_j = \varnothing$ for all $i \neq j$, then $\varphi(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \varphi(E_n)$.

However, in some situations, Conditions 1 and 4 are incompatible. In order to develop a measuring function which preserves countable additivity, the requirement that φ be well-defined for all subsets must be sacrificed. In spite of this shortcoming, outer measure provides the framework for building the Lebesgue measure, which is the key to Lebesgue's theory of integration. We consider the following definition.

DEFINITION 2.1.2 *A set E is said to be* Lebesgue measurable *(henceforth, simply* measurable*) if given $\varepsilon > 0$, there exists an open set G with $E \subset G$ and $m^*(G - E) < \varepsilon$. If E is measurable, its outer measure is simply called its measure, and is denoted mE.*

Carathéodory gives the following theorem, which we will use in proving several results. As we will see, this is often more useful than the definition for proving measurability.

THEOREM 2.1.1 *A set E is* measurable *if for each set A, we have*

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c).$$

Perhaps the first thing to notice about this theorem is that it provides an algebraic thrust to the theory of measurable sets. This theorem is very useful for proving some basic results about measurable sets. It is important to note that $m^*A \leq m^*(A \cap E) + m^*(A \cap E^c)$ is already guaranteed by the sub-additivity of outer measure. Therefore, showing $m^*A \geq m^*(A \cap E) + m^*(A \cap E^c)$ is the key step in using this theorem. Here we will show some key properties of measurable sets, beginning with the following lemma.

LEMMA 2.1.1 *If* $m^*E = 0$*, then* E *is measurable.*

Proof. For any given A, $m^*A \leq m^*(A \cap E) + m^*(A \cap E^c)$, since $A = (A \cap E) \cup (A \cap E^c)$ and sub-additivity holds for outer measure. Now since $m^*E = 0$, and $(A \cap E) \subset E$, we have $m^*(A \cap E) = 0$. Furthermore, since $(A \cap E^c) \subset A$, we have $m^*A \geq m^*(A \cap E^c)$. Thus $m^*A \geq m^*(A \cap E) + m^*(A \cap E^c)$. Hence we have $m^*A = m^*(A \cap E) + m^*(A \cap E^c)$, and thus E is measurable. $\square$

We will now show that countable additivity holds, as desired. We will first require the following lemma.

LEMMA 2.1.2 *If* A *is any set and* $E_1, E_2, \ldots$ *are of disjoint measurable sets, then*

$$m^*(A \cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Proof. We will prove this result by induction on n. We begin by letting $n = 1$. In this case, the result holds by Carathéodory's theorem. Now assume that the statement holds true for $n-1$; that is, assume

$$m^*(A \cap \bigcup_{k=1}^{n-1} E_k) = \sum_{k=1}^{n-1} m^*(A \cap E_i).$$

Since E_n is measurable, for any A', we have $m^*A' = m^*(A' \cap E_n) + m^*(A' \cap E_n^c)$. Then if we let $A' = A \cap (\bigcup_{i=1}^{n} E_i)$, we have

$$\begin{aligned}
m * [A \cap (\bigcup_{i=1}^{n} E_i)] &= m^*[A \cap (\bigcup_{i=1}^{n} E_i) \cap E_n] + m^*[A \cap (\bigcup_{i=1}^{n} E_i) \cap E_n^c] \\
&= m^*(A \cap E_n) + m^*[A \cap (\bigcup_{i=1}^{n-1} E_i)] \\
&= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) \\
&= \sum_{i=1}^{n} m^*(A \cap E_i).
\end{aligned}$$

$\square$

Note that if $A = \bigcup_{i=1}^{\infty} E_i$, then $m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*(E_i)$; that is, this lemma establishes finite additivity. To extend this to countable additivity, we present the following proposition.

PROPOSITION 2.1.2 *If $\{E_i\}$ is a sequence of pairwise disjoint measurable sets, then*

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} mE_i.$$

Proof. By the Lemma 2.1.2, we have $m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^*E_i$. Since $\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i$, we have $m^*(\bigcup_{i=1}^{n} E_i) \leq m^*(\bigcup_{i=1}^{\infty} E_i)$. Then

$$\sum_{i=1}^{n} m^*E_i \leq m^*(\bigcup_{i=1}^{\infty} E_i).$$

Now letting $n \to \infty$, we have

$$\sum_{i=1}^{\infty} m^*E_i \leq m^*(\bigcup_{i=1}^{\infty} E_i).$$

By the sub-additivity of the outer measure, we have

$$m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*E_i.$$

Therefore we have

$$\sum_{i=1}^{\infty} m^*E_i = m^*(\bigcup_{i=1}^{\infty} E_i).$$

□

We now consider a final result which gives some very useful properties of measurable sets. Here it is presented without proof; however, interested readers may refer to Royden for more information.

THEOREM 2.1.2 *If $\{E_1, E_2, \ldots\}$ is a countable collection of measurable sets, then*

1. E_i^c is measurable,

2. $\bigcup_{k=1}^{\infty} E_k$ is measurable, and

3. $\bigcap_{k=1}^{\infty} E_k$ *is measurable.*

That is, complements of measurable sets are measurable, as are intersections and unions of countable collections of measurable sets.

Most of the sets we need in analysis are measurable. In particular, all open and closed sets are measurable. However, using a cardinality argument, we can show that "measurable sets" are only a tiny minority among all the possible subsets of the real numbers. That does not mean that constructing a non-measurable set is easy. As a matter of fact, such a construction relies on the Axiom of Choice and some clever techniques. We refer interested readers to Royden for an example of a non-measurable set.

2.2 Measurable Functions

DEFINITION 2.2.1 *An extended (that is, possibly infinite) real-valued function* f *is said to be Lebesgue measurable if its domain is measurable and it satisfies one of the following (where* $\alpha \in \mathbb{R}$*):*

1. the set $\{x : f(x) > \alpha\}$ *is measurable,*

2. the set $\{x : f(x) \geq \alpha\}$ *is measurable,*

3. the set $\{x : f(x) < \alpha\}$ *is measurable, or*

4. the set $\{x : f(x) \leq \alpha\}$ *is measurable.*

In fact, these 4 statements are equivalent. Observe that $1 \Leftrightarrow 4$ and $2 \Leftrightarrow 3$, since the sets involved are complementary. We note that if E is the domain of f, then

$$\{x : f(x) > \alpha\} = E \setminus \{x : f(x) \leq \alpha\}.$$

A similar statement shows that $2 \Leftrightarrow 3$. To show that $1 \Rightarrow 2$, we observe that

$$\bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\} = \{x : f(x) \geq \alpha\}.$$

Now since the countable intersection of measurable sets is again measurable, we have that

$\{x : f(x) \geq \alpha\}$ is measurable. In order to show that $2 \Rightarrow 1$, we consider

$$\bigcup_{n=1}^{\infty}\{x : f(x) > \alpha + \frac{1}{n}\} = \{x : f(x) > \alpha\}.$$

Again, since the union of measurable sets is measurable, we have that $\{x : f(x) > \alpha\}$ is measurable.

The following propositions will illustrate some "nice" properties of measurable functions. First we will show that several properties of continuous functions can be carried over to measurable functions.

PROPOSITION 2.2.1 *If c is a constant and f, g are two measurable real-valued functions defined on a measurable set E, then the functions:*

1. $f + c$

2. cf

3. $f + g$

4. $f - g$

5. fg

are also measurable.

Proof. To prove part 1, assume f is measurable. Then $\{x : f(x) > \alpha\}$ is a measurable set. Hence a set $\{x : f(x) + c > \alpha\} = \{x : f(x) > \alpha - c\}$ is also measurable. Thus $f(x) + c$ is a measurable function. Part 2 can be proved similarly. To prove part 3, suppose $f(x) + g(x) > \alpha$. Then $f(x) > \alpha - g(x)$. Now since f and g are real-valued, there is a $q \in \mathbb{Q}$ with $f(x) > q > \alpha - g(x)$. Then we can write

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{q\in\mathbb{Q}}(\{x : f(x) > q\} \cap \{x : g(x) > \alpha - q\}).$$

The right-hand side is measurable, since it is a countable union of finite intersections of measurable sets. Thus $f + g$ is measurable. Part 4 can be proved in a similar fashion as part

3. To prove part 5, we write

$$\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) \le -\sqrt{\alpha}\}.$$

The set on the right-hand side is measurable, being the union of two measurable sets. Thus the set $\{x : f^2(x) > \alpha\}$ is measurable and hence f^2 is a measurable function. That fg is measurable follows from the equation $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$. $\square$

The term "almost everywhere" will have some significance in the sections to follow Here we present the definition.

DEFINITION 2.2.2 *A property is said to hold* almost everywhere *if its conditions are satisfied everywhere except on a set of measure zero.*

We will now apply this definition in the following proposition.

PROPOSITION 2.2.2 *If f is a measurable function and $f = g$ almost everywhere, then g is also measurable.*

Proof. Since $f = g$ a.e., there exists a set $F = \{x : f(x) \neq g(x)\}$ with $mF = 0$. Now we consider the set $\{x : g(x) > \alpha\}$. Since $g = f$ a.e., we can write this set as

$$\{x : g(x) > \alpha\} = (\{x : f(x) > \alpha\} \cup \{x \in F : g(x) > \alpha\}) \setminus \{x \in F : g(x) \le \alpha\}.$$

Now $\{x : f(x) > \alpha\}$ is measurable, since f is a measurable function. Furthermore, $\{x \in F : g(x) > \alpha\}$ and $\{x \in F : g(x) \le \alpha\}$ are also measurable, since they are subsets of F and $mF = 0$. Thus the right-hand side is a measurable set, and hence g is measurable. $\square$

Measurable functions will be our primary concern in studying the theory of integration. As with measurable sets, many of the functions we need in analysis are indeed measurable. However, nonmeasurable functions do exist. For example, any function whose domain is not measurable is a nonmeasurable function.

2.3 Convergence

Convergence is one of the most important tools available in the study of real analysis. This is particularly true in integration theory. Virtually all of the results we will consider rely on some type of convergence to determine which functions are to be considered "good". Here we will examine several types of convergence, each with an important role in building the theory of calculus; however, in the chapters on integration we will primarily be concerned with convergence almost everywhere and convergence in measure.

We begin with perhaps the most intuitively simple type of convergence, that of a sequence converging pointwise.

DEFINITION 2.3.1 *A sequence of functions f_n is said to converge* pointwise *to a function f on a set E if for every given $x \in E$ and $\varepsilon > 0$, there exists a natural number N, depending on x and ε, such that for all $n \geq N$ we have $|f(x) - f_n(x)| < \varepsilon$.*

As an example, consider the following.

EXAMPLE 2.3.1 Consider the sequence of functions $f_n(x) = x^n$ on [0,1]. It is clear that as $n \to \infty$, $f_n(x)$ converges to $f(x)$ given by:

$$f(x) = \begin{cases} 0 : x \in [0, 1) \\ 1 : x = 1 \end{cases}$$

So when $x \in [0, 1)$, the sequence converges to 0, but at $x = 1$ the limit of the sequence is 1. Thus we say that the sequence converges pointwise. We will further examine this sequence of functions in the context of uniform convergence.

The concept of pointwise convergence can be relaxed somewhat to convergence *almost everywhere* (a.e.), allowing many of the following results to be applied to a much broader class of functions. This type of convergence will also serve as the basis for the convergence theorems we present in Chapter 4.

DEFINITION 2.3.2 *A sequence of functions f_n is said to converge* almost everywhere *to a function f if f_n meets the requirements for pointwise convergence to f every-*

where on E except *a set of measure zero.*

EXAMPLE 2.3.2 We begin by observing that since the rational numbers are countable, they may be enumerated. Then we define

$$f_n(x) = \begin{cases} 1 : x = q_1, q_2, \ldots, q_n \\ 0 : \text{ otherwise} \end{cases}$$

where $\{q_1, q_2, \ldots\}$ is the set of rational numbers in the interval $[0, 1]$, denoted by $\mathbb{Q}_{[0,1]}$. Then f_n converges to 0 everywhere except when $x \in \mathbb{Q}_{[0,1]}$. Now since $\mathbb{Q}_{[0,1]} \subset \mathbb{Q}$, and as we have already seen, the measure of a countable set is zero, we have that f_n converges to 0 everywhere except on a set of measure zero; that is, f_n converges to 0 *almost everywhere.*

Pointwise convergence can be further extended to the concept of *uniform* convergence.

DEFINITION 2.3.3 *A sequence of functions is said to converge uniformly on E if for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that* for all $x \in E$ *and for every $n \geq N$, we have $|f(x) - f_n(x)| < \varepsilon$, or equivalently, if*

$$\lim_{n\to\infty} \sup\{\, |f_n(x) - f(x)| : x \in \boldsymbol{E}\,\} = 0.$$

This condition is much more stringent than pointwise convergence, so it is often the case that a sequence of functions converges in a pointwise sense without converging uniformly.

Recalling example 2.3.2, we can see that the convergence in this case is not uniform. In fact, for any given natural number N, we have, for $m > N$, that $|f(q_m) - f_m(q_m)| = 1$. Thus the criterion for uniform convergence cannot be satisfied if ε is chosen to be less than 1 beforehand.

The following result provides a useful connection between convergence almost everywhere and uniform convergence.

THEOREM 2.3.1 (Egorov's Theorem) *If f_n is a sequence of measurable functions* *that converge to a real-valued function f a.e. on a measurable set E, with $mE < \infty$, then given $\eta > 0$, there is a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \setminus A$.*

The proof of this result will be omitted. Interested readers may refer to Wheeden [**?**]. Essentially, this shows that convergence a.e. on a set implies uniform convergence on a slightly smaller set. Returning to our definition of convergence a.e., we see that this is intuitively clear, since the set where the sequence does not converge has measure zero.

We now turn our attention to convergence in measure.

DEFINITION 2.3.4 *A sequence f_n of measurable functions is said to converge to f in measure if for every given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have*

$$m\{x : |f(x) - f_n(x)| \geq \varepsilon\} < \varepsilon.$$

Our first result is almost immediate, and highlights the relationship between pointwise convergence and convergence in measure.

THEOREM 2.3.2 *Let f and f_n be measurable and finite a.e. on E, with $mE < \infty$. If f_n converges to f almost everywhere, then f_n converges to f in measure on E.*

Proof. Given $\varepsilon > 0$. By Egorov's Theorem, there exists a closed subset F of E and $N \in \mathbb{N}$ such that $m(E \setminus F) < \varepsilon$ and $|f(x) - f_n(x)| < \varepsilon$ for all $x \in F, n > N$. Then if $n > N$, we have that $\{x \in E : |f(x) - f_n(x)| > \varepsilon\} \subset E \setminus F$. It then follows that

$$m\{x \in E : |f(x) - f_n(x)| > \varepsilon\} \leq m(E \setminus F) < \delta,$$

and thus f_n converges to f in measure on E. □

The following example shows that the converse of Theorem 2.3.2 is not true.

EXAMPLE 2.3.3 We begin by defining a sequence of intervals $\{A_n\}$, where the A_i are subintervals of $[0, 1]$ and are constructed by halving the previous terms; that is, we begin with $A_1 = [0, 1]$, then generate A_2 and A_3 by taking the two halves of $[0, 1]$ so that

$A_2 = [0, \frac{1}{2}]$ and $A_3 = [\frac{1}{2}, 1]$. Subsequent terms are generated in the same method, so that the next terms are $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], \ldots$. Then we let $\{f_n\}$ be the sequence of characteristic functions of the members of $\{A_n\}$; that is,

$$f_n(x) = \begin{cases} 1 : x \in A_n \\ 0 : \text{ otherwise} \end{cases}$$

We claim that this sequence of functions converges in measure to 0. Observe that for a given n, the length of each interval is less than $\frac{1}{n}$ and $|f_n(x) - 0| > 0$ only for those x contained in the given interval. Then given $\varepsilon > 0$, there is a sufficiently large $N \in \mathbb{N}$ such that if $n \geq N$, then $m\{x : |f_n(x) - 0| > \varepsilon\} < \varepsilon$. Thus $\{f_n\}$ converges in measure to 0.

However, if we consider the same sequence of functions under pointwise convergence, we see that the sequence diverges for every $x \in [0, 1]$. Thus convergence in measure does not imply convergence almost everywhere.

Although convergence in measure does not directly imply pointwise convergence, we do have the following result which allows us to make a connection between the two types of convergence.

THEOREM 2.3.3 *If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere.*

Proof. For each given k, there exists N_k such that when $n \geq N_k$,

$$m\{x : |f(x) - f_n(x)| > 2^{-k}\} < 2^{-k}.$$

Now let $E_k = \{x : |f_{N_k}(x) - f(x)| > 2^{-k}\}$. Then $mE_k < 2^{-k}$. Now if $x \notin \bigcup_{i=k}^{\infty} E_i$, then

$$x \in (\bigcup_{i=k}^{\infty} E_i)^c = \bigcap_{i=k}^{\infty} E_i^c.$$

Then we have

$$|f_{N_i}(x) - f(x)| < 2^{-i},$$

for all $i \geq k$. Thus $f_{N_i}(x) \to f(x)$. Now we define $A = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i$. Then if $x \notin A$, then $f_{N_i}(x) \to f(x)$. For any k,

$$mA \leq m(\bigcup_{i=k}^{\infty} E_i) \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}.$$

Therefore $mA = 0$. □

This theorem, while interesting in its own right, is very useful in Chapter 4, in which we prove several convergence theorems. It will serve as the basis for developing another interesting property.

The following result is often called the Cauchy criterion for convergence in measure.

THEOREM 2.3.4 *A sequence of functions $\{f_n\}$, defined and finite almost everywhere on a set E, converges in measure on E if and only if for each $\varepsilon > 0$,*

$$\lim_{n,k\to\infty} m\{x \in E : |f_n(x) - f_k(x)| > \varepsilon\} = 0.$$

Proof. First, suppose that $\{f_n\}$ converges to f in measure. Then for a given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n > N$, then $m\{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2}\} < \frac{\varepsilon}{2}$. Now consider $|f_n(x) - f_k(x)|$. Applying the triangle inequality, we have

$$|f_n(x) - f_k(x)| \leq |f(x) - f_n(x)| + |f(x) - f_k(x)|.$$

If $|f_n(x) - f_k(x)| \geq \varepsilon$, then at least one of $|f(x) - f_n(x)|$ and $|f(x) - f_k(x)|$ must be greater than or equal to $\frac{\varepsilon}{2}$. Hence we have

$$\{x : |f_n(x) - f_k(x) \geq \varepsilon\} \subset \{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2}\} \cup \{x : |f(x) - f_k(x)| \geq \frac{\varepsilon}{2}\}.$$

Then when $k, n \geq N$, we have

$$m\{x : |f_n(x) - f_k(x) \geq \varepsilon\} \leq m\{x : |f(x) - f_n(x)| \geq \frac{\varepsilon}{2}\} + m\{x : |f(x) - f_k(x)| \geq \frac{\varepsilon}{2}\}$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then letting $n, k \to \infty$, we have the desired result. To prove the converse, suppose that

$$\lim_{n,k \to \infty} m\{x \in E : |f_n(x) - f_k(x)| > \varepsilon\} = 0$$

for each $\varepsilon > 0$. By Theorem 2.3.3, there exists a subsequence $\{f_{n_k}\}$ that converges to a measurable function f almost everywhere. Since $\{f_n\}$ is a Cauchy sequence, for the given $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n, n_k \geq N$, we have

$$m\{x : |f_n(x) - f_{n_k}(x)| \geq \varepsilon\} < \varepsilon.$$

Now since $\{f_{n_k}\} \to f$, if we fix n and let $n_k \to \infty$, we obtain

$$m\{x : |f_n(x) - f(x)| \geq \varepsilon\} < \varepsilon,$$

and thus $\{f_n\}$ converges in measure to f. □

We now provide an example of a sequence of functions which is convergent in measure.

EXAMPLE 2.3.4 We consider the subsets of $\mathbb{R}$ given by $E_n = (0, \frac{1}{n})$, where $n \in \mathbb{N}$ and let f_{E_n} be the characteristic function over E_n. We begin by observing that each f_{E_n} is finite and measurable. Consider the sequence $\{f_{E_n}\}_{n \in \mathbb{N}}$. We claim that this sequence converges in measure to 0. In fact, if we consider the n^{th} member of the sequence, f_{E_n}, it is clear that $f_{E_n}(x) > 0$ only when $x \in E_n$. Then for any ε with $0 < \varepsilon < 1$, we have

$$m\{x : |f_{E_n}(x) - 0| \geq \varepsilon\} = \frac{1}{n},$$

and thus

$$\lim_{n\to\infty} m\{x : |f_{E_n}(x) - 0| \geq \varepsilon\} = 0.$$

Therefore the given sequence of functions converges in measure to 0.

Convergence and measure are the fundamental elements upon which the theory of integration is constructed, particularly in the case of Lebesgue integration. As a result, we will often return to the results discussed in this chapter in order to study the integration theory in the next chapter.

CHAPTER 3

INTEGRATION

Beginning around the turn of the 19th century, several mathematicians began working on a rigorous footing for the calculus developed by Newton and Leibniz. Cauchy made important advances in the theory and provided a groundwork for others, in particular Riemann, to build upon. Later the theory of integration developed further with the work of Lebesgue. In this chapter, we will trace the development of the integral from Riemann to Lebesgue and provide examples to illustrate advantages and disadvantages of each.

3.1 Riemann Integrals

In 1854, Bernhard Riemann gave his definition of the integral that bears his name, building upon the work of Cauchy and others. This provided the primary theoretical background for integration for nearly 50 years, and is still very useful, particularly in practical settings and applications. Many students' first experience with integration consists of simple exercises using Riemann's method. In this section we will examine Riemann's definition and consider an example which is not Riemann-integrable.

DEFINITION 3.1.1 *Let f be a nonnegative, real-valued function defined on the interval $[a, b]$ and let*

$$P = \{\alpha_0, \alpha_1, \ldots, \alpha_n\},$$

where

$$a = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n = b,$$

be a partition of $[a, b]$. For each such partition, define the upper and lower Riemann sums S and s respectively, by

$$S = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) M_i$$

and

$$s = \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1}) m_i,$$

where

$$M_i = \sup_{\alpha_{i-1} < x \leq \alpha_i} f(x),$$

and

$$m_i = \inf_{\alpha_{i-1} < x \leq \alpha_i} f(x).$$

Then we define the upper and lower Riemann integrals by

$$\overline{\int_a^b} f(x) dx = \inf S,$$

and

$$\underline{\int_a^b} f(x) dx = \sup s,$$

where the infimum and supremum are taken over all possible partitions. If these two integrals agree, then we say that f is Riemann integrable.

Although Riemann's definition is not comprehensive, it is sufficient for many situations. For instance, all bounded continuous functions are integrable using Riemann's definition.

THEOREM 3.1.1 *Every continuous function f on $[a, b]$ is Riemann integrable.*

For the proof of this result, we will use the following lemma.

LEMMA 3.1.1 *(Extreme Value Theorem) A continuous real-valued function f on a closed interval $[a, b]$ assumes its maximum and minimum vales on $[a, b]$; that is, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a, b]$.*

Proof of the lemma. Let $M = \sup\{f(x) : x \in [a, b]\}$. Then M is finite. If M is not finite, then f is not bounded on $[a, b]$, and so for each $n \in \mathbb{N}$ there corresponds an $x_n \in [a, b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$

which converges to some real number, say x_0. Then since f is continuous, we have

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0).$$

However, since f is assumed to be unbounded, we also have

$$\lim_{k\to\infty} f(x_{n_k}) = \infty.$$

This is a contradiction. Thus, f is bounded, and hence M is finite. Now for each $n \in \mathbb{N}$ there exists $y_n \in [a, b]$ such that $M - \frac{1}{n} < f(y_n) \leq M$, and thus $\lim f(y_n) = M$. Then there is a subsequence $\{y_{n_k}\}$ which converges to some real number in $[a, b]$, say y_0. Since f is continuous, $\lim_{k\to\infty} f(y_{n_k}) = f(y_0)$. Now since $\{f(y_{n_k})\}$ is a subsequence of $\{f(y_n)\}$, we have

$$\lim_{k\to\infty} f(y_{n_k}) = \lim_{k\to\infty} f(y_n) = M.$$

That is, $f(y_0) = M$. Thus f assumes its maximum value at y_0. It can be shown in a similar fashion that f assumes its minimum value at x_0. □

Proof of the main theorem. Let $\varepsilon > 0$. Then since f is continuous, by definition there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. Now partition the x-axis as follows: let $P = \{a = x_0 < x_1 < \ldots < b = x_n\}$, where $\max\{x_k - x_{k-1}, k = 1, 2, \ldots, n\} < \delta$. By the preceding lemma, f assumes maximum and minimum values on each interval $[x_{k-1}, x_k]$. Then we have

$$M(f, [x_{k-1}, x_k]) - m(f, [x_{k-1}, x_k]) < \frac{\varepsilon}{b-a}$$

for each k. Then

$$S(f, P) - s(f, P) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(x_k - x_{k-1}) = \varepsilon,$$

and thus f is Riemann integrable. □

This theorem, along with the next, provides one of the most powerful results of Riemann integration; namely, that any bounded, almost everywhere continuous function is Riemann integrable. This allows Riemann's theory to be used in a wide array of applications and practical situations.

THEOREM 3.1.2 *A bounded function f on $[a, b]$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.*

Proof. We begin by noting that the sufficiency clause is simply an extension of Theorem 3.1.1 above, and can be shown by simply modifying the proof used there to account for continuity almost everywhere; in particular, since the set where f is not continuous has measure zero, it also has integral zero. To show the converse, suppose f is bounded and Riemann integrable on $[a, b]$. Let $\{P_k\}$ be a sequence of partitions of $[a, b]$, with the measure of the partitions tending to zero and the partitioning points of P_k given by $x_1^k < x_2^k < \ldots$. Define $\upsilon_k(x), \gamma_k(x)$ on each interval $[x_i^k, x_{i+1}^k)$ as the upper and lower bounds of f on that interval. Then since f is bounded on $[a, b]$, it is clear that $\upsilon_k(x), \gamma_k(x)$ are bounded. Furthermore, there exists a set F with $mF = 0$ and outside of which $\upsilon = f = \gamma$. Now if $x \neq x_i^k$ for any k, i and if $x \notin F$, we claim that f is continuous at x. Observe that if f is not continuous at x and if x is not a partitioning point of P_k, then there exists $\varepsilon > 0$ with $\upsilon_k(x) - \gamma_k(x) \geq \varepsilon$. But $x \notin F \Rightarrow \upsilon(x) - \gamma(x) < \varepsilon$, hence a contradiction. Thus f is continuous a.e. on $[a, b]$. $\square$

EXAMPLE 3.1.1 Let us consider the characteristic function of the rational numbers over the reals. It is clear that any attempt at partitioning the x-axis, no matter how fine the division, will result in each partition containing both rational and irrational numbers. Then the supremum over any partition is 1, while the infimum is 0, which in turn forces the upper and lower Riemann sums to 1 and 0, respectively. Then since the upper and lower sums do not agree, the Riemann integral does not exist.

We could also examine this function in light of Theorem 3.1.2. Recalling that "almost everywhere" means everywhere except on a set of measure zero, we see that this function cannot be Riemann integrable, since its set of discontinuities (i.e., the irrational numbers in $[0, 1]$) has measure 1.

We will return to this example in the next section and examine the function again in the context of Lebesgue integration.

Despite its elegance and utility in integrating continuous functions, the Riemann integral is far from being completely satisfactory. Even when extended to allow improper integrals, many important functions are not Riemann integrable. It is also difficult to develop the kind of convergence theorems that play such a significant role in Lebesgue integration. These shortcomings led to Lebesgue's further refinement of the theory of integration and the development of the Lebesgue integral.

3.2 Lebesgue Integrals

We begin our study of Lebesgue integration with some preliminary definitions, following standard conventions [R]. If E is a measurable set, we define $\chi_E(x)$ as the characteristic function of E. A linear combination of the form

$$\phi(x) = \sum_{i=1}^{n} d_i \chi_{E_i}(x),$$

is called a simple function, where E_i is measurable for all i and the d_i are constants. In fact, $\phi(x)$ is simple if and only if it takes a finite number of values. It is clear the ϕ is measurable. If $d_1, \ldots, d_n$ are the distinct non-zero values of ϕ, then $\phi(x)$ can be expressed as $\phi = \sum d_i \chi_{E_i}(x)$, where $E_i = \{x : \phi(x) = d_i\}$, and $E_i \cap E_j = \varnothing$ for $i \neq j$.

Our definition of the Lebesgue integral will be based on the integral of such simple functions, which we will now define. By the above, $\phi = \sum d_i \chi_{E_i}(x)$ and E_i is measurable for all i. Then we define the integral of ϕ as

$$\int \phi = \sum_{i=1}^{n} d_i m E_i.$$

We will now utilize the above information to define the Lebesgue integral.

DEFINITION 3.2.1 *If f is a bounded measurable function defined on a measurable set E of finite measure and ϕ is a simple function with $\phi \geq f$, then we define the Lebesgue integral of f over E as*

$$\int_E f(x)dx = \inf \int_E \phi(x)dx,$$

where the infimum is taken over all possible simple functions satisfying the above criteria.

Note that henceforth the term "integrable" will be understood to mean "Lebesgue integrable". We also will limit the term "integrable" to those functions with $\int f < \infty$.

In the general case, we use the following definition for the Lebesgue integral. Here we denote the positive and negative components of f by $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = -\min\{f(x), 0\}$, respectively. We will also suppress x and dx in the notation.

DEFINITION 3.2.2 *A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. When f is integrable, we define*

$$\int_E f = \int_E f^+ - \int_E f^-.$$

This definition expands the class of functions considered integrable, as well as making it much easier to handle signed functions. In addition, it allows us to develop several useful and familiar properties of the integral, presented without proof in the following proposition.

PROPOSITION 3.2.1 *If f and g are integrable over E, then so are cf and $f + g$, where c is a constant. Moreover, we have*

1. $\int_E cf = c \int_E f$;

2. $\int_E f + g = \int_E f + \int_E g$;

3. If $f \leq g$ a.e., then $\int_E f \leq \int_E g$;

4. If A and B are subsets of E with $A \cap B = \varnothing$, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

The following result shows an important connection between Riemann and Lebesgue integration; namely, any function that is Riemann integrable over an interval is also Lebesgue integrable. Here we will denote the Riemann integral as "$\mathcal{R}\int$" and the Lebesgue integral with the usual notation.

THEOREM 3.2.1 *If f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and*

$$\int_a^b f = \mathcal{R}\int_a^b f.$$

Proof. Suppose f is Riemann integrable on $[a, b]$. Let L, U denote the collections of lower and upper Riemann sums, respectively. Then we have $\inf U = \sup L$, where the infimum is taken over U and the supremum is taken over L. Now since each step function used in the calculation of the Riemann integral can also be considered as a simple function, we have

$$\sup L \leq \sup \int \varphi \leq \inf \int \psi \leq \inf U,$$

where the supremum in the middle is taken over all the allowable simple functions $\varphi \leq f$, and the infimum in the middle is taken over all allowable simple functions $\psi \geq f$. Since $\sup L = \inf U$, we have

$$\sup \int \varphi = \inf \int \psi.$$

Thus f is Lebesgue integrable. □

This result shows that any function which is Riemann integrable is also Lebesgue integrable, which is certainly to be expected. However, to understand the advantages of Lebesgue integration, we need to find an example of a function which is Lebesgue integrable but not Riemann integrable. Recall the characteristic function of the rationals over $[0, 1]$. We have already seen that this function is not Riemann integrable, since the upper and lower Riemann sums diverge for any possible partition. However, it is in fact Lebesgue integrable. If we consider the domain of this function as two disjoint sets, defined as

$$E_1 = \mathbb{Q} \cap [0, 1],$$

and

$$E_2 = [0,1] \setminus \mathbb{Q},$$

then we have $f(x) = (\chi_{E_1} \times 1) + (\chi_{E_2} \times 0) = \chi_{E_1}$. Then the Lebesgue integral is given by

$$\int_E \chi_{E_1} = mE_1 = 0,$$

since E_1 is countable.

We will consider another interesting example related to this theorem, but will require the following theorem. The theorem provides another useful result, particularly interesting for its use in determining whether a given function is integrable or not.

THEOREM 3.2.2 *Let f be measurable on E. Then f is integrable over E if and only if $|f|$ is integrable over E.*

Proof. Suppose f is integrable on E. Then by definition, $\int_E f = \int_E f^+ - \int_E f^-$ is finite. But then $\int_E f^+$ and $\int_E f^-$ must both be finite. Then clearly their sum is finite; that is,

$$\int_E f^+ + \int_E f^- = \int_E (f^+ + f^-).$$

But $f^+ + f^- = |f|$, and so we have that $|f|$ is integrable. To show the converse, suppose $|f|$ is integrable on E. That is, $\int_E |f|$ is finite. Then since

$$\int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^-,$$

and

$$\left| \int_E f \right| \leq \int_E f^+ + \int_E f^-,$$

we have that $\left|\int_E f\right| \leq \int_E |f|$, and thus f is integrable. $\square$

We may now demonstrate that the restriction in Theorem 3.2.1 requiring the domain of integration to be bounded is necessary, as we will see in the following example.

EXAMPLE 3.2.1 Consider the function $\frac{\sin x}{x}$ on the interval $[0, \infty)$. We claim that the improper Riemann integral of this function exists, but the Lebesgue integral does not. Here we would like to note that this does not contradict Theorem 3.2.1, since the domain of integration is not bounded above.

We begin by computing the Riemann integral. It is not hard to check that the usual tactics for computing integrals, such as integration by parts, will not work in this case. We will consider the following series method, proposed by Titchmarsh [T]. Consider

$$\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots.$$

We can write the sum of the first n terms of this series in the following manner.

$$s_n(x) = \int_0^x (\cos t + \cos 2t + \cos 3t + \ldots + \cos nt)dt.$$

Consider the sum

$$\sum_{k=0}^{n} e^{ikx} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}}.$$

Applying Euler's Formula, we have

$$\sum_{k=0}^{n} (\cos kx + i \sin kx) = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}}.$$

Multiplying the right-hand side by its conjugate yields

$$\sum_{k=0}^{n} \cos kx + i \sum_{k=0}^{n} \sin kx = \left(\frac{1 - e^{i(n+1)x}}{1 - e^{ix}}\right)\left(\frac{1 + e^{ix}}{1 + e^{ix}}\right).$$

Through algebraic manipulation, we find that the real part of $\sum_{k=0}^{n} e^{ikx}$ is

$$\sum_{k=0}^{n} \cos kx = \frac{\sin(n + \frac{1}{2})t - \sin \frac{1}{2}t}{2 \sin \frac{1}{2}t}.$$

Hence

$$\begin{aligned}
s_n(x) &= \int_0^x (\cos t + \cos 2t + \cos 3t + \ldots + \cos nt)dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t - \sin\frac{1}{2}t}{2\sin\frac{1}{2}t}dt \\
&= \int_0^x \left(\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t} + \frac{\sin(n+\frac{1}{2})t}{t} - \frac{\sin(n+\frac{1}{2})t}{t} - \frac{\sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\right)dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t}{t}dt + \int_0^x \left[\sin(n+\frac{1}{2})t\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\right]dt - \int_0^x \frac{\sin\frac{1}{2}t}{2\sin\frac{1}{2}t}dt \\
&= \int_0^{(n+\frac{1}{2})x} \frac{\sin u}{u}du + \int_0^x \left[\sin(n+\frac{1}{2})t\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\right]dt - \frac{x}{2}.
\end{aligned}$$

A straightforward computation shows that $\int_0^h \frac{\sin u}{u}du$ attains its absolute maximum at $h = \pi$. We can use the periodicity of the sine function to sum the series. If we let x have some fixed value, with $0 < x < 2\pi$, then we have

$$\lim_{n\to\infty} \int_0^{n+\frac{1}{2}} \frac{\sin u}{u}du = \int_0^\infty \frac{\sin u}{u}du.$$

We denote this integral by I. Now consider

$$\int_0^x \left[\sin(n+\frac{1}{2})t\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\right]dt.$$

Here we may use integration by parts to obtain

$$-\left(\frac{1}{2\sin\frac{1}{2}x} - \frac{1}{x}\right)\frac{\cos(n+\frac{1}{2})x}{n+\frac{1}{2}} + \frac{1}{n+\frac{1}{2}}\int_0^x \left[\frac{d}{dt}\left(\frac{1}{2\sin\frac{1}{2}t} - \frac{1}{t}\right)\cos(n+\frac{1}{2})t\right]dt. \quad (3.1)$$

But then as $n \to \infty$, the value of (3.4) tends to zero, as each of the summands contains an n term in the denominator. Thus if $s(x) = \lim_{n\to\infty} s_n(x)$, then we have

$$s(x) = I + 0 - \frac{x}{2}.$$

Evaluating at $x = \pi$, we have

$$I = \frac{\pi}{2}.$$

Thus the improper Riemann integral exists, with the value $\frac{\pi}{2}$.

Now to show the Lebesgue integral does not exist, we refer to Theorem 3.2.2, which states that the Lebesgue integral of a function f exists if and only if the Lebesgue integral of $|f|$ does. Then we can consider

$$\int_0^\infty \left|\frac{\sin x}{x}\right| dx.$$

It is well known that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and that $\frac{\sin(k\pi)}{k\pi} = 0$ for all $k \in \mathbb{N}$, so since we are considering $\left|\frac{\sin x}{x}\right|$ we can see that on each $(k\pi, (k+1)\pi)$ we will have a positive area enclosed by the curve and the x-axis. We claim that the sum of the areas of these regions is infinite. To illustrate this, we will construct a rectangular subset of each region and compute its area, then compute the sum of these areas.

On the interval $[0, \pi]$, we consider the subinterval $[\frac{\pi}{2}, \frac{3\pi}{4}]$ and construct a rectangle that has base $\frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}$, and height given by the function value at the right end point; that is, the function value at $\frac{3\pi}{4}$, which is $\frac{\frac{\sqrt{2}}{2}}{\frac{3\pi}{4}} = \frac{2\sqrt{2}}{3\pi}$. Then this rectangle has area $\frac{2\sqrt{2}}{3\pi} \times \frac{\pi}{4} = \frac{\sqrt{2}}{6}$. In the same manner, we construct a similar rectangle on the interval $[\frac{3\pi}{2}, \frac{7\pi}{4}]$. Then this rectangle has base $\frac{\pi}{4}$ and height $\frac{2\sqrt{2}}{7\pi}$, and thus it has area $\frac{\pi}{4} \times \frac{2\sqrt{2}}{7\pi} = \frac{\sqrt{2}}{14}$. In general, the n^{th} rectangle will have height given by $\frac{\sqrt{2}}{2} \times \frac{4}{(4n-1)\pi}$ and base $\frac{\pi}{4}$.

In considering the total area enclosed by the curve, we first point out that the above construction is merely a proper portion of the total area. Then we have

$$\begin{aligned}
\int_0^\infty \left|\frac{\sin x}{x}\right| dx &\geq \sum_{n=1}^{\infty} \frac{\pi}{4} \times \frac{\sqrt{2}}{2} \times \frac{4}{(4n-1)\pi} \\
&= \sum_{n=1}^{\infty} \frac{\sqrt{2}}{2(4n-1)} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{2(4n-1)} \\
&\geq \frac{1}{8}\sum_{n=1}^{\infty} \frac{1}{n} = \infty.
\end{aligned}$$

Thus the integral diverges, and thus the Lebesgue integral does not exist.

Lebesgue's theory of integration allows for a much larger class of functions to be integrated. We are no longer restricted to bounded, almost everywhere continuous functions. In the next chapter, we will explore the convergence theorems, arguably the most powerful results to come out of Lebesgue integration.

CHAPTER 4

CONVERGENCE THEOREMS

The convergence theorems we will discuss in this chapter provide powerful tools for Lebesgue integration. Each of the theorems gives conditions for interchanging the operations of computing the Lebesgue integral and taking the pointwise limit. These theorems are, in fact, one of the primary advantages of the Lebesgue integral over the Riemann integral. We will also see that they can be extended to sequences of functions which converge in measure.

4.1 The Bounded Convergence Theorem

We begin by examining the Bounded Convergence Theorem, which provides a method of interchanging integration and limiting for functions that are bounded by some real number.

THEOREM 4.1.1 (Bounded Convergence Theorem) *Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure, and suppose there exists $M \in \mathbb{R}$ such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$, then*

$$\int_E f = \lim \int_E f_n.$$

Proof. Since f_n converges pointwise to f on E, we may apply Egorov's Theorem to find a smaller set on which the convergence is uniform. That is, given $\varepsilon > 0$, there is an N and a measurable set $A \subset E$ with $mA < \frac{\varepsilon}{4M}$ such that for $n \geq N$ and $x \in E \setminus A$, we have

$|f_n(x) - f(x)| < \frac{\varepsilon}{2mE}$. Thus

$$\begin{aligned} \left| \int_E f_n - \int_E f \right| &= \left| \int_E f_n - f \right| \\ &\leq \int_E |f_n - f| \\ &= \int_{E \setminus A} |f_n - f| + \int_A |f_n - f| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore we have

$$\int_E f_n \to \int_E f.$$

□

4.2 The Monotone Convergence Theorem

We begin by introducing a powerful lemma. As its conditions are weaker than the other theorems, the result is also somewhat weaker. However, it can also be applied to a wider class of functions and establishes a bound for the integral.

LEMMA 4.2.1 (Fatou's Lemma) *If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then*

$$\int_E f \leq \liminf \int_E f_n.$$

Proof. First we note that since the integral over a set of measure zero is zero, we may assume the convergence is everywhere without loss of generality. Let g be a bounded, measurable function not greater than f defined on a set of finite measure, say E', and vanishing outside E'. Define $g_n = \min\{g(x), f_n(x)\}$. Then since g is bounded, g_n is bounded and vanishes outside E'. Furthermore, $g_n(x) \to g(x)$ for all $x \in E'$. Thus we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \liminf \int_E f_n.$$

Taking the supremum over h gives

$$\int_E f \leq \liminf \int_E f_n.$$

□

THEOREM 4.2.1 (Monotone Convergence Theorem) *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim_{n\to\infty} f_n$ almost everywhere. Then we have*

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

Proof. We begin by noting that

$$\int_E f \leq \liminf \int_E f_n,$$

by Fatou's Lemma. Now since $\{f_n\}$ is monotonically increasing and converges a.e. to f, we observe that $f_n \leq f$ for almost all n. Then $\int_E f_n \leq \int_E f$. Thus we have

$$\limsup \int_E f_n \leq \int_E f,$$

and hence,

$$\int_E f = \lim \int_E f_n.$$

□

So for any increasing convergent sequence of measurable functions, we may interchange integration and the limiting process.

It should be noted that the requirement in the hypothesis that $\{f_n\}$ be increasing is necessary to the validity of the theorem. To demonstrate this, we consider the following example.

EXAMPLE 4.2.1 Consider the sequence of functions $\{f_n\}$, where

$$f_n(x) = \begin{cases} 0 : x < n \\ 1 : x \geq n. \end{cases}$$

This sequence of functions converges pointwise to 0. However, each function in the sequence has a divergent integral. Thus we have

$$\lim_{n\to\infty} \int_E f_n(x)dx = \infty, \text{ and } \int_E f(x)dx = 0;$$

that is,

$$\lim_{n\to\infty} \int_E f_n \neq \int_E f.$$

4.3 The Lebesgue Dominated Convergence Theorem

Arguably the most powerful of the convergence theorems, the Dominated Convergence Theorem also requires a strong assumption. However, it allows us to integrate any sequence of functions that is bounded above by an integrable function. We present the theorem, along with its proof and an example to show that one of the assumptions is critical. We will then conclude the section by demonstrating that the convergence theorems do not hold for Riemann integration.

THEOREM 4.3.1 (Lebesgue Dominated Convergence Theorem) *Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E, and suppose that for almost all $x \in E$ we have $f(x) = \lim f_n(x)$. Then*

$$\int_E f = \lim \int_E f_n.$$

Proof. Consider the function $g - f_n$. By hypothesis, $|f_n| \leq g$, so $g - f_n$ is nonnegative.

Then applying Fatou's lemma, we have

$$\int_E (g - f_n) \leq \liminf \int_E (g - f_n).$$

Now since g is integrable and $|f| \leq g$, f is integrable as well and we have

$$\int_E g - \int_E f \leq \int_E g - \limsup \int_E (g - f_n),$$

and thus

$$\int_E f \geq \limsup \int_E f_n.$$

Using a similar method and considering $g + f_n$, we may obtain

$$\int_E f \leq \liminf \int_E f_n,$$

and thus we have

$$\int_E f = \lim \int_E f_n.$$

□

We may also observe that the Bounded Convergence Theorem introduced in Section 4.1 is simply a corollary to this result, where g is a constant function.

As mentioned previously, these convergence theorems are one of the key advantages of Lebesgue's theory over Riemann's. To illustrate this, we will now show that the Monotone Convergence and Dominated Convergence theorems do not hold for Riemann integration. We consider the following counterexample.

EXAMPLE 4.3.1 Recall the sequence of functions $\{f_n\}$ from example 2.3.2; that is, the sequence given by

$$f_n(x) = \begin{cases} 1 : x = q_1, q_2, \ldots, q_n \\ 0 : \text{ otherwise} \end{cases}.$$

Since each of these functions is bounded and discontinuous at a finite number of points, we see that each member of this sequence is Riemann integrable. Now for each $x \in [0, 1]$, it is clear that the sequence $\{f_n(x)\}$ is monotonically increasing and is bounded above by the constant function 1. Thus the sequence converges to the Dirichlet function (that is, the characteristic function of the rationals) on $[0, 1]$. But we have seen previously that this function is not Riemann integrable. Thus both the Monotone Convergence and Dominated Convergence theorems can not be applied to the Riemann integral.

4.4 Convergence in Measure

In the following section, we will show that Fatou's Lemma, the Monotone Convergence Theorem, and Lebesgue's Dominated Convergence Theorem remain valid under convergence in measure, allowing the further extension of Lebesgue integration. In order to prove this result, we first present a lemma which we will use to prove the main results. We will also rely on the result shown in Theorem 2.3.3.

LEMMA 4.4.1 *The sequence $\{f_n\}$ converges pointwise to f if and only if every subsequence of $\{f_n\}$ has in turn a subsequence that converges to f.*

Proof. Suppose $\lim f_n = f$ and let $\{f_{n_k}\}$ be a subsequence. Then let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n - f| < \varepsilon$. Now since $n_1 < n_2 < \ldots$, a simple induction argument shows that $n_k \geq k$ for any $k \in \mathbb{N}$. Then if $k \geq N$, it must be the case that $n_k \geq N$, and thus $|f_{n_k} - f| < \varepsilon$. Hence we have that $\lim f_{n_k} = f$. Now if $\{f_{n_{k_l}}\}$ is a subsequence of $\{f_{n_k}\}$, then we may use the same line of reasoning to show that this sub-subsequence converges to the same limit as the original subsequence, and thus, to f.

To show the converse, suppose that every subsequence has a sub-subsequence converging to f and recall the definition of a cluster point: l is called a *cluster point* of $\{g_n\}$ if given $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists $n \geq N$ such that $|g_n - l| < \varepsilon$ (that is, all limits are cluster points, but cluster points are not necessarily limits). Then f is a cluster point of every subsequence f_{n_k}, and thus a cluster point of f_n. Now we observe that if there exists another cluster point, say g, with $f \neq g$, then there would be some subsequence converging to g. But such a subsequence could not have a sub-subsequence converging to f, since g is its only cluster point. This contradicts the hypothesis, and thus f must be the only cluster point of f_n, and therefore $\lim f_n = f$. $\square$

We now consider the Dominated Convergence Theorem under convergence in measure.

THEOREM 4.4.1 (Dominated Convergence) *Let $\{f_n\}$ be as above defined a.e. on E. Suppose $f_n \to f$ in measure and $|f_n| \leq |g|$ a.e. (with g being integrable). Then*

$$\int_E f = \lim_{n\to\infty} \int f_n.$$

Proof. Consider a subsequence $\{f_{n_k}\}$ of $\{f_n\}$. Clearly $f_{n_k} \to f$ in measure, and so there exists a subsequence $\{f_{n_{k_l}}\}$ of $\{f_{n_k}\}$ which converges to f pointwise a.e. Since this subsequence converges almost everywhere and $|f_{n_k}| \leq |g|$, we may apply the dominated convergence theorem, which gives

$$\int_E f = \lim_{l\to\infty} \int_E f_{n_{k_l}}.$$

$\square$

Thus the Dominated Convergence theorem holds if we replace the assumption of convergence almost everywhere with convergence in measure. Additionally, the Bounded Convergence theorem can be applied since it is a corollary.

We now consider Fatou's Lemma and the Monotone Convergence theorem.

THEOREM 4.4.2 (Fatou's Lemma) *Let $\{f_n\}$ be a sequence of nonnegative measurable functions converging in measure to f on E. Then*

$$\int_E f \leq \liminf \int_E f_n.$$

Proof. Consider $\{f_{n_k}\}$, a subsequence of $\{f_n\}$ with

$$\liminf \int_E f_n = \lim_{k\to\infty} \int_E f_{n_k}.$$

Recall that a sequence converges if and only if for every subsequence, there exists a sub-subsequence which converges to the same limit. Since $\{f_{n_k}\}$ converges to f in measure, there exists a sub-subsequence $\{f_{n_{k_l}}\}$ which converges to f almost everywhere. Then Fatou's lemma (for convergence a.e.) gives

$$\int_E f \le \liminf_{l\to\infty} \int_E f_{n_{k_l}}.$$

Then since $\liminf \int_E f_n = \lim_{k\to\infty} \int_E f_{n_k}$, we have

$$\int_E f \le \liminf \int_E f_n.$$

$\square$

THEOREM 4.4.3 (Monotone Convergence Theorem) *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions and let f_n converge to f in measure. Then*

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Since $\{f_n\}$ converges to f in measure, there exists a subsequence of $\{f_n\}$, say $\{f_{n_k}\}$, which converges to f almost everywhere. Now since $\{f_n\}$ is monotonically increasing, $\{f_{n_k}\}$ is also monotonically increasing. Now we have an increasing sequence which converges almost everywhere to a measurable function f, and thus we may apply the Monotone Convergence theorem. Thus we have

$$\lim_{k\to\infty} \int f_{n_k} = \int f.$$

Now since $f_{n_k} \to f$, we have

$$\lim_{n\to\infty} \int f_n = \lim_{k\to\infty} \int f_{n_k} = \int f.$$

$\square$

Now we have shown that the convergence theorems presented in this chapter may be applied to sequences of functions which convergence pointwise, including convergence almost everywhere, as well as to those sequences of functions which converge in measure.

CHAPTER 5

CONCLUSION

As we have shown, a rigorous theory of integration was conceived by Cauchy, refined by Riemann, and then revolutionized by Lebesgue. This rigorous footing finally allowed many ideas, some dating back to Newton and Leibniz, to fully come to fruition. The rich theory that has developed out of the definitions proposed by Riemann and Lebesgue has provided the structure for integration theory for over a century. Lebesgue's definition in particular is still the most common notion of integration.

In this paper we have seen the close relationship between convergence and integration. The Convergence Theorems developed in Chapter 4 are among the strongest results found in the theory of Lebesgue integration, and one of the primary advantages of Lebesgue's method over the Riemann integral. These results expand the class of integrable functions and provide a convenient method for handling sequences of functions.

It is our hope that the preceding sections may serve as an effective and interesting introduction to the rich theory of Lebesgue measure and integration.

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