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Aspects of Fourier Analysis on Euclidean Space

Joseph William Roberts

ASPECTS OF FOURIER ANALYSIS ON EUCLIDEAN SPACE

A Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

Joseph W. Roberts

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Mathematics

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ABSTRACT

The field of Fourier analysis encompasses a vast spectrum of mathematics and has far reaching applications in all STEM fields. Here we introduce and study the Fourier transform and Fourier series on Euclidean space. After defining the Fourier transform, establishing its basic properties, and presenting some classical results we looked into what impact the smoothness of a function has on the growth and integrability of its Fourier transform. This endeavor also involved a brief study of Bessel functions and interpolation of operators. Having established several results indicating that the behavior of a function's Fourier transform is largely dictated by the smoothness of the function, the thesis concludes with a look into Fourier series and Bochner-Riesz means.

KEYWORDS: Fourier analysis, harmonic analysis, Fourier transform, Fourier series, Bessel functions, Schwartz space, convolution,

This abstract is approved as to form and content

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Dr. William Bray
Chairperson, Advisory Committee
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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

THE FOURIER TRANSFORM

The Fourier transform is widely considered the father of oscillatory integrals whose use is felt far beyond the realms of mathematics. It is a transformation that sends a function from its spatial domain to its frequency domain and if applied again gives back the function, albeit reflected. Almost everything in the world can be described via waveform making the Fourier transform a very useful tool. Just a few applications include solving partial differential equations, magnetic resonance imaging (MRI), spectroscopy, signal processing, quantum mechanics, and problems in neuroscience. Developed by Joseph Fourier while studying the equations of heat conduction, the Fourier transform would become a powerful tool for pioneering thinkers around the globe. If only Fourier could see the far-reaching applications of his work! Despite its appearance in a variety of fields, we shall limit our study of the Fourier transform to a purely mathematical perspective.

We begin by defining and introducing approximate identities. As you will soon see, convolution of a function with an approximate identity gives rise to a sequence of functions that converge in norm to the original function. This property will prove crucial, as one can not study harmonic analysis without coming across convolution operators. With approximate identities out of the way the Fourier transform is defined and studied on the space of Schwartz functions. The Schwartz functions give us an ideal space to study the Fourier transform without running into any snags. Thus we will establish the basic properties of the Fourier transform and use the Schwartz space as a launch pad to extend the Fourier transform to larger spaces.

One of the primary goals is to extend the Fourier transform to $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. To do this we will use the groundwork laid on the Schwartz space to catapult the Fourier transform to the space of integrable functions and square-integrable functions. From there a definition of Fourier transform can be constructed for $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Naturally, when leaving the well-behaved Schwartz functions there will be some complications to address. Luckily, many of the properties of the Fourier transform established on Schwartz space are preserved in the more general spaces.

1.1 Approximate Identities

Before introducing the Fourier transform, we briefly introduce the notion of approximate identities. After providing a definition we will prove a fundamental result used throughout the paper.

DEFINITION 1.1: An *approximate identity (as $\varepsilon \to 0$)* is a family of $L^1(\mathbb{R}^n)$ functions k_ε with the following three properties:

1. There exists a constant $c > 0$ such that $\|k_\varepsilon\|_1 \leq c$ for all $\varepsilon > 0$.

2. $\int_{\mathbb{R}^n} k_\varepsilon(x)dx = 1$ for all $\varepsilon > 0$.

3. For every neighborhood V about the origin we have $\int_{V^c} |k_\varepsilon|\, dx \to 0$ as $\varepsilon \to 0$.

One can think of an approximate identity as a sequence of positive functions k_ε whose integral is equivalent to one for every $\varepsilon > 0$ that spike near zero (or identity element) with the crucial property $k_\varepsilon * f \to f$ in L^p as $\varepsilon \to 0$. The property just described gives approximate identities their namesake. Proof of the described property follows.

THEOREM 1.2: Let k_ε be an approximate identity as $\varepsilon \to 0$ on $\mathbb{R}^n$. Then for $f \in L^p(\mathbb{R}^n)$ we have:

1. When $1 \leq p < \infty$ then $\|k_\varepsilon * f - f\|_p \to 0$ as $\varepsilon \to 0$.

2. When $p = \infty$ then the following is valid: If f is continuous in a neighborhood of a compact subset K of $\mathbb{R}^n$, then $\|k_\varepsilon * f - f\|_\infty \to 0$ as $\varepsilon \to 0$.

Proof. We begin by proving 1. Let $\eta > 0$ and $1 \leq p < \infty$. By applying the Minkowski integral inequality we have:

$$\begin{aligned}
\|f * k_\varepsilon - f\|_p &\leq \left\| \int_{\mathbb{R}^n} |f(x-y) - f(x)| \, k_\varepsilon(y) dy \right\|_p \\
&\leq \int_{\mathbb{R}^n} \|f(x-y) - f(x)\|_p \, k_\varepsilon(y) dy \\
&= \int_{\{|y| \leq \delta\}} \|f(x-y) - f(x)\|_p \, k_\varepsilon(y) dy \qquad (1.1.1) \\
&\quad + \int_{\{|y| > \delta\}} \|f(x-y) - f(x)\|_p \, k_\varepsilon(y) dy.
\end{aligned}$$

We will now show both the integrals in 1.1.1 get arbitrarily small as ε approaches zero. Note by continuity of translations on $L^p(\mathbb{R}^n)$ we can choose $\delta > 0$ such that for every $|y| \leq \delta$ we have:

$$\|f(x-y) - f(x)\|_p < \frac{\eta}{2}.$$

Thus for the chosen delta we have

$$\int_{\{|y| \leq \delta\}} \|f(x-y) - f(x)\|_p \, k_\varepsilon(y) dy < \frac{\eta}{2} \int_{\{|y| \leq \delta\}} k_\varepsilon(y) dy < \frac{\eta}{2}.$$

Next we show the second integral of 1.1.1 is arbitrarily small as $\varepsilon \to 0$. Note that

$$\|f(x-y) - f(x)\|_p \leq 2 \|f\|_p < \infty \text{ since } f \in L^p(\mathbb{R}^n).$$

However, by definition of approximate identity we know for small enough epsilon we have

$$\int_{\{|y| > \delta\}} k_\varepsilon(y) dy < \frac{\eta}{4 \|f\|_{L^p}}.$$

Thus, for small ε we have

$$\int_{\{|y| > \delta\}} \|f(x-y) - f(x)\|_p \, k_\varepsilon(y) dy \leq 2 \|f\|_p \int_{\{|y| > \delta\}} k_\varepsilon(y) dy < \frac{\eta}{2}.$$

Therefore, as $\varepsilon \to 0$ we have the desired result:

$$\|k_\varepsilon * f - f\|_p < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

implying $||k_\varepsilon * f - f||_p \to 0$ as $\varepsilon \to 0$. The proof of 2 is similar. For all x we have

$$\begin{aligned} |f * k_\epsilon - f| \;\; &\leq \;\; \int_{K \cap \{|y| \leq \delta\}} |f(x-y) - f(x)| \, k_\varepsilon(y) dy \\ &\quad + \int_{K \cap \{|y| > \delta\}} |f(x-y) - f(x)| \, k_\varepsilon(y) dy. \end{aligned}$$

Proving the second integral above becomes arbitrarily small as $\varepsilon \to 0$ is the same as before. To prove the first integral is arbitrarily small simply apply the continuity of f on K to finish the proof. $\square$

It is also worth clarifying that we say $\{k_N\}$ is an approximate identity as $N \to \infty$ if it meets the same first two conditions in Definition 1.1 and the slightly modified third condition:

$$\int_{V^c} |k_N| \, dx \to 0 \text{ as } N \to \infty,$$

where again V is some arbitrary neighborhood about the origin. Of course approximate identities are not limited to Euclidean space. It turns out that definition 1.1 and Theorem 1.2 are largely identical on any locally compact group with a left invariant Haar measure. Thus we may also use approximate identities when studying the torus, which will be key when introducing Fourier series. However, here we have sacrificed generality for simplicity and for the more generalized definiton please refer to Grafakos[1].

1.2 Fourier Transform on the Space of Schwartz Functions

The space of Schwartz functions is a wonderful space to introduce the Fourier transform. We will briefly discuss the Schwartz space then move on to our topic of interest, the Fourier transform. The Schwartz space on $\mathbb{R}^n$, denoted $S(\mathbb{R}^n)$, is the space of smooth functions on $\mathbb{R}^n$ all of whose derivatives are rapidly decreasing. Elements of $S(\mathbb{R}^n)$ are known as Schwartz functions. More precisely a Schwartz function and all of its derivatives decay faster than the reciprocal of any polynomial at infinity. A formal definition follows.

DEFINITION 1.3: A C^∞ complex-valued function f on $\mathbb{R}^n$ is called a *Schwartz function* if for every pair of multi-indices α and β there exists a positive constant $C_{\alpha,\beta}$ such that

$$\rho_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta f(x)\right| = C_{\alpha,\beta} < \infty.$$

Additionally, the quantities $\rho_{\alpha,\beta}(f)$ are called the *Schwartz seminorms* of f.[1]

One useful example of a Schwartz function is the Gaussian function $f(x) = e^{-\pi x^2}$. We will later see that this particular function is a fixed point of the Fourier transform on $S(\mathbb{R}^n)$. The one-dimensional Gaussian is plotted in Figure 1.1 below. Make note of how rapidly the function decays. Naturally, all bump functions are also Schwartz functions.

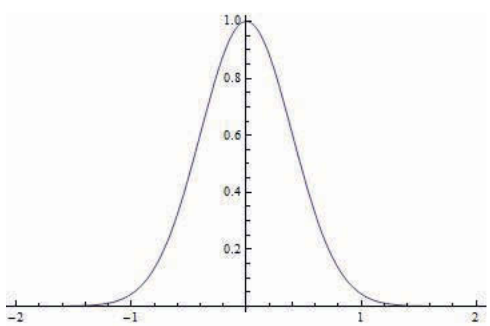

Figure 1.1: One-Dimensional Gaussian

The Schwartz seminorms defined above are used to define convergence of a sequence of Schwartz functions and can be used to construct a metric on $S(\mathbb{R}^n)$. It is known that $S(\mathbb{R}^n)$ is a Fréchet space and that convergence in S is stronger than convergence in L^p for all $0 < p \leq \infty$. The proofs of the properties mentioned are omitted in the interest of moving on to the topic of interest, the Fourier transform.[2]

[1] C^∞ denotes the set of all smooth functions.

[2] For more information on Schwartz spaces or for the proofs that were omitted please refer to L.

As you can see the Schwartz space is occupied by very well-behaved functions with no shortage of wonderful properties. However, why bring up Schwartz functions when introducing the Fourier transform? For starters the rapid decay of Schwartz functions at infinity allow us to study the Fourier transform without worrying about the convergence of the integral. As we will see in a future section, convergence issues arise when we no longer have the rapid decay. Additionally, all Schwartz functions are also smooth which we will show guarantees good behaviour from the Fourier transform. Additionally, the Fourier transform is a automorphism of the Schwartz space and Fourier inversion holds in it. In this environment we can define the Fourier transform, study a lot of its properties, then begin generalizing some of those concepts to larger sets of functions. Without any further adieu we introduce the most important tool in harmonic analysis, the Fourier transform.

DEFINITION 1.4: Given $f \in S(\mathbb{R}^n)$ we define $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$ the *Fourier transform* of f.

A specific example demonstrating the Fourier transform follows. Lets take the Fourier transform of the Gaussian function introduced earlier. For $f(x) = e^{-\pi x^2}$ we have

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x\xi} dx \\ &= \int_{\mathbb{R}} e^{-\pi(x^2+2i\xi x-\xi^2+\xi^2)} dx \\ &= e^{-\pi\xi^2} \int_{\mathbb{R}} e^{-\pi(x+i\xi)^2} dx.\end{aligned}$$

By using u-substitution we obtain

$$\hat{f}(\xi) = \frac{e^{-\pi\xi^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-u^2} du = e^{-\pi\xi^2} \text{ since } \int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}.$$

Thus we have calculated the Fourier transform and shown that $f(x) = e^{-\pi x^2}$ is a

Grafakos [1].

fixed point as previously claimed. We will now introduce some fundamental properties of the Fourier transform. However, before proving properties we need to define several transformations that will be used for the remainder of the paper.

DEFINITION 1.5: For a measurable function f on $\mathbb{R}^n$, $x, y \in \mathbb{R}^n$, and $a > 0$ we define

1. *translation* of f by $\tau^y(f)(x) = f(x-y)$,
2. *dilation* of f by $\delta^a(f)(x) = f(ax)$,
3. *reflection* of f by $\tilde{f}(x) = f(-x)$.

PROPOSITION 1.6: Let $f, g \in S(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, $b \in \mathbb{C}$, α a multi-index, and $t > 0$. Then

1. $||\hat{f}||_\infty \leq \|f\|_1$,
2. $\widehat{f+g} = \hat{f} + \hat{g}$,
3. $\widehat{bf} = b\hat{f}$,
4. $\hat{\tilde{f}} = \tilde{\hat{f}}$,
5. $\hat{\bar{f}} = \bar{\tilde{\hat{f}}}$,
6. $\widehat{\tau^y(f)}(\xi) = e^{-2\pi i y\cdot\xi}\hat{f}(\xi)$,
7. $\widehat{e^{2\pi i x\cdot y} f}(x)(\xi) = \tau^y(\hat{f})(\xi)$,
8. $\widehat{\delta^t(f)} = t^{-n}\delta^{\frac{1}{t}}(\hat{f})$,
9. $\widehat{\partial^\alpha f}(\xi) = (2\pi i\xi)^\alpha \hat{f}(\xi)$,
10. $\partial^\alpha \hat{f}(\xi) = (-\widehat{2\pi i x)^\alpha} f(x)(\xi)$,
11. $\hat{f} \in S$,

12. $\widehat{f * g} = \hat{f}\hat{g}$,

13. $\widehat{f \circ A}(\xi) = \hat{f}(A\xi)$ where A is an orthogonal matrix and ξ is a column vector.

Proof. The proofs of properties 1 through 5 are very straight forward. Note for all $\xi \in \mathbb{R}^n$ we have

$$\begin{aligned} |\hat{f}(\xi)| &\leq \int_{\mathbb{R}^n} \left| f(x) e^{-2\pi i x \cdot \xi} \right| dx \\ &\leq \int_{\mathbb{R}^n} |f(x)| \left| e^{-2\pi i x \cdot \xi} \right| dx \\ &\leq \int_{\mathbb{R}^n} |f(x)| \, dx = \|f\|_1 \,, \end{aligned}$$

establishing 1. Thus implying the Fourier transform is always bounded. Proving 2 is also quite simple:

$$\begin{aligned} \widehat{f+g}(\xi) &= \int_{\mathbb{R}^n} (f(x) + g(x)) e^{-2\pi i x \cdot \xi} dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx + \int_{\mathbb{R}^n} g(x) e^{-2\pi i x \cdot \xi} dx \\ &= \hat{f} + \hat{g}. \end{aligned}$$

We now establish 3,

$$\begin{aligned} \widehat{bf} &= \int_{\mathbb{R}^n} b f(x) e^{-2\pi i x \cdot \xi} dx \\ &= b \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \\ &= b\hat{f}. \end{aligned}$$

To prove 4 we simply use the substitution $u = -x$,

$$
\begin{aligned}
\hat{\tilde{f}} &= \int_{\mathbb{R}^n} f(-x) e^{-2\pi i x\cdot\xi} dx \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(-x) e^{-2\pi i x\cdot\xi} dx_1 \dots dx_n \\
&= (-1) \int_{\infty}^{-\infty} \cdots (-1) \int_{\infty}^{-\infty} f(u) e^{-2\pi i u\cdot(-\xi)} du_1 \dots du_n \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i u\cdot(-\xi)} du \\
&= \hat{f}(-\xi) \\
&= \tilde{\hat{f}}.
\end{aligned}
$$

Now we prove 5,

$$
\begin{aligned}
\bar{\tilde{\hat{f}}} &= \overline{\int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot(-\xi)} dx} \\
&= \int_{\mathbb{R}^n} \overline{f(x)}\ \overline{e^{-2\pi i x\cdot(-\xi)}} dx \\
&= \int_{\mathbb{R}^n} \overline{f(x)} e^{-2\pi i x\cdot\xi} dx \\
&= \hat{\bar{f}}.
\end{aligned}
$$

To prove 6, simply use the substitution $u = x - y$ in the following manner:

$$
\begin{aligned}
\widehat{\tau^y(f)}(\xi) &= \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i x\cdot\xi} dx \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i (u+y)\cdot\xi} du \\
&= e^{-2\pi i y\cdot\xi} \int_{\mathbb{R}^n} f(u) e^{-2\pi i u\cdot\xi} du \\
&= e^{-2\pi i y\cdot\xi} \hat{f}(\xi).
\end{aligned}
$$

Proving 7 is again elementary,

$$\begin{aligned} \widehat{e^{2\pi i x\cdot y} f}(x)(\xi) &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot y} f(x) e^{-2\pi i x\cdot\xi} dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i(\xi-y)\cdot x} dx \\ &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot(\xi-y)} dx \\ &= \tau^y(\hat{f})(\xi). \end{aligned}$$

To prove 8 we simply use the substitution $u = tx$,

$$\begin{aligned} \widehat{\delta^t(f)}(\xi) &= \int_{\mathbb{R}^n} f(tx) e^{-2\pi i x\cdot\xi} dx \\ &= \frac{1}{t^n} \int_{\mathbb{R}^n} f(u) e^{\frac{-2\pi i}{t} u\cdot\xi} du \\ &= t^{-n} \delta^{\frac{1}{t}}(\hat{f}). \end{aligned}$$

Property 9 is proved by integration by parts, which is justified by the rapid decay of the integrands:

$$\begin{aligned} \widehat{\partial^\alpha f}(\xi) &= \int_{\mathbb{R}^n} (\partial^\alpha f)(x) e^{-2\pi i x\cdot\xi} dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x)(-2\pi i\xi)^\alpha e^{-2\pi i x\cdot\xi} dx \\ &= (2\pi i\xi)^\alpha \hat{f}(\xi). \end{aligned}$$

To prove 10 we will use an induction argument. Let $\alpha_j = (0, \ldots, 1, \ldots 0)$, where all entries are zero except for the jth entry, which is 1. Then

$$\frac{e^{-2\pi i x\cdot(\xi+he_j)} - e^{-2\pi i x\cdot\xi}}{h} \to -2\pi i x_j e^{-2\pi i x\cdot\xi} \tag{1.2.2}$$

as $h \to 0$. Additionally, the left hand expression in 1.2.2 is bounded by $c\,|x|$ for all h and ξ where c is a constant. Application of the Lebesgue dominated convergence theorem yields

$$\lim_{h\to 0}\int_{\mathbb{R}^n}\frac{e^{-2\pi ix\cdot(\xi+he_j)}-e^{-2\pi ix\cdot\xi}}{h}f(x)dx=\int_{\mathbb{R}^n}-2\pi ix_je^{-2\pi ix\cdot\xi}f(x)dx.$$

$$\therefore\ \partial^{\alpha_j}\hat{f}(\xi)=\widehat{(-2\pi ix)^{\alpha_j}f(x)}(\xi).$$

Assume, for induction, that for $k\in\mathbb{N}$ we have $\partial^{k\alpha_j}\hat{f}(\xi)=\widehat{(-2\pi ix)^{k\alpha_j}f(x)}(\xi)$. Then we have

$$\begin{aligned}
\partial^{(k+1)\alpha_j}\hat{f}(\xi) &= \partial^{\alpha_j}\partial^{k\alpha_j}\hat{f}(\xi)\\
&= \partial^{\alpha_j}\widehat{(-2\pi ix)^{k\alpha_j}f(x)}(\xi)\\
&= \partial^{\alpha_j}\int_{\mathbb{R}^n}(-2\pi ix)^{k\alpha_j}f(x)e^{-2\pi ix\cdot\xi}dx\\
&= \int_{\mathbb{R}^n}(-2\pi ix)^{(k+1)\alpha_j}f(x)e^{-2\pi ix\cdot\xi}dx\\
&= \widehat{(-2\pi ix)^{(k+1)\alpha_j}f}(\xi).
\end{aligned}$$

Thus we have established 10 with respect to the jth variable. However, j is arbitrary and thus 10 is established. To prove 11 we apply properties 9, 10, and 1 in the following manner:

$$\begin{aligned}
\left\|x^\alpha(\partial^\beta\hat{f})(x)\right\|_\infty &= \left\|x^\alpha(\widehat{(-2\pi ix)^\beta f(x)})\right\|_\infty\\
&= \left\|x^\alpha(-2\pi i)^\beta(\widehat{x^\beta f(x)})\right\|_\infty\\
&= (2\pi)^\beta\left\|x^\alpha(\widehat{x^\beta f(x)})\right\|_\infty\\
&= \frac{(2\pi)^\beta}{(2\pi)^\alpha}\left\|\partial^\alpha\widehat{(x^\beta f(x))}\right\|_\infty.
\end{aligned}$$

So by applying property 1 and the fact that $f\in S$ we obtain the following result:

$$\frac{(2\pi)^\beta}{(2\pi)^\alpha}\left\|\partial^\alpha\widehat{(x^\beta f(x))}\right\|_\infty\le\frac{(2\pi)^\beta}{(2\pi)^\alpha}\left\|\partial^\alpha(x^\beta f(x))\right\|_1<\infty.$$

$$\therefore \quad \left\| x^\alpha (\partial^\beta \hat{f})(x) \right\|_\infty < \infty \text{ implying } \hat{f} \in S \text{ and property 11 is established.}$$

Property 12 is a simple application of Fubini's theorem:

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) e^{-2\pi i x \cdot \xi} dy dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) e^{-2\pi i (x-y) \cdot \xi} e^{-2\pi i y \cdot \xi} dy dx \\
&= \int_{\mathbb{R}^n} g(y) e^{-2\pi i y \cdot \xi} \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i (x-y) \cdot \xi} dx dy \\
&= \hat{f}(\xi) \hat{g}(\xi).
\end{aligned}$$

Lastly,

$$\begin{aligned}
\widehat{f \circ A}(\xi) &= \int_{\mathbb{R}^n} f(Ax) e^{-2\pi i x \cdot \xi} dx \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i A^{-1} u \cdot \xi} du \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i A^t u \cdot \xi} du \\
&= \int_{\mathbb{R}^n} f(u) e^{-2\pi i u \cdot A\xi} du \\
&= \hat{f}(A\xi),
\end{aligned}$$

where we used the substitution $u = Ax$ and the fact that $|\det A| = 1$. Thus 13 is established and all properties have been proven. □

It turns out that many of the properties above are not limited to the space of Schwartz functions and are quite fundamental for the Fourier transform, holding in a variety of spaces. Recall in the introduction it was mentioned that taking the Fourier transform of a function twice will return a reflection of the function. To counter-act the reflection a simple modification is made to create what is known as the inverse Fourier transform.

DEFINITION 1.7: Given $f \in S$ we define the *inverse Fourier transform* of f, denoted $\check{f}$, as follows

$$\check{f}(x) = \hat{f}(-x) = \int_{\mathbb{R}^n} f(\xi)e^{2\pi i\xi\cdot x}d\xi.$$

Naturally, the inverse Fourier transform also has all the properties outlined in Proposition 1.6. Establishing these properties for the inverse Fourier transform is the same as before and thus the proof made unnecessary and omitted. So, ideally, the Fourier transform and inverse Fourier transform are inverses of each other. Unfortunately, for particular functions the integral in either definition may be undefined resulting in complications. We will later see that there exists integrable functions whose Fourier transform is not integrable which prevents use of the inverse Fourier transform to return the original function. Thus, we say a set or space of functions have the property of Fourier inversion if for all functions f in the space $(\hat{f})\check{} = f = (\check{f})\hat{}$. We will now prove that Fourier inversion holds on S.

LEMMA 1.8: Let $f, g \in S(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(x)g(x)dx$. Additionally, $\int_{\mathbb{R}^n} f(x)\check{g}(x)dx = \int_{\mathbb{R}^n} \check{f}(x)g(x)dx$.

Proof. The proof is a straightforward application of Fubini's theorem:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx &= \int_{\mathbb{R}^n} f(x)\int_{\mathbb{R}^n} g(y)e^{-2\pi iy\cdot x}dydx \\
&= \int_{\mathbb{R}^n} g(y)\int_{\mathbb{R}^n} f(x)e^{-2\pi ix\cdot y}dxdy \\
&= \int_{\mathbb{R}^n} g(y)\hat{f}(y)dy \\
&= \int_{\mathbb{R}^n} \hat{f}(x)g(x)dx.
\end{aligned}$$

The same approach is used to prove the case involving the inverse Fourier transform. □

THEOREM 1.9: (Fourier Inversion) Let $f \in S(\mathbb{R}^n)$. Then $(\hat{f})\check{} = f = (\check{f})\hat{}$.

Proof. Let $g(\xi) = e^{2\pi i\xi\cdot t}e^{-\pi|\varepsilon\xi|^2}$.Then $\hat{g}(x) = \frac{1}{\varepsilon^n}e^{-\pi\left|\frac{(x-t)}{\varepsilon}\right|^2}$, which is an approximate identity as $\varepsilon \to 0$. Applying Lemma 1.8 gives us

$$\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} f(x)e^{-\pi\left|\frac{x-t}{\varepsilon}\right|^2}dx = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\xi\cdot t}e^{-\pi|\varepsilon\xi|^2}d\xi. \tag{1.2.3}$$

Now apply Theorem 1.2,

$$\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} f(x)e^{-\pi\left|\frac{x-t}{\varepsilon}\right|^2}dx \to f(t) \text{ uniformly as } \varepsilon \to 0.$$

Thus the left-hand expression of 1.2.3 converges to $f(t)$ as $\varepsilon \to 0$. Lets look at the right-hand expression of 1.2.3. Note that

$$\lim_{\varepsilon\to 0}\hat{f}(\xi)e^{2\pi i\xi\cdot t}e^{-\pi|\varepsilon\xi|^2} = \hat{f}(\xi)e^{2\pi i\xi\cdot t}.$$

Additionally, for every $\varepsilon > 0$ we know the right-hand integrand of 1.2.3 is absolutely bounded by $\left|\hat{f}(\xi)e^{2\pi i\xi\cdot t}\right|$, which is integrable. Hence, by the Lebesgue dominated convergence theorem

$$\int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\xi\cdot t}e^{-\pi|\varepsilon\xi|^2}d\xi \to \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i\xi\cdot t}d\xi = (\hat{f})\check{}(t) \text{ as } \varepsilon \to 0.$$

$$\therefore f(t) = (\hat{f})\check{}(t).$$

To show $(\check{f})\hat{} = f$ simply replace f by $\tilde{f}$ and repeat the same argument above. □

It comes as no surprise that the property of Fourier inversion is critical in many applications. Without Fourier inversion, the Fourier transform works in only one direction and thus its usefulness is much more limited. For example, in signal processing a particular signal needs to be examined in both the spatial and frequency domains. Thus one need to be able freely move between the two, which requires Fourier inversion. With Fourier inversion established on $S(\mathbb{R}^n)$ we will now prove a few more classical results.

LEMMA 1.10: (Parseval's Relation) Let $f, h \in S(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} f(x)\bar{h}(x)dx =$

$\int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi$. Additionally, $\int_{\mathbb{R}^n} f(x)\bar{h}(x)dx = \int_{\mathbb{R}^n} \check{f}(\xi)\overline{\check{h}(\xi)}d\xi$.

Proof. Note that by applying 4 and 5 from Proposition 1.6 we know $\hat{\bar{\hat{h}}} = \overline{\tilde{\hat{\hat{\hat{h}}}}} = \overline{\hat{\hat{\hat{\tilde{h}}}}}$. Now apply the definition of the inverse Fourier transform and Theorem 1.9 to get $\overline{\hat{\hat{\hat{\tilde{h}}}}} = \overline{\hat{\check{h}}} = \overline{h}$.

$$\therefore \hat{\bar{\hat{h}}} = \overline{h}. \tag{1.2.4}$$

By application of Lemma 1.8 we know

$$\int_{\mathbb{R}^n} f(x)\hat{\bar{\hat{h}}}(x)dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi.$$

Thus by combining the above result with 1.2.4 we obtain the desired result:

$$\int_{\mathbb{R}^n} f(x)\overline{h}(x)dx = \hat{f}(\xi)\overline{\hat{h}(\xi)}d\xi.$$

Since the inverse Fourier transform shares the same properties used above the same approach is applied to show the relation also holds for the inverse Fourier transform. □

We will now use Parseval's Relation (Lemma 1.10) to prove Plancherel's Identity.

THEOREM 1.11: (Plancherel's Identity) Let $f \in S(\mathbb{R}^n)$. Then $\|f\|_2 = ||\hat{f}||_2 = ||\check{f}||_2$.

Proof. It is a well known fact that $|f|^2 = f\bar{f}$. Combining this fact with Lemma 1.10 we obtain

$$\|f\|_2^2 = \int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_{\mathbb{R}^n} f(x)\overline{f(x)}dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi)}d\xi = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = ||\hat{f}||_2^2.$$

$$\therefore \|f\|_2 = ||\hat{f}||_2.$$

Similarly, $\|f\|_2 = ||\check{f}||_2$. □

Our study of the Fourier transform on the space of Schwartz functions concludes with the following result.

THEOREM 1.12: The Fourier transform is a homeomorphism from $S(\mathbb{R}^n)$ onto itself. Additionally, the Fourier transform is a automorphism of $S(\mathbb{R}^n)$.

Proof. Let $F : S(\mathbb{R}^n) \mapsto S(\mathbb{R}^n)$ be given by $F(f)(\xi) := \hat{f}(\xi)$. Fourier inversion on $S(\mathbb{R}^n)$ yields that the mapping F is bijective. To show continuity of F we need only show $f_k \to f$ in $S(\mathbb{R}^n)$ as $k \to \infty$ implies $\hat{f}_k \to \hat{f}$ in $S(\mathbb{R}^n)$ as $k \to \infty$. Assume $f_k \to f$ in $S(\mathbb{R}^n)$ and let α, β be any multi-indices. Then for every $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $\forall k > N$ we have

$$\left\|\partial^\alpha(x^\beta(f_k - f))\right\|_1 < \frac{(2\pi)^\alpha \varepsilon}{(2\pi)^\beta}. \tag{1.2.5}$$

Additionally, $\widehat{f_k - f} = \hat{f}_k - \hat{f}$. By applying the previous identity and properties 1,9, 10 of Proposition 1.6 we obtain:

$$\begin{aligned}
\rho_{\alpha,\beta}(\hat{f}) &= \sup_{x \in \mathbb{R}^n} \left|x^\alpha \partial^\beta \hat{f}\right| \\
&\leq \left\|x^\alpha \partial^\beta \hat{f}\right\|_\infty \\
&= \left\|x^\alpha (\widehat{(-2\pi i x)^\beta f(x)})\right\|_\infty \\
&= (2\pi)^\beta \left\|x^\alpha (\widehat{x^\beta f(x)})\right\|_\infty \\
&= \frac{(2\pi)^\beta}{(2\pi)^\alpha} \left\|\partial^\alpha \widehat{(x^\beta f(x))}\right\|_\infty \\
&\leq \frac{(2\pi)^\beta}{(2\pi)^\alpha} \left\|\partial^\alpha (x^\beta f(x))\right\|_1 .
\end{aligned}$$

$$\therefore \rho_{\alpha,\beta}(\hat{f}_k - \hat{f}) = \rho_{\alpha,\beta}(\widehat{f_k - f}) \leq \frac{(2\pi)^\beta}{(2\pi)^\alpha} \left\|\partial^\alpha(x^\beta(f_k(x) - f(x)))\right\|_1 .$$

Thus by 1.2.5 for $k > N$ we have $\rho_{\alpha,\beta}(\hat{f}_k - \hat{f}) < \varepsilon$. Therefore showing $\hat{f}_k \to \hat{f}$ in $S(\mathbb{R}^n)$ and establishing the continuity of F. By using the same arguments one can prove continuity for the inverse Fourier transform. None the less, we have proven that the Fourier transform is a homeomorphism from $S(\mathbb{R}^n)$ onto itself. We have already shown that the Fourier transform is bijective and proved it to be homomorphic in Proposition 1.6, thus establishing the automorphism claim. $\square$

1.3 Fourier Transform on L^1

Now that we have introduced the Fourier transform on the space of Schwartz functions we will study the Fourier transform on the space of integrable functions. It is clear that for $f \in L^1(\mathbb{R}^n)$ we have the following inequality:

$$\left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \right| \leq \int_{\mathbb{R}^n} |f(x)| \, dx.$$

Thus the Fourier transform given in Definition 1.4 is an absolutely convergent integral when $f \in L^1$. Hence Definition 1.4 also works for the integrable functions. With this definition it is easy to prove that properties (1)-(8), (12), and (13) from Proposition 1.6 still hold for $f, g \in L^1$. As you can see the Fourier transform and many of its properties easily extend to the space of integrable functions. However, it was previously mentioned that Fourier inversion does not hold on all of $L^1(\mathbb{R}^n)$. Consider the following example.

EXAMPLE 1.13: Let's look at $f(x) = \chi_{(-\frac{1}{2},\frac{1}{2})}$. Taking the Fourier transform of f here gives us

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}} \chi_{(-\frac{1}{2},\frac{1}{2})} e^{-2\pi i x \xi} d\xi \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x \xi} d\xi \\ &= \frac{\sin(\pi \xi)}{\pi \xi}. \end{aligned}$$

Thus $\hat{f}(\xi) \notin L^1(\mathbb{R})$ implying that $(\hat{f})^{\vee}$ is undefined and Fourier inversion does not hold for all integrable functions.[3]

[3] Throughout the paper χ denotes the characteristic function.

Unfortunately we will need to place more restrictions on $L^1(\mathbb{R}^n)$ functions to obtain Fourier inversion. We will now see under what conditions Fourier inversion holds on $L^1(\mathbb{R}^n)$.

LEMMA 1.14: If $f \in L^1(\mathbb{R}^n)$ then $\hat{f}$ is continuous on $\mathbb{R}^n$.

Proof. Note that

$$\begin{aligned}\left|\hat{f}(x-y) - \hat{f}(x)\right| &= \left|\int_{\mathbb{R}^n} (e^{-2\pi i t\cdot(x-y)} - e^{-2\pi i t\cdot x}) f(t) dt\right| \\ &\leq \int_{\mathbb{R}^n} \left|e^{-2\pi i t\cdot(x-y)} - e^{-2\pi i t\cdot x}\right| |f(t)|\, dt.\end{aligned}$$

Since f is integrable we can apply the Lebesgue dominated convergence theorem and pull the limit as $y \to 0$ inside the integral. Thus

$$\lim_{y\to 0}\left|\hat{f}(x-y) - \hat{f}(x)\right| \leq \int_{\mathbb{R}^n} \lim_{y\to 0}\left|e^{-2\pi i t\cdot(x-y)} - e^{-2\pi i t\cdot x}\right| |f(t)|\, dt = 0.$$

Therefore $\left|\hat{f}(x-y) - \hat{f}(x)\right| \to 0$ as $y \to 0$ proving the claim. □

We will now use the lemma above to establish a sufficient condition that guarantees Fourier inversion.

THEOREM 1.15: (Fourier Inversion on Integrable Functions) If f and $\hat{f}$ are both integrable then $(\hat{f})\check{} = f$ a.e.

Proof. We know from Lemma 1.14 that $\hat{f}$ is continuous and we can thus apply Fubini's theorem to establish the following identity:

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(x)g(x)dx, \text{ where } g \in S(\mathbb{R}^n).$$

Now let $\hat{g}(x)$ be the same approximate identity used in the proof of Theorem 1.9 and follow the same approach to obtain the desired result. □

As you can see Fourier inversion holds on L^1 when $\hat{f}$ is also integrable. Now that we have extended the Fourier transform to L^1, studied its basic properties, and established under what conditions Fourier inversion holds we will conclude our study on integrable functions with the Riemann-Lebesgue lemma.

THEOREM 1.16: (Riemann-Lebesgue Lemma) Let $f \in L^1(\mathbb{R}^n)$. Then

$$\left|\hat{f}(\xi)\right| \to 0 \text{ as } |\xi| \to \infty.$$

Proof. We will first show that the Fourier transform of any simple function decays to zero. Note that

$$\begin{aligned}
\left|\widehat{\chi_{[a,b]}}(\xi)\right| &\leq \int_a^b \left|e^{-2\pi i x \cdot \xi}\right| dx \\
&= \left|\frac{1}{2\pi i \xi}\right| \left|e^{-2\pi i \xi a} - e^{-2\pi i \xi b}\right| \\
&\leq \left|\frac{1}{\xi}\right| \to 0 \text{ as } \xi \to \infty.
\end{aligned}$$

Similarly, for $g = \prod_{j=1}^n \chi_{[a_j, b_j]}$ on $\mathbb{R}^n$ we have

$$\hat{g}(\xi) = \prod_{j=1}^n \frac{e^{-2\pi i \xi_j a_j} - e^{-2\pi i \xi_j b_j}}{2\pi i \xi_j},$$

which also tends to zero as $|\xi| \to \infty$ in $\mathbb{R}^n$. Thus we can conclude that the Fourier transform of any simple function in $\mathbb{R}^n$ tends to zero as $|\xi| \to \infty$. By the simple approximation theorem there exist a finite sum h of simple functions such that $\|f - h\|_{L^1} < \frac{\varepsilon}{2}$ for any $\varepsilon > 0$. Thus for large $|\xi|$ we have

$$\left|\hat{f}(\xi)\right| \leq \left|\hat{f}(\xi) - \hat{h}(\xi)\right| + \left|\hat{h}(\xi)\right| \leq \|f - h\|_1 + \left|\hat{h}(\xi)\right| < \varepsilon.$$

$\square$

1.4 Fourier Transform On L^2

Extending the Fourier transform to the space of square integrable functions is not as straight forward. The integral defining the Fourier transform does not necessarily converge absolutely for functions in $L^2(\mathbb{R}^n)$. However there is a way to overcome this problem and develope a natural definition for the Fourier transform on the space of square integrable functions. Note that the Schwartz space S is a dense subset of L^2. Thus for any $f \in L^2$ there exists a sequence $\{f_n\}$ in S such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. Now by applying Theorem 1.11 to this sequence we can see $||\hat{f}_n - \hat{f}_m||_2 = ||\widehat{f_n - f_m}||_2 = \|f_n - f_m\|_2 \to 0$ as $m, n \to \infty$. Thus $\{\hat{f}_n\}$ is a Cauchy sequence in L^2 and we know its limit exist in L^2 since L^2 is complete. We define this limit as the Fourier transform of $f \in L^2$.

DEFINITION 1.17: Let $f \in L^2$ and $\{f_n\}$ be the sequence of Schwartz functions that converges to f in L^2. Then we define the *Fourier transform of* f to be

$$\hat{f} = \lim_{n\to\infty} \hat{f}_n.$$

Since we our definining the Fourier transform on L^2 as the limit of a sequence of Schwartz functions its easy to check that properties (1)-(8), (12) and (13) from Proposition 1.6 also extend to L^2. The density of the Schwartz functions in L^2 also gives us Fourier inversion in L^2 where the inverse Fourier transform is still $\check{f} = \hat{f}(-x)$. We conclude our study of the Fourier transform on the space of square-integrable functions with the following theorem and corollary.

THEOREM 1.18: Let $f, g \in L^2(\mathbb{R}^n)$. Then $\|f\|_2 = ||\hat{f}||_2$.

Proof. Let $\{f_n\}$ be the sequence of functions described in definition 1.17. Then by Theorem 1.11 we have

$$||f||_2 = \lim_{n\to\infty} ||f_n||_2 = \lim_{n\to\infty} ||\hat{f}_n||_2 = ||\hat{f}||_2.$$

$\square$

COROLLARY 1.19: Let $f, g \in L^2(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} f(x)\bar{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi.$$

Proof. It is well-known that an inner-product on $L^2(\mathbb{R}^n)$ is given by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\bar{g}(x)dx.$$

With this in mind, apply Theorem 1.18 and the polarization identity in the following manner:

$$\begin{aligned}
\langle f, g\rangle &= \frac{1}{4}\left(||f+g||_2^2 - ||f-g||_2^2 + i||f+ig||_2^2 - i||f-ig||_2^2\right)\\
&= \frac{1}{4}\left(||\widehat{f+g}||_2^2 - ||\widehat{f-g}||_2^2 + i||\widehat{f+ig}||_2^2 - i||\widehat{f-ig}||_2^2\right)\\
&= \frac{1}{4}\left(||\hat{f}+\hat{g}||_2^2 - ||\hat{f}-\hat{g}||_2^2 + i||\hat{f}+i\hat{g}||_2^2 - i||\hat{f}-i\hat{g}||_2^2\right)\\
&= \left\langle \hat{f}, \hat{g}\right\rangle.
\end{aligned}$$

Therefore $\int_{\mathbb{R}^n} f(x)\bar{g}(x)dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi$ as desired. $\square$

CHAPTER 2

L^p-THEORY

Everything is now ready to define the Fourier transform on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. However, we wish to do far more than just provide a definition. Due to generality, $L^p(\mathbb{R}^n)$ is the primary space of interest in studying the Fourier transform. Thus we will switch gears from extending previously established properties to establishing new results that hold on all spaces discussed thus far. To do so we will need some fundamental interpolation theorems and several properties of Bessel functions. Thus our study of the Fourier transform shall be placed on the sideline for a short time while we prove some crucial interpolation theorems and briefly introduce Bessel functions.

Of particular use in our studies will be the Riesz-Thorin interpolation theorem. It will require some complex analysis to prove, specifically Hadamard's three lines lemma. While on the topic of interpolation theorems another fundamental result, the Marcinkiewicz interpolation theorem, will also be presented. The Marcinkiewicz interpolation came some time after the Riesz-Thorin result and requires less in the assumption and requires no complex analysis to prove. Both results are classical interpolation results and foundational in the theory of interpolation of operators. Interpolation theorems give rise to the Hausdorff-Young inequality which will prove instrumental in deriving further results.

It turns out that Bessel functions play a key part in studying both Fourier series and the Fourier transform. Recall that it was mentioned that the smoothness of the Schwartz functions was one reason for introducing the Fourier transform on Schwartz space. So what impact does the smoothness of a function have on the behavior of its Fourier transform, if any at all? Firstly, we must develope a rigorous way to measure the smoothness of a function. It is in this endeavor that a particular spherical mean operator comes into play. An identity directly links spherical

Bessel functions with the afforementioned spherical mean operator. With Bessel functions in play we will need to take the time to establish some of their properties and, more importantly, their asymptotics. The Hausdorff-Young inequality and the asymptotics of Bessel functions will provide all that is necessary to establish some truly interesting results about the Fourier transform on $L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$.

2.1 Interpolation Theorems

There has been much research in the interpolation of operators. Interpolation theorems allow us to establish boundedness for linear, or sublinear, operators by looking at an "upper" and "lower" bound estimate. We will take the time to prove two fundamental results: the Marcinkiewicz interpolation theorem and the Riesz-Thorin interpolation theorem. The Riesz-Thorin interpolation theorem in particular will be crucial in our study of the Fourier transform on L^p where $1 \leq p \leq 2$. We begin by defining linear and sublinear operators.

DEFINITION 2.1: Let $\mathbf{T}$ be an operator defined on a linear space of complex-valued measurable functions on a measure space (X, μ) and taking values in the set of all complex-valued finite almost everywhere measurable functions on a measure space (Y, ν). Then $\mathbf{T}$ is said to be *linear* if for all f, g and all $\lambda \in \mathbb{C}$, we have

$$\mathbf{T}(f+g) = \mathbf{T}(f) + \mathbf{T}(g) \text{ and } \mathbf{T}(\lambda f) = \lambda \mathbf{T}(f).$$

$\mathbf{T}$ is *sublinear* if for all f, g and all $\lambda \in \mathbb{C}$, we have

$$|\mathbf{T}(f+g)| \leq |\mathbf{T}(f)| + |\mathbf{T}(g)| \text{ and } |\mathbf{T}(\lambda f)| = |\lambda| \, |\mathbf{T}(f)| \, .$$

First we will take a look at the Riesz-Thorin interpolation theorem. One can not study interpolation of operators without coming across the Riesz-Thorin interpolation theorem. The theorem was originally known as the Riesz convexity theorem until Riesz's student, G. Olof Thorin, made several improvements giving rise to the interpolation theorem we know it as today. The elegant proof given for the Riesz-Thorin interpolation theorem in this paper was originally conceived by J.D. Tamarkin and A. Zygmund in 1944 and is drawn from L. Grafakos[1]. Tamarkin's and Zygmund's proof of the Riesz-Thorin interpolation theorem will require some tools from complex analysis, notably Hadamard's three lines lemma. For more history on the result please refer to L. Grafakos[1].

LEMMA 2.2: (Hadamard's three lines lemma) Let F be analytic in the open strip $S = \{z \in \mathbb{C} : 0 < Re(z) < 1\}$, continuous and bounded on its closure, such that $|F(z)| \leq B_0$ when $Re(z) = 0$ and $|F(z)| \leq B_1$ when $Re(z) = 1$, where $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-\theta} B_1^{\theta}$ when $Re(z) = \theta$, for any $0 \leq \theta \leq 1$.

Proof. Let

$$G(z) = F(z)(B_0^{1-z} B_1^z)^{-1} \text{ and } G_n(z) = G(z)e^{\frac{(z^2-1)}{n}}.$$

It is immediate that both are analytic functions and $G(z)$ is bounded by some constant M on the closed unit strip given that $F(z)$ is bounded on the strip while $B_0^{1-z} B_1^z$ is bounded from below on the unit strip. The hypothesis also implies that G is bounded by one on the boundaries. For the remainder of the proof let $z = x + iy$. Note that

$$|G_n(x + iy)| = |G(x + iy)|e^{\frac{-y^2}{n}} e^{\frac{x^2-1}{n}} \leq Me^{\frac{-y^2}{n}}.$$

Thus $G_n(z)$ converges uniformly to zero in $0 < x < 1$ as $|y| \to \infty$. This allows us to choose $y(n) > 0$ such that for $|y| \geq y(n)$, $|G(z)| \leq 1$ uniformly in the rectangle $[0, 1] \times [-y(n), y(n)]$. Thus $|G_n(z)| \leq 1$ everywhere in the strip. By letting $n \to \infty$ it follows that $|G(z)| \leq 1$ in the strip. Therefore, in the strip we know $|F(z)(B_0^{1-z} B_1^z)^{-1}| \leq 1$ and the desired result is reached. □

With Hadamard's three lines lemma proven, we have all that is necessary to prove the Riesz-Thorin interpolation theorem.

THEOREM 2.3: (Riesz-Thorin Interpolation Theorem) Let (X, μ) and (Y, ν) be two measure spaces. Let T be a linear operator defined on the set of all simple functions on X and taking values in the set of measurable functions on Y. Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and assume that

$$\begin{aligned} \|T(f)\|_{q_0} &\le M_0 \|f\|_{p_0}, \\ \|T(f)\|_{q_1} &\le M_1 \|f\|_{p_1}, \end{aligned} \tag{2.1.1}$$

for all simple functions f on X. Then for all $0 < \theta < 1$ we have

$$\|T(f)\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p}$$

for all simple functions f on X, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

By density, T has a unique extension as a bounded operator from $L^p(X, \mu)$ to $L^q(Y, \nu)$ for all p and q as described above.

Proof. Let

$$f = \sum_{k=1}^{m} a_k e^{i\alpha_k} \chi_{A_k}$$

be a simple function on X, where $a_k > 0, \alpha_k \in \mathbb{R}$, and A_k are pairwise disjoint subsets of X with finite measure. Note that

$$\|T(f)\|_{L^q} = \sup_{g \in G} \left| \int_Y T(f)(x) g(x) d\nu(x) \right|,$$

where $G = \left\{ g \in L^{q'} : g \text{ is simple and } \|g\|_{L^{q'}} \le 1 \right\}$. Thus we need to control the supremum above. Write

$$g = \sum_{j=1}^{n} b_j e^{i\beta_j} \chi_{B_j},$$

where ofcourse $b_j > 0, \beta_j \in \mathbb{R}$, and B_j are pairwise disjoint subsets of Y with finite measure[1]. Let

$$P(z) = \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \text{ and } Q(z) = \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z.$$

For z in the closed unit strip $\bar{S} = \{z \in \mathbb{C} : 0 \le Re(z) \le 1\}$, define

$$F(z) = \int_Y T(f_z)(x) g_z(x) d\nu(x),$$

where

$$f_z = \sum_{k=1}^{m} a_k^{P(z)} e^{i\alpha_k} \chi_{A_k}, g_z = \sum_{j=1}^{n} b_j^{Q(z)} e^{i\beta_j} \chi_{B_j}.$$

By the linearity of T,

$$F(z) = \sum_{k=1}^{m} \sum_{j=1}^{n} a_k^{P(z)} e^{i\alpha_k} b_j^{Q(z)} e^{i\beta_j} \int_Y T(\chi_{A_k}) \chi_{B_j}(x) d\nu(x).$$

As you can see F is analytic in z. We will now look at z on the boundaries of the unit strip $\bar{S}$. Consider $z \in \bar{S}$ such that $Re(z) = 0$. Thus

$$\left| a_k^{P(z)} \right| = \left| e^{P(z) ln(a_k)} \right| = \left| e^{\frac{p}{p_0} ln(a_k)} e^{i(Im(P(z)) ln(a_k)} \right| = a_k^{\frac{p}{p_0}}.$$

Thus by the disjointness of the sets A_k and the previous equation we have $\|f_z\|_{p_0}^{p_0} = \|f\|_p^p$. Similarly, $\|g_z\|_{q'_0}^{q'_0} = \|g\|_{q'}^{q'}$. Now by applying Hölders inequality and conditions

[1]Throughout the paper we use the following notation: $q' = \frac{q}{q-1}$.

in the hypothesis we obtain

$$\begin{aligned}
|F(z)| &\leq \|T(f_z)\|_{q_0} \|g_z\|_{q_0'} \\
&\leq M_0 \|f_z\|_{p_0} \|g_z\|_{q_0'} \\
&= M_0 \|f\|_p^{\frac{p}{p_1}} \|g\|_{q'}^{\frac{q'}{q_0'}} . \\
\therefore |F(z)| &\leq M_0 \|f\|_p^{\frac{p}{p_1}} \|g\|_{q'}^{\frac{q'}{q_0'}} \text{ when } Re(z) = 0. \qquad (2.1.2)
\end{aligned}$$

By the same train of thought, when $Re(z) = 1$ we have

$$|F(z)| \leq M_1 \|f\|_p^{\frac{p}{p_1}} \|g\|_{q'}^{\frac{q'}{q_1'}} . \qquad (2.1.3)$$

The stage is now set to apply Hadamard's three lines lemma. Applying Lemma 2.2 gives us

$$|F(z)| \leq \left(M_0 \|f\|_p^{\frac{p}{p_1}} \|g\|_{q'}^{\frac{q'}{q_0'}} \right)^{1-\theta} \left(M_1 \|f\|_p^{\frac{p}{p_1}} \|g\|_{q'}^{\frac{q'}{q_1'}} \right)^{\theta} \text{ when } Re(z) = \theta.$$

The conditions on p and q mentioned above can now be used to clean up the inequality above and give us

$$|F(z)| \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p \|g\|_{q'} \text{ when } Re(z) = \theta.$$

Note that $P(\theta) = Q(\theta) = 1$, implying $F(\theta) = \int_Y T(f) g d\nu$ which brings us to the desired result:

$$\begin{aligned}
\|T(f)\|_q &= \sup_{g \in G} \left| \int_Y T(f)(x) g(x) d\nu(x) \right| = \sup_{g \in G} |F(\theta)| \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p . \\
\therefore \|T(f)\|_q &\leq M_0^{1-\theta} M_1^{\theta} \|f\|_p .
\end{aligned}$$

$\square$

The Riesz-Thorin interpolation theorem is a wonderful result that will be used to establish the Hausdorff-Young inequality later. However, it does require somewhat strong endpoint estimates and is restricted to linear operators. So can an interpolation result be established for sublinear operators with looser endpoint restrictions? It is here we introduce the Marcinkiewicz interpolation theorem. The Marcinkiewicz interpolation theorem was first announced by mathematician Józef Marcinkiewicz in 1939, some years after the Riesz-Thorin interpolation result. Proving the Marcinkiewicz interpolation theorem uses a real-method and does not require tools from complex analysis. Instead the result is proved by a divide and conquer strategy where the arbitrary function f is divided into two components, large and small f. Each component of the function is then estimated and the two results combined at the end to produce the desired result. Before presenting the Marcinkiewicz interpolation theorem we establish the following simple lemma.

LEMMA 2.4: Let X be a metric space with measure μ and $\alpha > 0$. For $f \in L^p(X)$, $0 < p < \infty$ we have

$$\|f\|_p^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha \text{ where } d_f \text{ is the distribution function.}$$

Proof. The proof is a simple application of Fubini's theorem and definition of distribution function.

$$\begin{aligned} p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha &= p \int_0^\infty \alpha^{p-1} \int_X \chi_{\{x:|f(x)|>\alpha\}} d\mu(x) d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) = \|f(x)\|_p^p . \end{aligned}$$

□

The elementary result above gives all the tools necessary to move forward.

Without any further adieu, we conclude our study of interpolation theorems with the Marcinkiewicz interpolation theorem.

THEOREM 2.5: (Marcinkiewicz Interpolation Theorem) Let (X,μ) and (Y,ν) be measure spaces and let $0 < p_0 < p_1 \leq \infty$. Let $\mathbf{T}$ be a sublinear operator defined on the space $L^{p_0}(X) + L^{p_1}(X)$ and taking values in the space of measurable functions on Y.[2] Assume that there exist two positive constants A_0 and A_1 such that

$$\|T(f)\|_{p_0,\infty} \leq A_0 \|f\|_{p_0} \text{ for all } f \in L^{p_0}(X), \tag{2.1.4}$$

$$\|T(f)\|_{p_1,\infty} \leq A_1 \|f\|_{p_1} \text{ for all } f \in L^{p_1}(X). \tag{2.1.5}$$

Then for every $p_0 < p < p_1$ and $f \in L^p(X)$ we have $\|T(f)\|_p \leq A\|f\|_p$,where constant

$$A = 2\left(\frac{p}{p-p_0} + \frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

Proof. We begin by proving the estimate for finite p_1. Fix $f \in L^p(X)$ and $\alpha > 0$. Split $f = f_0^\alpha + f_1^\alpha$, where $f_0^\alpha \in L^{p_0}$ and $f_1^\alpha \in L^{p_1}$ by cutting $|f|$ at height $\delta\alpha$ for some $\delta > 0$. So

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \delta\alpha, \\ 0 & \text{for } |f(x)| \leq \delta\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} 0 & \text{for } |f(x)| > \delta\alpha, \\ f(x) & \text{for } |f(x)| \leq \delta\alpha. \end{cases}$$

Clearly such a split is possible. Since $p_0 < p$ we have

$$\begin{aligned} \|f_0\|_{p_0}^{p_0} &= \int_{|f|>\delta\alpha} |f|^p \, |f|^{p_0-p} \, d\mu(x) \\ &\leq (\delta\alpha)^{p_0-p} \|f\|_p^p. \end{aligned}$$

[2]Here $||\cdot||_{p,\infty}$ denotes the weak L^p norm.

Similarly,

$$\begin{aligned}\|f_1^\alpha\|_{p_1}^{p_1} &= \int_{|f|\le\delta\alpha} |f|^p\,|f|^{p_1-p}\,d\mu(x)\\ &\le (\delta\alpha)^{p_1-p}\,\|f\|_p^p\,.\end{aligned}$$

Thus by the sublinearity of T we have

$$|T(f)| = |T(f_0^\alpha + f_1^\alpha)| \le |T(f_0^\alpha)| + |T(f_1^\alpha)|\,,\ \text{implying}$$

$$\{x : |T(f)| > \alpha\} \subseteq \left\{x : |T(f_0^\alpha(x))| > \frac{\alpha}{2}\right\} \cup \left\{x : |T(f_1^\alpha(x))| > \frac{\alpha}{2}\right\}.$$

$$\therefore d_{T(f)}(\alpha) \le d_{T(f_0^\alpha)}(\frac{\alpha}{2}) + d_{T(f_1^\alpha)}(\frac{\alpha}{2}),$$

where $d_{T(f)}$ denotes the distribution function of $T(f)$. Now apply Chebyshev's inequality to obtain the following estimate:

$$\begin{aligned}d_{T(f)}(\alpha) &\le (\frac{\alpha}{2})^{-p_0}\int_{|T(f_0^\alpha)|>\frac{\alpha}{2}} |T(f_0^\alpha)|^{p_0}\,d\mu(x) + (\frac{\alpha}{2})^{-p_1}\int_{|T(f_1^\alpha)|>\frac{\alpha}{2}} |T(f_1^\alpha)|^{p_1}\,d\mu(x)\\ &\le (\frac{2}{\alpha})^{p_0}\,\|T(f_0^\alpha)\|_{p_0}^{p_0} + (\frac{2}{\alpha})^{p_1}\,\|T(f_1^\alpha)\|_{p_1}^{p_1}\,.\end{aligned}$$

Application of 2.1.4 and 2.1.5 from the hypothesis to the estimate above yield

$$d_{T(f)}(\alpha) \le (2A_0)^{p_0}\alpha^{-p_0}\,\|f_0^\alpha\|_{p_0}^{p_0} + (2A_1)^{p_1}\alpha^{-p_1}\,\|f_1^\alpha\|_{p_1}^{p_1}\,.$$

With the result above, Lemma 2.4, and a little bit of massaging we obtain the following estimate

$$\begin{aligned}
\|T(f)\|_p^p &\leq p(2A_0)^{p_0}\int_0^\infty \alpha^{p-1-p_0}\int_{|f|>\delta\alpha}|f(x)|^{p_0}d\mu(x)d\alpha \\
&\quad +p(2A_1)^{p_1}\int_0^\infty \alpha^{p-1-p_1}\int_{|f|\leq\delta\alpha}|f(x)|^{p_1}d\mu(x)d\alpha \\
&= p(2A_0)^{p_0}\int_X |f(x)|^{p_0}\int_0^{\frac{1}{\delta}|f(x)|}\alpha^{p-1-p_0}d\alpha d\mu(x) \\
&\quad +p(2A_1)^{p_1}\int_X |f(x)|^{p_1}\int_{\frac{1}{\delta}|f(x)|}^\infty \alpha^{p-1-p_1}d\alpha d\mu(x) \\
&= \frac{p(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}\int_X |f(x)|^{p_0}|f(x)|^{p-p_0}d\mu(x) \\
&\quad +\frac{p(2A_1)^{p_1}}{p_1-p}\frac{1}{\delta^{p-p_1}}\int_X |f(x)|^{p_1}|f(x)|^{p-p_1}d\mu(x) \\
&= p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}+\frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\|f\|_p^p. \\
\therefore \|T(f)\|_p^p &\leq p\left(\frac{(2A_0)^{p_0}}{p-p_0}\frac{1}{\delta^{p-p_0}}+\frac{(2A_1)^{p_1}}{p_1-p}\delta^{p_1-p}\right)\|f\|_p^p. \qquad (2.1.6)
\end{aligned}$$

The convergence of the integrals in α above are justified since $p_0 < p < p_1$. Given p, p_1, p_0, A_0, A_1 are all constants we can choose

$$\delta = \frac{(2A_0)^{\frac{p_0}{p_1-p_0}}}{(2A_1)^{\frac{p_1}{p_1-p_0}}} > 0.$$

This choice for δ and 2.1.6 give us the desired result:

$$\|T(f)\|_p \leq A\|f\|_p \text{ where } A = 2\left(\frac{p}{p-p_0}+\frac{p}{p_1-p}\right)^{\frac{1}{p}} A_0^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}.$$

Now that we have proven the case for finite p_1, let us consider $p_1 = \infty$. The proof here is similar to the finite case. As before write $f = f_0^\alpha + f_1^\alpha$, where

$$f_0^\alpha(x) = \begin{cases} f(x) & \text{for } |f(x)| > \gamma\alpha, \\ 0 & \text{for } |f(x)| \leq \gamma\alpha, \end{cases}$$

$$f_1^\alpha(x) = \begin{cases} 0 & \text{for } |f(x)| > \gamma\alpha, \\ f(x) & \text{for } |f(x)| \leq \gamma\alpha. \end{cases}$$

As before,

$$\{x : |T(f)| > \alpha\} \subseteq \left\{x : |T(f_0^\alpha(x))| > \frac{\alpha}{2}\right\} \cup \left\{x : |T(f_1^\alpha(x))| > \frac{\alpha}{2}\right\}.$$

Choose $\gamma = (2A_1)^{-1}$. Thus, $\|T(f_1^\alpha)\|_\infty \leq A_1 \|f_1^\alpha\|_\infty \leq A_1\gamma\alpha = \frac{\alpha}{2}$ implying that the set $\left\{x : |T(f_1^\alpha)(x)| > \frac{\alpha}{2}\right\}$ has measure zero. Therefore,

$$d_{T(f)}(\alpha) \leq d_{T(f_0^\alpha)}(\frac{\alpha}{2}).$$

Since T maps L^{p_0} to $L^{p_0,\infty}$, it follows that

$$\begin{aligned} d_{T(f_0^\alpha)}(\frac{\alpha}{2}) &\leq \frac{(2A_0)^{p_0} \|f_0^\alpha\|_{L^{p_0}}^{p_0}}{\alpha^{p_0}} = \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\gamma\alpha} |f(x)|^{p_0} d\mu(x). \\ \therefore d_{T(f)}(\alpha) &\leq \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\gamma\alpha} |f(x)|^{p_0} d\mu(x). \end{aligned}$$

As in the first case we now apply Lemma 2.4 to the estimate above to obtain the desired result:

$$\begin{aligned} \|T(f)\|_p^p &= p\int_0^\infty \alpha^{p-1} d_{T(f)}(\alpha) d\alpha \\ &\leq p\int_0^\infty \alpha^{p-1} d_{T(f_0^\alpha)}(\frac{\alpha}{2}) d\alpha \\ &\leq p\int_0^\infty \alpha^{p-1} \frac{(2A_0)^{p_0}}{\alpha^{p_0}} \int_{|f|>\gamma\alpha} |f(x)|^{p_0} d\mu(x) d\alpha \\ &= p(2A_0)^{p_0} \int_X |f(x)| \int_0^{2A_1|f(x)|} \alpha^{p-p_0-1} d\alpha d\mu(x) \\ &= \frac{p(2A_1)^{p-p_0}(2A_0)^{p_0}}{p-p_0} \int_X |f(x)|^p d\mu(x) \\ &= \left(2\left(\frac{p}{p-p_0}\right)^{\frac{1}{p}} A_1^{\frac{p-p_0}{p}} A_0^{\frac{p_0}{p}} \|f(x)\|_{L^p}\right)^p. \end{aligned}$$

$\square$

2.2 Bessel Functions

To say the study of Bessel functions is both vast and extensive would certainly be an understatement. Indeed one need look no farther than *A Treatise On The Theory of Bessel Functions* by G.N. Watson[7] for a very detailed and extensive study of Bessel functions. Since their conception by Daniel Bernoulli and generalization by Friedrich Bessel, Bessel functions have had no shortage of uses in a variety of mathematical fields and problems. Our study on Bessel functions will be brief, where we will introduce a few useful properties used throughout this paper. Bessel functions of order ν, denoted $J_\nu(t)$, are solutions of Bessel's differential equation:

$$t^2\frac{d^2}{dt^2}(J_\nu(t)) + t\frac{d}{dt}(J_\nu(t)) + (t^2 - \nu^2)J_\nu(t) = 0.$$

However, we will define Bessel Function $J_\nu(t)$ by its Poisson representation formula.

DEFINITION 2.6: Let $\nu > -\frac{1}{2}$ and $t \geq 0$. Then we define the *Bessel function $J_\nu(t)$ of order ν* as

$$J_\nu(t) = \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1 - s^2)^{\nu - \frac{1}{2}} ds.$$

With a little bit of work one can see that $J_\nu(t)$ certainly solves Bessels differential equation. Several Bessel functions of the first kind are graphed in Figure 2.1. Note the oscillation and rapid decay.

Armed with our definition, we will now introduce several basic properties of Bessel functions in the proposition below.

PROPOSITION 2.7: Let $t > 0$, $\nu > \frac{1}{2}$, and $\mu > -\frac{1}{2}$. Then we have the following properties:

1. $\frac{d}{dt}(t^{-\nu}J_\nu(t)) = -t^{-\nu}J_{\nu+1}(t)$,

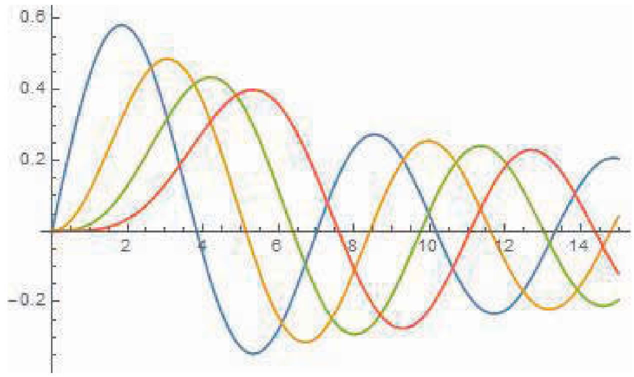

Figure 2.1: Bessel functions of the first kind.

2. $\frac{d}{dt}(t^\nu J_\nu(t)) = t^\nu J_{\nu-1}(t)$,

3. $J_\nu(t) = \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{t}{2}\right)^\nu \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\nu+1)}\frac{t^{2j}}{(2j)!}$,

4. $\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds = \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}} J_{\mu+\nu+1}(t)$.

Proof. A few simple, if tedious, calculations show

$$\begin{aligned} t^{-\nu}J_\nu'(t) &= \frac{\nu}{t2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 e^{its}(1-s^2)^{\nu-\frac{1}{2}}ds \\ &\quad + \frac{i}{2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 se^{its}(1-s^2)^{\nu-\frac{1}{2}}ds. \end{aligned}$$

Now apply the result above, integration by parts, a little bit of massaging and the fact that $\Gamma(x+1) = x\Gamma(x)$:

$$\begin{aligned} \frac{d}{dt}(t^{-\nu}J_\nu(t)) &= -\nu t^{-\nu-1}J_\nu(t) + t^{-\nu}J_\nu'(t) \\ &= \frac{i}{2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 se^{its}(1-s^2)^{\nu-\frac{1}{2}}ds \\ &= \frac{i}{2^\nu\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 \frac{it}{2}e^{its}\frac{(1-s^2)^{\nu+\frac{1}{2}}}{\nu+\frac{1}{2}}ds \\ &= \frac{-t}{2^{\nu+1}(\Gamma((\nu+1)+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 e^{its}(1-s^2)^{(\nu+1)-\frac{1}{2}}ds \\ &= -t^{-\nu}J_{\nu+1}(t). \end{aligned}$$

Similarly, $\frac{d}{dt}(t^\nu J_\nu(t)) = t^\nu J_{\nu-1}(t)$. Thus we have established the first two identities listed above. Establishing the third identity will take a little algebraic handiwork and the following idenitity:

$$e^{its} = \sum_{j=0}^{\infty} (-1)^j \frac{(ts)^{2j}}{(2j)!} + i\sin(ts).$$

Applying the identity above and a simple u-substitution yield

$$\begin{aligned} J_\nu(t) &= \frac{\left(\frac{t}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1-s^2)^{\nu-\frac{1}{2}}ds \\ &= \frac{(\frac{t}{2})^\nu}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}(-1)^j \frac{1}{\Gamma(\nu+\frac{1}{2})}\frac{t^{2j}}{(2j)!}2\int_0^1 s^{2j-1}(1-s^2)^{\nu-\frac{1}{2}}sds \\ &= \frac{(\frac{t}{2})^\nu}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}(-1)^j \frac{1}{\Gamma(\nu+\frac{1}{2})}\frac{t^{2j}}{(2j)!}\int_0^1 u^{j-\frac{1}{2}}(1-u)^{\nu-\frac{1}{2}}ds. \end{aligned}$$

We now apply the beta function identity $\int_0^1 t^{w-1}(1-t)^{z-1}dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$ to obtain the desired result:

$$\begin{aligned} J_\nu(t) &= \frac{(\frac{t}{2})^\nu}{\Gamma(\frac{1}{2})}\sum_{j=0}^{\infty}(-1)^j \frac{1}{\Gamma(\nu+\frac{1}{2})}\frac{t^{2j}}{(2j)!}\frac{\Gamma(j+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+j+1)} \\ &= \frac{1}{\Gamma(\frac{1}{2})}\left(\frac{t}{2}\right)^\nu \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(j+\frac{1}{2})}{\Gamma(j+\nu+1)}\frac{t^{2j}}{(2j)!}. \end{aligned}$$

To establish the last identity we use the series representation of $J_\nu(t)$ established above and the beta function identity again:

$$\begin{aligned}
\int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds &= \frac{(\frac{t}{2})^\mu}{\Gamma(\frac{1}{2})}\int_0^1 \sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}s^{2j+\mu+\mu}(1-s^2)^\nu s ds \\
&= \frac{1}{2}\frac{(\frac{t}{2})^\mu}{\Gamma(\frac{1}{2})}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\int_0^1 u^{j+\mu}(1-u)^\nu du \\
&= \frac{1}{2}\frac{(\frac{t}{2})^\mu}{\Gamma(\frac{1}{2})}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+1)(2j)!}\frac{\Gamma(u+j+1)\Gamma(\nu+1)}{\Gamma(j+\mu+\nu+2)} \\
&= \frac{2^\nu\Gamma(\nu+1)}{t^{\nu+1}}\frac{(\frac{t}{2})^{\mu+\nu+1}}{\Gamma(\frac{1}{2})}\sum_{j=0}^\infty \frac{(-1)^j\Gamma(j+\frac{1}{2})t^{2j}}{\Gamma(j+\mu+\nu+2)(2j)!} \\
&= \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t). \\
\therefore \int_0^1 J_\mu(ts)s^{\mu+1}(1-s^2)^\nu ds &= \frac{\Gamma(\nu+1)2^\nu}{t^{\nu+1}}J_{\mu+\nu+1}(t) \text{ as desired.}
\end{aligned}$$

Therefore all four properties above have been proven. □

With the Bessel function well-defined and some of its more basic properties established, let us now look at some interesting identities involving Bessel functions. Surprisingly Bessel functions are crucial when studying the Fourier transform and spheres. For example, what happens when you take the Fourier transform of surface measure on the unit sphere S^{n-1} in $\mathbb{R}^n$? Let $d\sigma$ denote surface measure on S^{n-1} for $n \geq 2$. Then we have the following identity[3]:

$$\hat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i\xi\cdot\theta}d\theta = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}J_{\frac{n-2}{2}}(2\pi|\xi|). \tag{2.2.7}$$

An interesting identity indeed! Another useful identity similar to the one above follows. Let S_{n-1} denote the unit sphere in $\mathbb{R}^n$ and ω_{n-1} its measure. Then we have the following identity:

$$\frac{1}{\omega_{n-1}}\int_{S^{n-1}} e^{2\pi i\xi\cdot\theta}d\theta = 2^\nu\Gamma(\nu+1)|\xi|^{-\nu}J_\nu(|\xi|), \text{ where } \nu = \frac{n-2}{2}. \tag{2.2.8}$$

[3]The proof of this identity can be found in Grafakos[1].

The identity above will prove crucial when studying the Fourier transform of a particular spherical mean operator used later. Bessel functions also arise when studying the Fourier transform of radial functions. The following theorem establishes what part Bessel functions play when taking the Fourier transform of a radial function on $\mathbb{R}^n$.

THEOREM 2.8: Let $f(x) = f_0(|x|)$ be a radial function defined on $\mathbb{R}^n$, where f_0 is defined on $[0, \infty)$. Then

$$\hat{f}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{\frac{n}{2}} dr.$$

Proof. The identity above is obtained by switching to polar coordinates and applying 2.2.7:

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \\
&= \int_0^\infty \int_{S^{n-1}} f_0(r) e^{-2\pi i \xi \cdot r\theta} d\theta r^{n-1} dr \\
&= \int_0^\infty f_0(r) r^{n-1} \left[\int_{S^{n-1}} e^{-2\pi i \xi r \cdot \theta} d\theta \right] dr \\
&= \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty f_0(r) J_{\frac{n-2}{2}}(2\pi r|\xi|) r^{\frac{n}{2}} dr.
\end{aligned}$$

□

Let us now present an interesting application of Theorem 2.8 and property 4 of Proposition 2.7. Consider

$$m_\alpha(\xi) = \begin{cases} (1 - |\xi|^2)^\alpha & \text{for } |\xi| \le 1, \\ 0 & \text{for } |\xi| > 1, \end{cases} \tag{2.2.9}$$

where α is nonnegative. Then Theorem 2.8 and Proposition 2.7 yield

$$\begin{aligned}(m_\alpha)\check{}(x) &= \frac{2\pi}{|x|^{\frac{n-2}{2}}}\int_0^1 J_{\frac{n}{2}-1}(2\pi|x|r)r^{\frac{n}{2}}(1-r^2)^\alpha dr \\ &= \frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{J_{\frac{n}{2}+\alpha}(2\pi|x|)}{|x|^{\frac{n}{2}+\alpha}}.\end{aligned} \tag{2.2.10}$$

The identity above will prove crucial when studying Bochner-Riesz summability. With so many useful identities now established, let us now look at the asymptotics of Bessel functions. Let us first consider the behavior of a Bessel function at infinity. A quick glance at Figure 2.1 shows us that a Bessel function certainly decays as it approaches infinity. The question is how fast is the decay and can it be controlled? We answer this question with the following theorem.

THEOREM 2.9: Let $r > 0$ and $\nu > -\frac{1}{2}$. Then

$$J_\nu(r) = \sqrt{2/\pi r}\cos(r - \pi\nu/2 - \pi/4) + O(r^{-\frac{3}{2}}) \text{ as } r \to \infty.$$

Implying $J_\nu(r) = O(r^{-\frac{1}{2}})$ as $r \to \infty$.[4]

Proof. The proof here is drawn from Grafakos[1] and Stein[2]. Fix $0 < \delta < \frac{1}{10} < 10 < R < \infty$. Consider the region $\Omega_{\delta,R}$ in the complex plane pictured in Figure 2.2.

Note that $\Omega_{\delta,R}$ is simply a rectangular region in which the bottom corners have been removed by quarter-circles of radius δ, centered at 1 and -1. Since $\log(1-z^2)$ is well defined and holomorphic in $\Omega_{\delta,R}$ we may define the holomorphic function

$$(1-z^2)^{\nu-\frac{1}{2}} = e^{(\nu-\frac{1}{2})\ln(1-z^2)},\ z \in \Omega_{\delta,R}.$$

Application of the Cauchy-Goursat theorem yields

$$\int_{\Omega_{\delta,R}} e^{irz}(1-z^2)^{\nu-\frac{1}{2}}dz = 0.$$

[4] $g(x) = O(h(x))$ if and only if there exists constant c such that $g(x) \le ch(x)$.

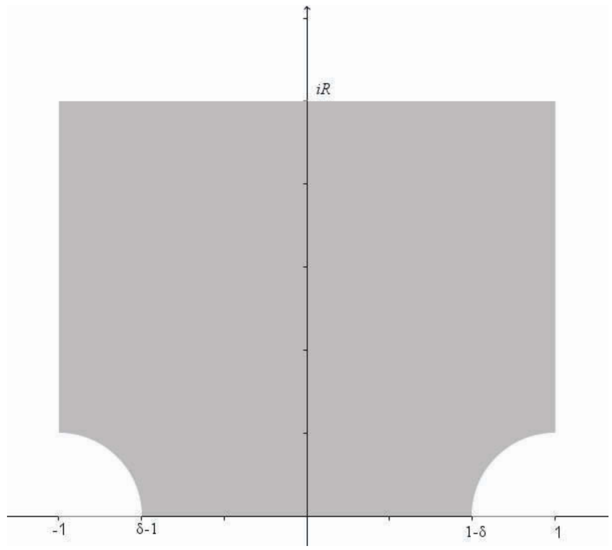

Figure 2.2: Region $\Omega_{\delta,R}$ in the complex plane.

Thus,

$$\begin{aligned} 0 &= \int_{-1+\delta}^{1-\delta} e^{irt}(1-t^2)^{\nu-\frac{1}{2}}\,dt + ie^{ir}\int_{\delta}^{R} e^{-rt}(t^2-2it)^{\nu-\frac{1}{2}}\,dt \\ &\quad -ie^{-ir}\int_{\delta}^{R} e^{-rt}(t^2+2it)^{\nu-\frac{1}{2}}\,dt + E(\delta,R), \end{aligned}$$

where $E(\delta, R)$ is the sum of the integrals over the two small quarter circles and the line segment from $1 + iR$ to $-1 + iR$. Let $\delta \to 0$ and $R \to \infty$. Then $E(\delta, R) \to 0$ and we have

$$\begin{aligned} \int_{-1}^{1} e^{irt}(1-t^2)^{\nu-\frac{1}{2}}\,dt &= ie^{-ir}\int_{0}^{\infty} e^{-rt}(t^2+2it)^{\nu-\frac{1}{2}}\,dt - ie^{ir}\int_{0}^{\infty} e^{-rt}(t^2-2it)^{\nu-\frac{1}{2}}\,dt \\ &\coloneqq I_1 - I_2. \end{aligned}$$

Let us now look at I_1 and I_2 respectively. It is known that

$$(t^2+2it)^{\nu-\frac{1}{2}} = \begin{cases} (2it)^{\nu-\frac{1}{2}} + O(t^{\nu+\frac{1}{2}}), & 0 \le t \le 1 \\ (2it)^{\nu-\frac{1}{2}} + O(t^{2\nu-1}), & 1 \le t < \infty \end{cases}$$

$$(t^2-2it)^{\nu-\frac{1}{2}} = \begin{cases} (-2it)^{\nu-\frac{1}{2}} + O(t^{\nu+\frac{1}{2}}), & 0 \le t \le 1 \\ (-2it)^{\nu-\frac{1}{2}} + O(t^{2\nu-1}), & 1 \le t < \infty. \end{cases}$$

Application of the identities above yield

$$\begin{aligned} I_1 &= ie^{-ir}\int_0^\infty e^{-rt}(t^2+2it)^{\nu-\frac{1}{2}}dt \\ &= ie^{-ir}\left[\int_0^\infty e^{-rt}(2it)^{\nu-\frac{1}{2}}dt + \int_0^1 e^{-rt}O(t^{\nu+\frac{1}{2}})dt + \int_1^\infty e^{-rt}O(t^{2\nu-1})dt\right] \\ &= ie^{-ir}\int_0^\infty e^{-rt}(2it)^{\nu-\frac{1}{2}}dt + O\left(\int_0^1 e^{-rt}t^{\nu+\frac{1}{2}}dt\right) + O\left(\int_1^\infty e^{-rt}t^{2\nu-1}dt\right). \end{aligned}$$

When we let $r \to \infty$ we have

$$I_1 = \frac{i(2i)^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})e^{-ir}}{r^{\nu+\frac{1}{2}}} + O(r^{-\nu-\frac{3}{2}}) + O(e^{-r}).$$

By the same approach,

$$I_2 = \frac{i(-2i)^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})e^{ir}}{r^{\nu+\frac{1}{2}}} + O(r^{-\nu-\frac{3}{2}}) + O(e^{-r}), \text{ as } r \to \infty.$$

Note that

$$J_\nu(r) = \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 e^{irt}(1-t^2)^{\nu-\frac{1}{2}}dt = \frac{\left(\frac{r}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}(I_1 - I_2).$$

Thus by collecting terms we obtain the desired result:

$$J_\nu(r) = \sqrt{2/\pi r}\cos(r - \pi\nu/2 - \pi/4) + O(r^{-\frac{3}{2}}) \text{ as } r \to \infty.$$

$\square$

Thus we may now conclude that for $r \geq 1$ we have $|J_\nu(r)| \leq cr^{-1/2}$ where c is a constant dependent only on ν. A good result but what about when $|r| < 1$? A quick glance at figure 2.1 seems to suggest that Bessel functions are bounded for small arguments, but bounded by what? The answer lies in the following theorem.

THEOREM 2.10: Let $0 < r \leq 1$ and $\nu > -1/2$. Then we have the following estimate

$$|J_\nu(r)| \leq cr^\nu,$$

where c is a constant dependent only on ν.

Proof. Let

$$S_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (e^{irt}-1)(1-t^2)^{\nu-\frac{1}{2}} dt.$$

Then we have

$$\begin{aligned} J_\nu(r) &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} dt + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{0}^{\pi} (\sin^2(\phi))^{\nu-\frac{1}{2}} \sin(\phi) d\phi + S_\nu(r). \end{aligned}$$

Now by applying a well known property of gamma functions to the integral above we obtain

$$\begin{aligned} J_\nu(r) &= \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\nu+1)} + S_\nu(r) \\ &= \frac{(r/2)^\nu}{\Gamma(\nu+1)} + S_\nu(r). \end{aligned} \tag{2.2.11}$$

Let us now look at the size of $S_\nu(r)$. Since $|e^{irt}-1| \leq r|t|$ for $0 < r \leq 1$ then

$$|S_\nu(r)| \leq \left| \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \right| r \int_{-1}^{1} |t|(1-t^2)^{\nu-\frac{1}{2}} dt = c_\nu r^{\nu+1}$$

where c_ν is a constant dependent only on ν. Combining the inequality above with

2.2.11 leads to

$$\begin{aligned}|J_\nu(r)| &\leq \left|\frac{(r/2)^\nu}{\Gamma(\nu+1)}\right| + |S_\nu(r)| \\ &\leq c_\nu r^\nu.\end{aligned}$$

$\square$

With the asymptotics of Bessel functions now known, we are ready to return to our study of the Fourier transform.

2.3 L^p-Theory

At last we have established everything necessary to begin our study of the Fourier transform on L^p functions! With the Fourier transform well-defined on L^1 and L^2 we may now extend the definition to L^p, where $1 \leq p \leq 2$. Let $f \in L^p$, $1 \leq p \leq 2$. Then we can split f such that $f = f_1 + f_2$ where $f_1 \in L^1$ and $f_2 \in L^2$. Note that such a split is possible, for instance $f_1 = f\chi_{|f|>1}$ and $f_2 = f\chi_{|f|\leq 1}$ will suffice. We can thus define $\hat{f} := \hat{f}_1 + \hat{f}_2$. However, is this definition well-defined? Suppose we have $f_1, h_1 \in L^1$ and $f_2, h_2 \in L^2$ such that $f = f_1 + f_2 = h_1 + h_2$. Then clearly $f_1 - h_1 = f_2 - h_2 \in L^1 \cap L^2$. Since $f_1 - h_1 = f_2 - h_2$ are equivalent on $L^1 \cap L^2$ then their Fourier transforms are equivalent and we may conclude $\widehat{f_1 + f_2} = \widehat{h_1 + h_2}$. Thus the Fourier transform is now well-defined on L^p and a formal definition follows.

DEFINITION 2.11: Let $f \in L^p(\mathbb{R}^n), 1 \leq p \leq 2$. Then the *Fourier transform of* f, denoted $\hat{f}$, is

$$\hat{f} = \hat{f}_1 + \hat{f}_2$$

where $f_1 \in L^1, f_2 \in L^2$ such that $f = f_1 + f_2$.

With this definition it is easy to show that properties (1)-(8), (12) and (13) from Proposition 1.6 are maintained. Now that we have extended the Fourier transform to L^p, we will establish the boundedness of the Fourier transform from L^p to

$L^{p'}$, $1 \leq p \leq 2$, in the following theorem.

THEOREM 2.12: (Hausdorff-Young inequality) For every $f \in L^p(\mathbb{R}^n)$ we have the estimate

$$||\hat{f}||_{p'} \leq \|f\|_p$$

whenever $1 \leq p \leq 2$.

Proof. The theorem is an immediate result of the Riesz-Thorin interpolation theorem (Theorem 2.3). We know that $||\hat{f}||_\infty \leq ||f||_1$ and $||\hat{f}||_2 = ||f||_2$ from basic properties of the Fourier transform and Corollary 1.18. Simply interpolate between the estimates $||\hat{f}||_\infty \leq ||f||_1$ and $||\hat{f}||_2 = ||f||_2$ to obtain $||\hat{f}||_{p'} \leq \|f\|_p$. □

The following corollary is immediate.

COROLLARY 2.13: If $f \in L^p$, where $1 \leq p \leq 2$, then $\hat{f} \in L^{p'}$.

Now that we have extended the Fourier transform to L^p and thus established it in all spaces of interest it's time to delve deeper. Understanding the relationship between the smoothness of a function and the integrability of its Fourier transform is a fundamental theme in Fourier analysis. It was the smoothness properties of the Schwartz functions that made $S(\mathbb{R}^n)$ such a natural space to introduce the Fourier transform. To begin understanding this relationship we will need a way to measure quantitatively the smoothness of a function. It is here that we introduce moduli of continuity. Let us begin with a formal definition.

DEFINITION 2.14: Let $(X, \mu), (Y, \nu)$ be metric spaces. A *modulus of continuity* is any real-extended valued function $\mathit{\Omega} : [0, \infty] \to [0, \infty]$, vanishing at zero and continuous at zero, such that

$$\lim_{t \to 0} \mathit{\Omega}(t) = \mathit{\Omega}(0) = 0.$$

We say $f : X \to Y$ admits $\mathit{\Omega}$ as modulus of continuity at the point $c \in X$ if and only if,

$$\forall x \in X \text{ we have } \nu(f(c), f(x)) \leq \Omega(\mu(x, c)).$$

Additionally, we say $f : X \to Y$ admits Ω as modulus of continuity if and only if,

$$\forall x, c \in X \text{ we have } \nu(f(c), f(x)) \leq \Omega(\mu(x, c)).$$

When considering the classical ε, δ definion of uniform continuity one could think the role of a modulus of continuity is to fix some explicit functional dependence of ε on δ. Moduli of continuity provide a gauge, in some sense, of the closeness of function values to function values nearby. As such they can be used to implement smoothness conditions. For example the modulus of continuity $\omega(t) = kt$ describes the k-Lipschitz functions. We now present the following theorem from Bray[6, 8].

THEOREM 2.15: Let $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R})$. Additionally, let $\Omega_p[p](t)$ be the L^p-modulus of continuity based on symmetric differences:

$$\Omega_p[f](t) = \sup_{0<h<t} ||f(\cdot + h) + f(\cdot - h) - 2f(\cdot)||_p.$$

Then there is a positive constant c_p independent of f such that

- $\sup_{\lambda} \left[min\{1, (\xi t)^2\} |\hat{f}(\xi)| \right] \leq c_1 \Omega_1[f](t)$ for $p = 1$,
- $\left[\int_{\mathbb{R}} min\{1, (\xi t)^{2p'}\} |\hat{f}(\xi)|^{p'} d\lambda \right]^{\frac{1}{p'}} \leq c_p \Omega_p[f](t)$ for $1 < p \leq 2$.

Proof. For simplicity let $Z_h[f] = f(\cdot + h) + f(\cdot - h) - 2f(\cdot)$. Applying property 6 from Proposition 1.6 we obtain the following identity:

$$\begin{aligned}
\widehat{Z_h[f]}(\xi) &= \widehat{\tau^h f}(\xi) + \widehat{\tau^{-h} f}(\xi) - 2\hat{f}(\xi) \\
&= e^{-2\pi i h\xi}\hat{f}(\xi) + e^{2\pi i h\xi}\hat{f}(\xi) - 2\hat{f}(\xi) \\
&= \hat{f}(\xi)[2\cos(2\pi h\xi) - 2] \\
&= -4\sin^2(\pi h\xi)\hat{f}(\xi). \\
\therefore \widehat{Z_h[f]}(\xi) &= -4\sin^2(\pi h\xi)\hat{f}(\xi). \qquad (2.3.12)
\end{aligned}$$

We will first prove the case $1 < p \leq 2$. By application of Theorem 2.12 we know $||\widehat{Z_h[f]}(\xi)||_{p'} \leq ||Z_h[f]||_p$. From this result and equation 2.3.12 we have

$$\begin{aligned}
\sup_{0\leq h\leq t} ||Z_h[f]||_p^{p'} &\geq \sup_{0\leq h\leq t} ||\widehat{Z_h[f]}(\xi)||_{p'}^{p'} \\
&= \sup_{0\leq h\leq t} \int_{\mathbb{R}} |-4\sin^2(\pi h\xi)\hat{f}(\xi)|^{p'} d\xi \\
&= \sup_{0\leq h\leq t} 4^{p'} \int_{\mathbb{R}} \sin^{2p'}(\pi h\xi)|\hat{f}(\xi)|^{p'} d\xi \\
&= \sup_{0\leq z\leq 1} 4^{p'} \int_{\mathbb{R}} \sin^{2p'}(\pi z t\xi)|\hat{f}(\xi)|^{p'} d\xi \\
&\geq 4^{p'} \int_{\mathbb{R}} \left[\int_0^1 sin^{2p'}(\pi z t\xi) dz\right] |\hat{f}(\xi)|^{p'} d\xi.
\end{aligned}$$

It is known that

$$\int_0^1 \sin^{2p'}(\pi z t\xi) dz \geq c \min(1, (\xi t)^{2p'})$$

for some positive constant c. Thus by collecting constants we have the desired result:

$$\left[\int_{\mathbb{R}} min\{1, (\lambda t)^{2p'}\} |\hat{f}(\lambda)|^{p'} d\lambda\right]^{\frac{1}{p'}} \leq c_p \varOmega_p[f](t).$$

Proving the case for $p = 1$ is very similar. Once again we use Theorem 2.12 to obtain $||\widehat{Z_h[f]}(\xi)||_\infty \leq ||Z_h[f]||_1$. As before there exist some positive constant c such that

$$\begin{aligned}
\sup_{0\le h\le t}||Z_h[f]||_1 &\ge \sup_{0\le h\le t}||\widehat{Z_h[f]}(\xi)||_\infty \\
&= \sup_{0\le h\le t}\left(\sup_{\xi}|-4\sin^2(\pi h\xi)\hat{f}(\xi)|\right) \\
&= \sup_{\xi}\left(4\left[\sup_{0\le z\le 1}\sin^2(\pi zt\xi)\right]|\hat{f}(\xi)|\right) \\
&\ge 4c\sup_{\xi}\left(\min(1,(t\xi)^2)|\hat{f}(\xi)|\right).
\end{aligned}$$

Which gives us the desired result when $p = 1$. □

Take a moment to refer to the placement of the minimum function in the result above. The minimum function gives control of the Fourier transform for small and large ξ. Indeed the previous result can be written as

$$\begin{aligned}
\int_{|\xi|\ge\frac{1}{t}}|\hat{f}(\xi)|^{p'}d\xi + t^{2p'}\int_{|\xi|<\frac{1}{t}}\xi^{2p'}|\hat{f}(\xi)|^{p'} &\le c_p^{p'}\mathit{\Omega}_p^{p'}[f](t) \text{ for } 1<p\le 2, \\
\max\left(\sup_{|\xi|\ge\frac{1}{t}}|\hat{f}(\xi)|,\ \sup_{|\xi|<\frac{1}{t}}(\xi t)^2|\hat{f}(\xi)|\right) &\le c_1\mathit{\Omega}_1[f](t) \text{ for } p=1.
\end{aligned}$$

As you can see Theorem 2.15 gives us an estimate for both small and large ξ. The estimate for large ξ provides a qualitative Riemann-Lebesgue lemma outlined in the following corollary, which is immediate from Theorem 2.15.

COROLLARY 2.16: Let $1 \le p \le 2$ and $f \in L^p(\mathbb{R})$. Then

$$\begin{aligned}
\left(\int_{|\xi|>\frac{1}{t}}|\hat{f}(\xi)|^{p'}d\xi\right)^{\frac{1}{p'}} &\le c_p\mathit{\Omega}_p[f](t) \text{ for } 1<p\le 2, \\
\sup_{|\xi|>\frac{1}{t}}|\hat{f}(\xi)| &\le c_1\mathit{\Omega}_1[f](t) \text{ for } p=1.
\end{aligned}$$

Thus the growth of the Fourier transform of $f \in L^p(\mathbb{R})$ is gauged by the modulus of continuity introduced in Theorem 2.15. Recall that Fourier inversion is dependent on the integrability of the Fourier transform. So any integrability results will give us insight into what conditions give us Fourier inversion. By using the estimate for small ξ and implementing a smoothness condition we have the following integrability result.

THEOREM 2.17: Let $f \in L^p(\mathbb{R})$ for some $1 \leq p \leq 2$. If $\exists \alpha \in (0, 2]$ such that

$$||f(\cdot + t) + f(\cdot - t) - 2f(\cdot)||_p = O(t^\alpha),$$

then $\hat{f} \in L^\beta(\mathbb{R})$ provided

$$\frac{p}{p + \alpha p - 1} < \beta \leq p'.$$

In particular, if $||f(\cdot+t)+f(\cdot-t)-2f(\cdot)||_p = O(t^\alpha)$ holds for $\alpha > \frac{1}{p}$ then $\hat{f} \in L^1(\mathbb{R})$ and Fourier inversion holds almost entirely.

As you can see if the symmetric differences of a function in $L^p(\mathbb{R})$ are bounded by t^α, for some $0 < \alpha \leq 2$, then we have a useful integrability result on the function's Fourier transform and thus establish a condition in which Fourier inversion holds on $L^p(\mathbb{R})$. However, its proof is fairly involved and thus omitted. To view the proof please refer to Bray[6].

It is now made abundantly clear by Theorem 2.15 that the growth of the Fourier transform of a function in $L^p(\mathbb{R})$ is qualitatively gauged by the modulus of continuity previously introduced. Additionally, we now have a wonderful integrability theorem and Riemann-Lebesgue lemma type result for $L^p(\mathbb{R})$ functions. Thus far, however, our results have been limited to L^p functions on the real line. Naturally, one wonders if these results could be extended to higher dimensions? In short, the answer is yes. Extending these results will be largely dependent on basing the modulus of continuity on the spherical mean operator:

$$M^t f(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + t\omega) d\omega,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$ and ω_{n-1} its measure. Historically, it seems that a Russian mathematician Vladimir Platonov was the first to define a modulus of continuity using the spherical mean operator in [9]. Let us now focus on extending Theorem 2.15 to higher dimensions.

To extend Theorem 2.15 this paper will follow Bray's[6, 8] approach. To accomplish our goals we will need to establish the following:

1. The identity

   $$\widehat{M^t f}(\xi) = \hat{f}(\xi) j_\nu(t|\xi|),$$

   where j_ν is the normalized spherical Bessel function of order $\nu = \frac{n-2}{2}$,

   $$j_\nu(r) = 2^\nu \Gamma(\nu+1) r^{-\nu} J_\nu(r),$$

   J_ν being a Bessel function of the first kind previously defined.

2. For $\alpha > -\frac{1}{2}$ we have $1 - j_\alpha(\lambda) \asymp \min\{1, \lambda^2\}$.[5]

As you can see Bessel functions play a key role for the remainder of this section. We now apply what we know of Bessel functions to prove the previous two claims.

LEMMA 2.18: Let $\nu = \frac{n-2}{2}$ and $n \geq 2$. Then

$$\widehat{M^t f}(\xi) = \hat{f}(\xi) j_\nu(t|\xi|),$$

where $j_\nu(r)$ is the normalized spherical Bessel function previously mentioned.

[5] Here $\asymp$ means the left hand side is bounded above and below by a constant times the right hand side.

Proof. Application of Fubini's theorem and the substitution $u = x + t\omega$ provide

$$\begin{aligned}
\widehat{M^t f}(\xi) &= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \int_{S^{n-1}} e^{-2\pi i \xi \cdot x} f(x + t\omega) d\omega dx \\
&= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \int_{S^{n-1}} e^{-2\pi i \xi \cdot u} e^{2\pi i \xi t \cdot \omega} f(u) d\omega du \\
&= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} e^{2\pi i \xi t \cdot \omega} \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot u} f(u) du d\omega \\
&= \hat{f}(\xi) \frac{1}{\omega_{n-1}} \int_{S^{n-1}} e^{2\pi i \xi t \cdot \omega} d\omega
\end{aligned}$$

To proceed we will apply identity 2.2.8:

$$\begin{aligned}
\widehat{M^t f}(\xi) &= \hat{f}(\xi) 2^\nu \Gamma(\nu + 1)(t|\xi|)^{-\nu} J_\nu(t|\xi|) \\
&= \hat{f}(\xi) j_\nu(t|\xi|).
\end{aligned}$$

□

Thus the first is established! Let us now prove the second.

LEMMA 2.19: Let $\alpha > -\frac{1}{2}$. Then $1 - j_\alpha(\lambda) \asymp \min\{1, \lambda^2\}$.

Proof. Let us first prove the lemma for $\lambda > 1$. Under this assumption the claim follows almost immediately from Theorem 2.9. We know from Theorem 2.9 when $\lambda > 1$ there exists a constant c_α dependent only on α such that

$$|J_\alpha(\lambda)| \leq c_\alpha \lambda^{-\frac{1}{2}}. \tag{2.3.13}$$

Application of the triangle inequality and 2.3.13 above provides

$$\begin{aligned}
|1 - j_\alpha(\lambda)| &\leq 1 + 2^\alpha \Gamma(\alpha + 1) \lambda^{-\alpha} |J_\alpha(\lambda)| \\
&\leq 1 + c_\alpha \lambda^{-(\alpha + \frac{1}{2})}.
\end{aligned}$$

However, $\alpha + \frac{1}{2} > 0$ and $\lambda > 1$ implying $|1 - j_\alpha(\lambda)| \leq c_\alpha$. Therefore $1 - j_\alpha(\lambda) \asymp \min\{1, \lambda^2\}$ when $\lambda > 1$. Let us now consider the case $0 < \lambda \leq 1$. Under this assumption we will follow the Bray-Pinsky[8] approach and use the Mehler type representation for j_α :

$$j_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)x^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^x (x^2-y^2)^{\alpha-\frac{1}{2}}\cos(\lambda y)dy.$$

Using the representation above, it's immediate that

$$\begin{aligned}
1-j_\alpha(\lambda x) &= \frac{2\Gamma(\alpha+1)x^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^x (x^2-y^2)^{\alpha-\frac{1}{2}}(1-\cos(\lambda y))dy \\
&= \frac{4\Gamma(\alpha+1)x^{-2\alpha}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^x (x^2-y^2)^{\alpha-\frac{1}{2}}\sin^2(\frac{\lambda y}{2})dy. \\
\therefore 1-j_\alpha(\lambda) &= \frac{4\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-y^2)^{\alpha-\frac{1}{2}}\sin^2(\frac{\lambda y}{2})dy.
\end{aligned}$$

It is known if $u \leq \pi/2$ then $2u/\pi \leq \sin(u) \leq u$. Applying this fact with the identity for $1-j_\alpha(\lambda)$ above gives us

$$\frac{4\Gamma(\alpha+1)\lambda^2}{\pi^{\frac{5}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-y^2)^{\alpha-\frac{1}{2}}y^2dy \leq 1-j_\alpha(\lambda) \leq \frac{\Gamma(\alpha+1)\lambda^2}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-y^2)^{\alpha-\frac{1}{2}}y^2dy.$$

A simple substitution gives rise to the compound inequality

$$\frac{2\Gamma(\alpha+1)\lambda^2}{\pi^{\frac{5}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t)^{(\alpha+\frac{1}{2})-1}t^{\frac{3}{2}-1}dt \leq A \leq \frac{\Gamma(\alpha+1)\lambda^2}{2\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-t)^{(\alpha+\frac{1}{2})-1}t^{\frac{3}{2}-1}dt,$$

where $A = 1-j_\alpha(\lambda)$. Now simplify the Beta function integrals above to obtain the desired inequality:

$$\frac{1}{\pi^2(\alpha+1)}\lambda^2 \leq 1-j_\alpha(\lambda) \leq \frac{1}{4(\alpha+1)}\lambda^2.$$

Thus, in any case, $1-j_\alpha(\lambda) \asymp \min\{1,\lambda^2\}$ as desired. □

The stage is now set! We conclude this section with the following theorem and its implications.

THEOREM 2.20: Let $n \geq 2$, $1 \leq p \leq 2$, and $f \in L^p(\mathbb{R}^n)$. Additionally, let $\Omega_p[f](t)$ be the spherical modulus of continuity:

$$\Omega_p[f](t) = ||M^t f - f||_p.$$

Then there exists a constant $c_{p,n} > 0$ dependent only on p and n such that for

- $p = 1$ we have $\sup_\xi \left[\min\{1, (t|\xi|)^2\}|\hat{f}(\xi)|\right] \leq c_{1,n}\Omega_1[f](t)$;
- $1 < p \leq 2$ we have $\left[\int_{\mathbb{R}^n} \min\{1, (t|\xi|)^{2p'}\}|\hat{f}(\xi)|^{p'} d\xi\right]^{\frac{1}{p'}} \leq c_{p,n}\Omega_p[f](t)$.

Proof. Let $\nu = \frac{n-2}{2}$ and $n \geq 2$. We will first prove the claim for $1 < p \leq 2$. Recall from Lemma 2.18,

$$\widehat{M^t} f(\xi) = \hat{f}(\xi) j_\nu(t|\xi|).$$

Application of the Hausdorff-Young inequality yields

$$||M^t f - f||_p^{p'} \geq \int_{\mathbb{R}^n} |\hat{f}(\xi)|^{p'} |1 - j_\nu(t|\xi|)|^{p'} d\xi.$$

From Lemma 2.19 there exists constant c such that

$$||M^t f - f||_p^{p'} \geq c \int_{\mathbb{R}^n} |\hat{f}(\xi)|^{p'} \min\{1, (t|\xi|)^{p'}\} d\xi.$$

Thus we have the desired result

$$||M^t f - f||_p^{p'} \geq c_\nu \int_{\mathbb{R}^n} |\hat{f}(\xi)|^{p'} \min\{1, (r|\xi|)^{2p'}\} d\xi$$

for $1 < p \leq 2$. The approach to proving the claim for $p = 1$ is analogous and thus omitted. $\square$

As in the one-dimensional case, if we focus on large ξ a Reimann-Lebesgue type result is immediate. The result described is presented in the following corollary.

COROLLARY 2.21: Let $1 \leq p \leq 2$ and $f \in L^p(\mathbb{R}^n)$. Then there exists a constant $c_{p,n}$, dependent on only p and n, such that

$$\left(\int_{|\xi| \geq \frac{1}{r}} |\hat{f}(\xi)|^{p'} d\xi\right)^{\frac{1}{p'}} \leq c_{p,n} \mathit{\Omega}_p[f](r) \text{ for } 1 < p \leq 2;$$

$$\sup_{|\xi| \geq \frac{1}{r}} |\hat{f}(\xi)| \leq c_{p,n} \mathit{\Omega}_p[f](r) \text{ for } p = 1.$$

Hence, on Euclidean space we may conclude that the growth of a function's Fourier transform is gauged by the moduli of continuity $\mathit{\Omega}_p[f](t)$. But does Theorem 2.20 give rise to an integrability result as in the one dimension case? It turns out that Theorem 2.17 has a parallel result in higher dimensions. This result can be found in Bray [6] and is left to the reader. It is on this note that our study of the Fourier transform comes to a close and we turn our focus to Fourier series.

CHAPTER 3

FOURIER SERIES

Many great thinkers have pondered if a periodic function can be decomposed into a sum of sinusoids, which in time led to the developement of Fourier series. Principles of Fourier series go back as far as 3rd century BC where ancient astronomers proposed a model of planetary motions based on the theory epicycles, which in itself rose from the work of Appolonius. The modern theory of Fourier series can be traced to the work of d'Alembert, Bernoulli, Euler, and Clairaut on the vibrating string problem. It was while studying the differential equations associated with a vibrating string that these mathematicians began to believe that an arbitrary periodic function could be represented by a sum of sines and cosines. However, Fourier series are named after Joseph Fourier when he introduces them in his 1807 *Mémoire sur la propagation de la chaleur dans les corps solides* (Treatise on the propagation of heat in solid bodies). Unfortunately, in the days of Fourier there was no precise notion of function or integral. Thus it would be some time before mathemeticians such as Dirichlet, Laplace and Riemann would distill the work of Fourier into the results that are so familiar and widely studied today.

Dirichlet was the first to discover that the partial sum of a function's Fourier series could be represented by convolution of the function with a trigonometric polynomial. The concept of describing Fourier series in terms of convolution operators would prove paramount in the field. Mathematicians Fejér and Cesàro would then develope what are known as the Fejér (or Cesàro) means. We will spend some time showing just how useful the Fejér means are in establishing some classical results in the field. Some time later the mathematician Salomon Bochner would modify the famous Riesz means to derive what we know today as the Bochner-Riesz means. The Bochner-Riesz mean is a convolution operator whose convolution kernel is an approximate identity, implying the Bochner-Riesz means of a function converge to the function in norm. After a brief look at the Bochner-Riesz means we conclude the paper with an introduction of an operator very similar in structure to the Bochner-Riesz means.

3.1 The Dirichlet And Fejér Kernels

As mentioned the notion that a periodic function could be represented as a sum of sines and cosines is quite ancient. However, it was some time before the notion developed some rigidity. Mathematicians such as Laplace and Dirichlet made massive contributions on the validity of a function's Fourier series representation. Since we are studying periodic functions, our focus shall be restricted to the n-torus $\mathbb{T}^n$ constructed on $[-\frac{1}{2}, \frac{1}{2}]^n$. All functions on $\mathbb{T}^n$ are 1-periodic functions on $\mathbb{R}^n$, which makes $\mathbb{T}^n$ an ideal place to study Fourier series. Additionally, there are transference results that allow us to automatically extend many results from $\mathbb{T}^n$ to $\mathbb{R}^n$. Without any further adieu, let us define the Fourier series of a function.

DEFINITION 3.1: Let $f \in L^1(\mathbb{T}^n)$, where $\mathbb{T}^n$ denotes the n-torus, and $m \in \mathbb{Z}^n$. Then

$$\hat{f}(m) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i m \cdot x} dx$$

is called the *mth Fourier coefficient of* f. The *Fourier series of* f at $x \in \mathbb{T}^n$ is

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x}.$$

It turns out that the partial sums of a function's Fourier series can be written as convolution between the function and a trigonometric polynomial. We demonstrate this crucial concept in the following theorem.

THEOREM 3.2: Let $f \in L^1(\mathbb{T}^n)$ and $m \in \mathbb{Z}^n$. For trigonometric polynomial

$P(x) = \sum\limits_{|m| \leq N} a_m e^{2\pi i m \cdot x}$ we have

$$(f * P)(x) = \sum_{|m| \leq N} a_m \hat{f}(m) e^{2\pi i m \cdot x},$$

where $|m| = \sqrt{m_1^2 + \ldots + m_n^2}$.

Proof. The proof is fairly straightforward. Note that

$$(e^{2\pi i m \cdot x} * f)(x) = \int_{\mathbb{T}^n} e^{2\pi i m \cdot (x-u)} f(u) du = e^{2\pi i m \cdot x} \hat{f}(m).$$

Application of the above identity yields

$$\begin{aligned} (f * P)(x) &= \sum_{|m| \leq N} a_m (e^{2\pi i m \cdot x} * f)(x) \\ &= \sum_{|m| \leq N} a_m \hat{f}(m) e^{2\pi i m \cdot x}. \end{aligned}$$

□

Using the same arguments it is easy to show for $P(x) = \sum\limits_{|m_j| \leq N} e^{2\pi i m \cdot x}$ we have

$$(f * P)(x) = \sum_{|m_j| \leq N} \hat{f}(m) e^{2\pi i m \cdot x}.$$

Therefore the square partial sums of a function's Fourier series can be constructed by convolving the function with the trigonometric polynomial

$$D_N(x) = \sum_{|m_j| \leq N} e^{2\pi i m \cdot x}.$$

The trigonometric polynomials above would be named after Dirichlet and give rise to the Dirichlet kernel on $\mathbb{T}^n$ defined below.

DEFINITION 3.3: Let $0 \leq R < \infty$. The *square Dirichlet kernel* on $\mathbb{T}^n$ is the function

$$D(n,R)(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le R}} e^{2\pi i m\cdot x}.$$

In dimension 1, the function $D(1,R)(x)$ is denoted $D_R(x)$ and called the *Dirichlet kernel*.

The one-dimensional Dirichlet kernel is plotted in Figure 3.1.

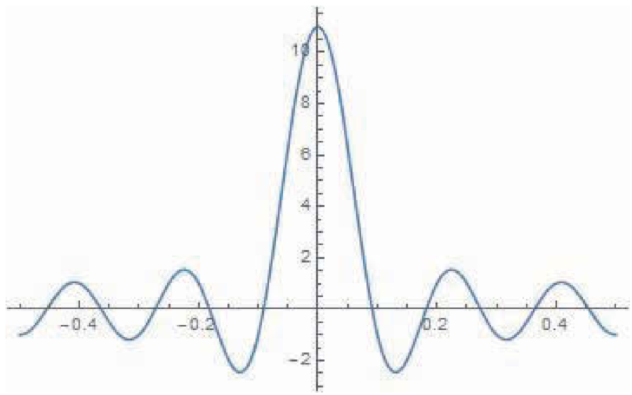

Figure 3.1: The Dirichlet kernel D_5 plotted on $[-\frac{1}{2},\frac{1}{2}]$

The one-dimensional case has special notation because the square Dirichlet kernel on $\mathbb{T}^n$ is equal to a product of one-dimensional Dirichlet kernels, that is,

$$D(n,R)(x_1,\ldots,x_n) = D_R(x_1)\cdots D_R(x_n).$$

There is also a very useful identity for the Dirichlet kernel established in the following proposition.

PROPOSITION 3.4: Let $x\in\mathbb{T}$, $m\in\mathbb{Z}$, and $0\le R<\infty$.Then

$$D_R(x) = \frac{\sin(\pi x(2\lfloor R\rfloor+1))}{\sin(\pi x)},$$

where $\lfloor\cdot\rfloor$ denotes the floor function.

Proof. Note that

$$\begin{aligned}
\sin(\pi x)D_R(x) &= \sin(\pi x)\left[1+\sum_{m=1}^{\lfloor R\rfloor} e^{2\pi imx}+\sum_{m=1}^{\lfloor R\rfloor} e^{-2\pi imx}\right] \\
&= \sin(\pi x)+\sin(\pi x)\sum_{m=1}^{\lfloor R\rfloor}\left(e^{2\pi imx}+e^{-2\pi imx}\right) \\
&= \sin(\pi x)+\sum_{m=1}^{\lfloor R\rfloor} 2\sin(\pi x)\cos(2\pi mx).
\end{aligned}$$

Now apply the trigonometric identity $2\cos(a)\sin(b) = \sin(a+b) - \sin(a-b)$ to obtain

$$\begin{aligned}
\sin(\pi x)D_R(x) &= \sin(\pi x)+\sum_{m=1}^{\lfloor R\rfloor}\left[\sin(\pi x(2m+1)) - \sin(\pi x(2m-1))\right] \\
&= \sin(\pi x(2\lfloor R\rfloor+1)).
\end{aligned}$$

$$\therefore D_R(x) = \frac{\sin(\pi x(2\lfloor R\rfloor+1))}{\sin(\pi x)} \text{ as desired.}$$

□

Unfortunately, the Dirichlet kernel is not a good kernel and is quite limited in application. It turns out that $||D_R||_1$ grows logarithmically as $R \to \infty$. Thus the Dirichlet kernel is not an approximate identity. Keep in mind that these kernels are being used as convolution kernels, and thus the ideal kernel is an approximate identity. Luckily mathematicians Féjer and Cesàro would independently develope a convolution kernel better behaved than the Dirichlet kernel. Their idea was actually quite simple and intuitive. Throughout analysis the mean of a sequence typically behaves better than the original sequence. By implementing an averaging process, Féjer and Cesàro would resolve a lot of the shortcomings of the Dirichlet kernels. We now present their notions in the following definition.

DEFINITION 3.5: Let N be a nonnegative integer. The *Fejér kernel* $F(n, N)$ on

$\mathbb{T}^n$ is

$$F(n,N)(x_1,\ldots,x_n) = \prod_{j=1}^{n} F_N(x_j), \text{ where}$$
$$F_N(x) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(x).$$

Similar to the Dirichlet case, $F_N(x)$ denotes the Fejér kernel in one-dimension.

The one-dimensional Fejér kernel is plotted in Figure 3.2.

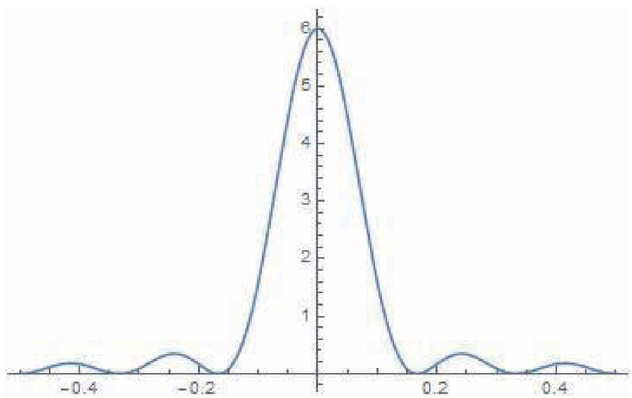

Figure 3.2: The Fejér kernel F_5 plotted on $[-\frac{1}{2}, \frac{1}{2}]$.

As you can see, the Féjer kernel is a trigonometric polynomial and simply the arithmetic mean of the product of the Dirichlet kernels in each variable[1]. So does the Fejér kernel succeed where the Dirichlet kernel failed and give rise to approximate identity? The answer is yes, but to prove that we will need to set up the following useful identity for the Fejér kernel.

LEMMA 3.6: Let N be a nonnegative integer, then

$$F_N(x) = \sum_{j=-N}^{N}\left(1 - \frac{|j|}{N+1}\right)e^{2\pi ijx} = \frac{1}{N+1}\left(\frac{\sin(\pi x(N+1))}{\sin(\pi x)}\right)^2.$$

[1]The Féjer kernel is also known as the Cesàro kernel.

Proof. We will first use a technique called summing by parts. Note that

$$\begin{aligned}\sum_{k=0}^{N} D_k &= D_0 + 2D_1 - D_1 + 3D_2 - 2D_2 + \ldots + (N+1)D_N - ND_N \\ &= (N+1)D_N + \sum_{k=0}^{N-1} (k+1)(D_k - D_{k+1}) \\ &= (N+1)D_N - \sum_{k=0}^{N-1} (k+1)(D_{k+1} - D_k).\end{aligned}$$

Now apply the result above and definition of the Dirichlet kernel to obtain

$$\begin{aligned}F_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) \\ &= D_N - \frac{1}{N+1}\sum_{k=0}^{N-1} (k+1)(D_{k+1} - D_k) \\ &= D_N - \frac{1}{N+1}\sum_{k=0}^{N-1} (k+1)(e^{2\pi i(k+1)x} + e^{-2\pi i(k+1)x}) \\ &= D_N - \frac{1}{N+1}\sum_{k=1}^{N} ke^{2\pi ikx} - \frac{1}{N+1}\sum_{k=1}^{N} ke^{-2\pi ikx} \\ &= \sum_{k=-N}^{N} e^{2\pi ikx} - \frac{1}{N+1}\sum_{k=-N}^{N} |k|e^{2\pi ikx} \\ &= \sum_{k=-N}^{N}\left(1 - \frac{|k|}{N+1}\right)e^{2\pi ikx}.\end{aligned}$$

Thus the first identity is established. Establishing the second identity will require the trigonometric identity $2\sin(a)\sin(b) = \cos(a-b) - \cos(a+b)$ and Proposition

3.4. Let us proceed,

$$
\begin{aligned}
(N+1)\sin^2(\pi x)F_N(x) &= \sum_{k=0}^{N} \frac{\sin(\pi x(2k+1))}{\sin(\pi x)} \cdot \sin^2(\pi x) \\
&= \sum_{k=0}^{N} \sin(\pi x(2k+1))\sin(\pi x) \\
&= \frac{1}{2}\sum_{k=0}^{N} (\cos(2\pi x k) - \cos(2\pi x(k+1))) \\
&= \frac{1-\cos(2\pi x(N+1))}{2} \\
&= \sin^2(\pi x(N+1)).
\end{aligned}
$$

$$
\therefore F_N(x) = \frac{1}{N+1}\left(\frac{\sin(\pi x(N+1))}{\sin(\pi x)}\right)^2.
$$

$\square$

With Lemma 3.6 it is immediate that the Fejér kernel can be represented in the two following ways:

$$
F(n,N)(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_2|}{N+1}\right) e^{2\pi i m\cdot x}, \tag{3.1.1}
$$

$$
F(n,N)(x) = \frac{1}{(N+1)^n}\prod_{j=1}^{n}\left(\frac{\sin(\pi(N+1)x_j)}{\sin(\pi x_j)}\right)^2. \tag{3.1.2}
$$

With the two identities above it is now possible to show that the Fejér kernel is an approximate identity.

THEOREM 3.7: Let N be a nonnegative integer and $x \in \mathbb{T}^n$. Then $\{F(n,N)\}_{N\in\mathbb{Z}^+}$ is an approximate identity as $N \to \infty$.

Proof. By application of Euler's identity it is known that

$$
\int_{\mathbb{T}^n} e^{2\pi i m x}dx = \begin{cases} 0, & m \neq 0, \\ 1, & m = 0. \end{cases}
$$

Therefore, with the result above and 3.1.1 we obtain

$$\begin{aligned}
\int_{\mathbb{T}^n} F(n,N)(x)dx &= \int_{\mathbb{T}^n} \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_2|}{N+1}\right) e^{2\pi i m\cdot x} \\
&= \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le N}} \left(1-\frac{|m_1|}{N+1}\right)\cdots\left(1-\frac{|m_2|}{N+1}\right) \int_{\mathbb{T}^n} e^{2\pi i m\cdot x}dx \\
&= 1.
\end{aligned}$$

Hence, the second condition in Definition 1.1 is fulfilled. From identity 3.1.2 it is clear that $F(n,N)(x) \geq 0$. Since $F(n,N)(x) \geq 0$ and $\int_{\mathbb{T}^n} F(n,N)(x)dx = 1$ we may conclude that $||F(n,N)(x)||_1 < 2$ always. Therefore the first condition in Definition 1.1 is satisfied. Now we need only show that the tails of the Fejér kernel go to zero. Thanks to identity 3.1.2, all that is needed to satisfy the final condition is to show for $\delta > 0$ and $y \in \mathbb{T}$ we have

$$\int_{|y|>\delta} \frac{1}{N+1}\left(\frac{\sin(\pi y(N+1))}{\sin(\pi y)}\right)^2 dy \to 0 \text{ as } N\to\infty.$$

Note that on $[-\frac{1}{2},\frac{1}{2}]$, $|\sin(\pi y)| \geq |\frac{y}{2}|$. Therefore,

$$\frac{1}{N+1}\left(\frac{\sin(\pi y(N+1))}{\sin(\pi y)}\right)^2 \leq \frac{1}{N+1}\cdot\frac{4}{|y|^2}.$$

With this estimate we may conclude

$$\begin{aligned}
\int_{|y|>\delta} \frac{1}{N+1}\left(\frac{\sin(\pi y(N+1))}{\sin(\pi y)}\right)^2 dy &= 2\int_{\delta}^{\frac{1}{2}} \frac{1}{N+1}\left(\frac{\sin(\pi y(N+1))}{\sin(\pi y)}\right)^2 dy \\
&\leq \frac{8}{N+1}\int_{\delta}^{\frac{1}{2}} y^{-2}dy.
\end{aligned}$$

Therefore,

$$\int_{|y|>\delta} \frac{1}{N+1}\left(\frac{\sin(\pi y(N+1))}{\sin(\pi y)}\right)^2 dy \to 0 \text{ as } N\to\infty.$$

$\square$

With the previous theorem it is now abundantly clear that the Fejér kernel is a better behaved convolution kernel than the Dirichlet kernel. Since the Fejér kernel is an approximate identity, Theorem 1.2 establishes the following convergence result.

COROLLARY 3.8: Let $f \in L^p(\mathbb{T}^n)$. Then the following are true:

1. If $1 \leq p < \infty$ then $||F(n, N) * f - f||_p \to 0$ as $N \to \infty$.

2. If $p = \infty$ and f is continuous then $||F(n, N) * f - f||_\infty \to 0$ as $N \to \infty$.

With the Fejér and Dirichlet kernels introduced and their more basic properties established it is time to return to the Fourier series. Any calculus student knows that a series is often studied by examining its partial sums. We now define partial sums of Fourier series with the kernels previously introduced.

DEFINITION 3.9: For $R \geq 0$ and $N \in \mathbb{Z}^+ \cup \{0\}$ the expressions

$$(f * D(n, R))(x) = \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq R}} \hat{f}(m) e^{2\pi i m \cdot x}$$

are called the *square partial sums of the Fourier series of* f, and the expressions

$$(f * F(n, N))(x) = \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq N}} \left(1 - \frac{|m_1|}{N+1}\right) \cdots \left(1 - \frac{|m_n|}{N+1}\right) \hat{f}(m) e^{2\pi i m \cdot x}$$

are called the *square Fejér means of* f.

As you can see the square Fejér means are constructed by convolving the function with the Fejér kernel. With the square Fejér means defined we are now ready to establish some classical results. We began the section by discussing function representation via Fourier series. So do the Fourier coefficients of a function uniquely determine the function? The answer is straightforward and simple.

LEMMA 3.10: If $f, g \in L^1(\mathbb{T}^n)$ satisfy $\hat{f}(m) = \hat{g}(m)$ for all $m \in \mathbb{Z}^n$, then $f = g$ almost everywhere.

Proof. By linearity, it suffices to assume $g = 0$. If $\hat{f}(m) = 0$ for all $m \in \mathbb{Z}^n$ then $F(n, N) * f = 0$ for all $N \in \mathbb{Z}^+$. Since the sequence $\{F(n, N)\}_{N \in \mathbb{Z}^+}$ is an approximate identity as $N \to \infty$, then

$$||f - F(n, N) * f||_1 \to 0$$

as $N \to \infty$. Therefore $||f||_1 = 0$ and we may conclude $f = 0$ almost everywhere. □

The following Fourier inversion result is almost immediate.

THEOREM 3.11: Suppose $f \in L^1(\mathbb{T}^n)$ and

$$\sum_{m \in \mathbb{Z}^n} |\hat{f}(m)| < \infty.$$

Then

$$f(x) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} \ a.e. \tag{3.1.3}$$

Proof. Note that both functions in 3.1.3 are well defined. To prove the claim we need only show that both functions have the same Fourier coefficients. For now let

$$g(x) = \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x}.$$

So,

$$\begin{aligned} \hat{g}(k) &= \int_{\mathbb{T}^n} \sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} e^{-2\pi i k \cdot x} dx \\ &= \int_{\mathbb{T}^n} \lim_{R \to \infty} e^{-2\pi i k \cdot x} (f * D(n, R))(x) dx \\ &= \int_{\mathbb{T}^n} \lim_{R \to \infty} \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \le R}} \hat{f}(m) e^{2\pi i x \cdot (m - k)} dx. \end{aligned}$$

We will now show the square partial sums in the integrand above are bounded by an integrable function for all $R > 0$:

$$\begin{aligned}
\left| \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq R}} \hat{f}(m) e^{2\pi i x \cdot (m-k)} \right| &\leq \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq R}} |\hat{f}(m) e^{2\pi i x \cdot (m-k)}| \\
&\leq \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq R}} |\hat{f}(m)| \\
&\leq \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)| \\
&< c,
\end{aligned}$$

where c is a constant whose existence is justified from the hypothesis. Recall all constant functions are integrable on the torus. Thus we can use the Lebesgue dominated convergence theorem to pull the limit outside of the integral which yields

$$\hat{g}(k) = \lim_{R \to \infty} \sum_{\substack{m \in \mathbb{Z}^n \\ |m_j| \leq R}} |\hat{f}(m)| \int_{\mathbb{T}^n} e^{2\pi i x \cdot (m-k)} dx.$$

Note that

$$\int_{\mathbb{T}^n} e^{2\pi i x \cdot (m-k)} dx = \begin{cases} 1, & m = k, \\ 0, & m \neq k. \end{cases}$$

Therefore the Fourier coefficients are equal and by the previous lemma the identity 3.1.3 is established. □

With the previous theorem it is now clear that a periodic function can be reproduced from its Fourier coefficients. Granted the result did require the function to have absolutely summable Fourier coefficients, which is not always guaranteed. Ofcourse when a function has absolutely summable Fourier coefficients then we can say the function has an absolutely convergent Fourier series. So when does a function's Fourier series converge absolutely? Naturally, convergence of the Fourier series is dependent upon the decay of the Fourier coefficients. Recall in the last chapter we established that the decay of a function's Fourier transform is governed by the smoothness of the function. Well it comes as no surprise then that the decay of a function's Fourier coefficients is governed by the smoothness of the function. Therefore convergence of a function's Fourier series is dependent upon the smoothness of the function. To measure smoothness in this case we will need the following definition.

DEFINITION 3.12: For $0 \leq \gamma < 1$ define

$$||f||_{\dot{\Lambda}_\gamma} = \sup_{x,h \in \mathbb{T}^n} \frac{|f(x+h) - f(x)|}{|h|^\gamma}$$

and

$$\dot{\Lambda}_\gamma(\mathbb{T}^n) = \{f : \mathbb{T}^n \to \mathbb{C} \text{ with } ||f||_{\dot{\Lambda}_\gamma} < \infty\}.$$

We call $\dot{\Lambda}_\gamma(\mathbb{T}^n)$ the *homogeneous Lipschitz space of order* γ on the torus. Functions in this space are known as *homogeneous Lipschitz functions of order* γ.

With the previous definition we are now ready to introduce a sufficient condition for a convergent Fourier series.

THEOREM 3.13: Let s be a nonnegative integer and let $0 \leq \alpha < 1$. Assume that f is a function defined on $\mathbb{T}^n$ all of whose partial derivatives of order s lie in the space $\dot{\Lambda}_\alpha$. Suppose that $s + \alpha > n/2$. Then

$$\sum_{m \in \mathbb{Z}^n} |\hat{f}(m)| < \infty$$

and

$$\sum_{m \in \mathbb{Z}^n} |\hat{f}(m)| \leq c \sup_{|\beta|=s} ||\partial^\beta f||_{\dot{\Lambda}_\alpha},$$

where c is a constant dependent on n, α, and s.

The proof is somewhat technical and omitted in the interest of moving on. Nonetheless, we have another exciting result in which the smoothness of the function is paramount. It is becoming abundantly clear that the behaviour of a function's Fourier transform or series is largely governed by the smoothness of the function. We now move on to our next topic of interest: convergence of the square Fejér means.

3.2 Convergence Of The Square Fejér Means

In the last section we proved that the square Fejér means of a periodic function converge in L^p-norm to the function. This convergence result allowed us to prove Theorem 3.11 showing that an integrable, periodic function and its Fourier series are equivalent almost everywhere assuming the Fourier series is absolutely convergent. The question now is, are there any more convergence results for the square Fejér means? Convergence in norm is well and good but what about almost everywhere convergence or pointwise convergence? From the second result in Corollary 3.8 we may conclude that if f is continuous at x_0 then the square Fejér means $(F(n,N)*f)(x_0)$ converge to $f(x_0)$ as $N \to \infty$. Thus we have pointwise convergence of the square Fejér means for continuous functions. Can pointwise convergence of the square Fejér means be extended to more general functions? The answer is yes and is established in the following theorem.

THEOREM 3.14: Let $f \in L^1(\mathbb{T})$. If f has left and right limits at a point x_0, denoted $f(x_0-)$ and $f(x_0+)$, respectively, then

$$(F_N * f)(x_0) \to \frac{1}{2}(f(x_0+) + f(x_0-)) \text{ as } N \to \infty.$$

Proof. Recall we identify $\mathbb{T}$ with $[-\frac{1}{2}, \frac{1}{2}]$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that for $0 < t < \delta$ we have

$$\left| \frac{f(x_0+t) + f(x_0-t)}{2} - \frac{f(x_0+) + f(x_0-)}{2} \right| < \frac{\varepsilon}{4}. \tag{3.2.4}$$

Since $F_N(x)$ is an approximate identity we know $\exists N_0 > 0$ such that for all $N \geq N_0$

we have

$$\sup_{t\in[\delta,\pi]} F_N(t) < c\varepsilon, \tag{3.2.5}$$

where constant $c = 2\max(||f - f(x_0-)||_1, ||f - f(x_0+)||_1)$. Note that

$$\begin{aligned}
(F_N * f)(x_0) - f(x_0+) &= \int_{\mathbb{T}} F_N(-t)(f(x_0+t) - f(x_0))dt, \\
(F_N * f)(x_0) - f(x_0-) &= \int_{\mathbb{T}} F_N(t)(f(x_0-t) - f(x_0))dt.
\end{aligned}$$

Averaging the two identities above and taking advantage of the even integrand yields

$$\begin{aligned}
(F_N * f)(x_0) \quad - &\quad \frac{f(x_0+) + f(x_0-)}{2} \\
= &\quad \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(t)\left(\frac{f(x_0-t) + f(x_0+t)}{2} - \frac{f(x_0-) + f(x_0+)}{2}\right)dt \\
= &\quad 2\int_{0}^{\frac{1}{2}} F_N(t)\left(\frac{f(x_0-t) + f(x_0+t)}{2} - \frac{f(x_0-) + f(x_0+)}{2}\right)dt \\
= &\quad 2\int_{0}^{\delta} F_N(t)\left(\frac{f(x_0-t) + f(x_0+t)}{2} - \frac{f(x_0-) + f(x_0+)}{2}\right)dt \\
&\quad +2\int_{\delta}^{\frac{1}{2}} F_N(t)\left(\frac{f(x_0-t) + f(x_0+t)}{2} - \frac{f(x_0-) + f(x_0+)}{2}\right)dt \\
:= &\quad I_1 + I_2.
\end{aligned}$$

Therefore we need only show I_1 and I_2 get arbitrarily small as $N \to \infty$ to prove the claim. By 3.2.4,

$$I_1 < \frac{\varepsilon}{2}\int_0^{\delta} F_N(t)dt \leq \frac{\varepsilon}{2}.$$

From 3.2.5 we know for N large enough we have

$$\begin{aligned}
I_2 &= \int_{\delta}^{\frac{1}{2}} F_N(t)(f(x_0-t) - f(x_0-))dt + \int_{\delta}^{\frac{1}{2}} F_N(t)(f(x_0+t) - f(x_0+))dt \\
&< c\varepsilon(||f - f(x_0-)||_1 + ||f - f(x_0+)||_1) \\
&\leq \varepsilon.
\end{aligned}$$

Therefore we may conclude $(F_N * f)(x_0) \to \frac{1}{2}(f(x_0+) + f(x_0-))$ as $N \to \infty$. $\square$

So at this point we may conclude that the square Fejér means have pointwise convergence on $L^1(\mathbb{T})$. It is unfortunate that the last result was limited to dimension one. However, keep in mind that Fejér kernel is the arithmetic mean of the Dirichlet kernel. Thus we may extend the pointwise convergence results for the square Fejér means to the square partial sums of the Fourier series.

COROLLARY 3.15: Let $f \in L^1(\mathbb{T}^n)$, $x_0 \in \mathbb{T}^n$ and assume the square partial sums of the Fourier series of f converge at x_0. Then the following are true:

1. If f is continuous then the square partial sums of the Fourier series $(f*D(n, R))(x_0)$ converge to $f(x_0)$ as $N \to \infty$.

2. In dimension one, if f has left and right limits at a point x_0, denoted $f(x_0-)$ and $f(x_0+)$, respectively, then $(f*D(n, R))(x_0) \to \frac{1}{2}(f(x_0+)+f(x_0-))$ as $N \to \infty$.

Proof. It is already established for continuous $f \in L^1(\mathbb{T}^n)$,

$$(F(n, N) * f)(x_0) \to f(x_0) \text{ as } N \to \infty.$$

If $(D(n, N) * f)(x_0) \to A$ as $N \to \infty$, then the arithmetic means of this sequence must also converge to A. Thus $A = f(x_0)$. Proof of the second claim is identical.

$\square$

At the beginning of the section it was hinted that the square Fejér means may have almost everywhere convergence instead of only the limited pointwise convergence that has been proven thus far. Our study of the square Fejér means concludes with the following theorem.

THEOREM 3.16: Let $f \in L^1(\mathbb{T}^n)$. Then

$$(F(n, N) * f)(x) \to f(x)$$

as $N \to \infty$ for almost all $x \in \mathbb{T}^n$.

Unfortunately, in analysis proving almost everywhere convergence is typically quite difficult. Proof of the theorem above is no exception. Establishing Theorem 3.16 requires the use of maximal operators, a topic not covered in this thesis. However, to learn of maximal operators and how they are used to prove the theorem above please refer to Grafakos[1].

3.3 Bochner-Riesz Means

Perhaps you noticed that up until this point we have been focused on square partial sums and means. Let us now consider spherical partial sums. At the beginning of the chapter we defined the square Dirichlet kernel,

$$D(n,R)(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m_j|\le R}} e^{2\pi i m\cdot x}.$$

Let us now introduce the spherical Dirichlet kernel, $\tilde{D}(n,R)$, which is defined as

$$\tilde{D}(n,R)(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\le R}} e^{2\pi i m\cdot x},$$

where $|m| = \sqrt{m_1^2 + \ldots + m_n^2}$. Comparing the two you can see where their namesakes come from. Convolving a function with the spherical Dirichlet kernel gives rise to the spherical partial sums of the Fourier series. Let us now formally define the spherical partial sums and spherical Fejér means.

DEFINITION 3.17: Let $R \ge 0$. Then the expressions

$$(f * \tilde{D}(n,R))(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\le R}} \hat{f}(m) e^{2\pi i m\cdot x}$$

are called the *spherical partial sums of the Fourier series of* f. The expressions

$$(f * \tilde{F}(n,R))(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\le R}} (1 - \frac{|m|}{R})\hat{f}(m)e^{2\pi i m\cdot x}$$

are called the *spherical Fejér means of* f.

Recall that the square Fejér means were constructed from the arithmetic means of the square dirichlet kernel. Similarly, the spherical Fejér means are derived by implementing an averaging process on the spherical partial sums, $(f * \tilde{D}(n,R))(x)$, in the following sense:

$$(f * \tilde{F}(n,R))(x) = \frac{1}{R}\int_0^R (\tilde{D}(n,r) * f)(x)dr.$$

At this point, any reasonable individual may begin to ponder if there is any notable difference between the rectangular and spherical partial sums in application? The answer is a resounding yes and one need look no farther then the work of Salomon Bochner[4]. In his paper *Summation of Multiple Fourier Series by Spherical Means,* Bochner goes on to show that the spherical partial sums have a great advantage over the rectangular partial sums. In the interest of pressing on, this is a topic we shall not delve into further. None the less, the reader is highly encouraged to look at the works of Bochner.

The spherical Féjer means of integrable functions on $\mathbb{T}^2$ always converge in L^1. A very nice result, but what about dimensions higher than 2? Unfortunately, spherical Féjer means may fail in dimension 3 or higher. In fact Grafakos[1] provides an example of an integrable function on $\mathbb{T}^3$ whose spherical Féjer mean diverges almost everywhere. To overcome the limitations in higher dimensions let us study the slightly modified means below:

$$\sum_{\substack{m\in\mathbb{Z}^n \\ |m|\le R}} \left(1 - \frac{|m|}{R}\right)^{1+\varepsilon} \hat{f}(m)e^{2\pi i m\cdot x},\ \varepsilon > 0. \tag{3.3.6}$$

Note how similar the expressions in 3.3.6 are to the spherical Féjer means. Does modifying the exponent provide any advantage? It turns out the expressions in 3.3.6 are better behaved as ε increases. Studies have shown that for fixed ε the expressions above get worse as the dimension increases. To better study this phenomenom we replace $1+\varepsilon$ by nonnegative index α and arrive at our topic of interest, the Bochner-Riesz means.

DEFINITION 3.18: Let $\alpha \geq 0$. The *Bochner-Riesz means of order* α of $f \in L^1(\mathbb{T}^n)$ are defined as follows:

$$B_R^\alpha(f)(x) = \sum_{\substack{m\in\mathbb{Z}^n \\ |m|\leq R}} \left(1 - \frac{|m|^2}{R^2}\right)^\alpha \hat{f}(m)e^{2\pi i m\cdot x}.$$

It turns out that there is no different behaviour of the means above if the expression $\left(1 - \frac{|m|^2}{R^2}\right)$ is replaced by $\left(1 - \frac{|m|}{R}\right)$. With that in mind the Bochner-Riesz means of order zero coincide with the spherical partial sums of the Fourier series while Bochner-Riesz means of order one coincide with the spherical Féjer means. As previously mentioned, the stability of the Bochner-Riesz means is determined by the size of order with respect to the dimension. The million dollar question is where does one draw the line in the sand? For what minimum value of α do we have good behaviour? The solution lies in the following theorem originally proven by Bochner.

THEOREM 3.19: Let $n > 1$. There exists an integrable function f on $\mathbb{T}^n$ such that

$$\limsup_{R\to\infty} \left| B_R^{\frac{n-1}{2}}(f)(x)\right| = \infty$$

for almost all $x \in \mathbb{T}^n$. Additionally, such a function can be constructed such that it is supported in an arbitrarily small neighborhood of the origin.

The proof of the theorem above is lengthy and involved. Thus it is omitted, but can be found in Grafakos[1]. None the less, Theorem 3.19 makes it abundantly clear that the spherical Féjer means are limited at higher dimensions and Bochner-Riesz means are badly behaved on $L^1(\mathbb{T}^n)$ for $\alpha \leq \frac{n-1}{2}$. Thus the value $\alpha = \frac{n-1}{2}$ is known as the critical index and the Bochner-Riesz means provide many wonderful results under the assumption $\alpha > \frac{n-1}{2}$.

Throughout this chapter almost all the partial sums and means have been written as convolution operators. What made the square Fejér means so useful was the fact that they are constructed by convolving the function with the approximate identity $F(n, N)$. So we must ask can the Bochner-Riesz means be represented by convolving f with some function? If so, is the convolution kernel in question an approximate identity? Well if the order of the Bochner-Riesz means is greater than the critical index, then yes B_R^α may be written as a convolution operator whose convolution kernel is an approximate identity. Before proving this we will need to introduce the Poisson Summation formula.

THEOREM 3.20: (Poisson Summation Formula) Suppose that $f, \hat{f} \in L^1(\mathbb{R}^n)$ such that

$$|f(x)| + |\hat{f}(x)| \leq c(1 + |x|)^{-n-\delta}$$

for some $c, \delta > 0$. Then f and $\hat{f}$ are both continuous, and for all $x \in \mathbb{R}^n$ we have

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} = \sum_{m \in \mathbb{Z}^n} f(x + m).$$

In particular,

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) = \sum_{m \in \mathbb{Z}^n} f(m).$$

Proof. Here we present the proof from Grafakos[1]. Since $\hat{f} \in L^1(\mathbb{R}^n)$ we know inversion holds by Theorem 1.15. Thus f can be identified with a continuous function. Define a 1-periodic function on $\mathbb{T}^n$ by setting

$$F(x) = \sum_{m \in \mathbb{Z}^n} f(x + m).$$

Then we have

$$\hat{F}(m) = \int_{\mathbb{T}^n} F(x)e^{-2\pi i m\cdot x}dx = \sum_{k\in\mathbb{Z}^n}\int_{[-\frac{1}{2},\frac{1}{2}]^n-k} f(x)e^{-2\pi i m\cdot x}dx = \hat{f}(m),$$

which shows that the sequence of Fourier coefficients of F coincide with the restriction of the Fourier transform of f on $\mathbb{Z}^n$. Application of the hypothesis yields

$$\sum_{m\in\mathbb{Z}^n}|\hat{F}(m)| = \sum_{m\in\mathbb{Z}^n}|\hat{f}(m)| \le c\sum_{m\in\mathbb{Z}^n}\frac{1}{(1+|m|)^{n+\delta}} < \infty.$$

With the result above the desired result follows from application of Theorem 3.11.

□

With the Poisson Summation formula now established we are ready to prove the following fundamental property of Bochner-Riesz means. The following proof is from Grafakos[1].

THEOREM 3.21: Let $\alpha > \frac{n-1}{2}$ and $f \in L^p(\mathbb{T}^n)$ where $1 \le p < \infty$. Then we have the following two results:

1. $B_R^\alpha(f)$ converges to f in L^p as $R \to \infty$.

2. If f is continuous on $\mathbb{T}^n$ then $B_R^\alpha(f)$ converges to f uniformly as $R \to \infty$.

Proof. The proof is largely dependent on the asymptotics of Bessel functions. First we must link Bessel functions and the Bochner-Riesz means. Consider the function

$$m_\alpha(\xi) = (1-|\xi|^2)_+^\alpha \text{ where } \xi \in \mathbb{R}^n.$$

Then

$$B_R^\alpha(f)(x) = \sum_{l\in\mathbb{Z}^n} m_\alpha(\frac{l}{R})\hat{f}(l)e^{2\pi i l\cdot x}.$$

We know from identity 2.2.10 that

$$(m_\alpha)\check{}(x) := K^\alpha(x) = \frac{\Gamma(\alpha+1)}{\pi^\alpha}\frac{J_{\frac{n}{2}+\alpha}(2\pi|x|)}{|x|^{\frac{n}{2}+\alpha}}. \tag{3.3.7}$$

Application of theorems 2.9 and 2.10 yield

$$|K^\alpha(x)| \le c_{n,\alpha}(1+|x|)^{-n-(\alpha-\frac{n-1}{2})} \tag{3.3.8}$$

where $c_{n,\alpha}$ is a constant dependent only on n and α. Now apply basic properties of the Fourier transform and identity 3.3.7 to obtain

$$\begin{aligned} B_R^\alpha(f)(x) &= \sum_{l\in\mathbb{Z}^n} m_\alpha(\frac{l}{R})\hat{f}(l)e^{2\pi il\cdot x} \\ &= \sum_{l\in\mathbb{Z}^n} \widehat{K^\alpha}(\frac{l}{R})\hat{f}(l)e^{2\pi il\cdot x} \\ &= \sum_{l\in\mathbb{Z}^n} (f * \widehat{R^n\delta^R(K^\alpha)})(l)e^{2\pi il\cdot x}, \end{aligned}$$

where the delta above denotes dilation from definition 1.5. Now apply the Poisson Summation formula:

$$B_R^\alpha(f)(x) = \sum_{l\in\mathbb{Z}^n}(f * R^n\delta^R(K^\alpha))(x+2\pi l) = (f * R^n\delta^R(L^\alpha))(x),$$

where $L^\alpha(x) = \sum_{k\in\mathbb{Z}^n} K^\alpha(x+k)$ which is a 1-periodic function on $\mathbb{T}^n$. Thus to prove the claim we need only establish the family $\{R^n\delta^R(L^\alpha)\}$ is an approximate identity on $\mathbb{T}^n$ as $R\to\infty$ and Theorem 1.2 will establish the desired result. It is immediate from 3.3.8 that L^α is integrable on $\mathbb{T}^n$ and the first condition of definition 1.1 satisfied. Since K^α is a normalized spherical Bessel function then

$$\int_{\mathbb{T}^n} R^n\delta^R(L^\alpha)(t)dt = \int_{\mathbb{R}^n} R^n\delta^R(K^\alpha)(x)dx = \int_{\mathbb{R}^n} K^\alpha(x)dx = 1.$$

Now we need only establish the third condition in definition 1.1 to complete the proof. Once again application of 3.3.8 yields that for $0 < \varepsilon < \frac{1}{2}$ we have

$$\begin{aligned}\int_{\varepsilon \le |x_j| \le \frac{1}{2}} |R^n \delta^R(L^\alpha)| dx &= R^n \int_{\varepsilon \le |x_j| \le \frac{1}{2}} |L^\alpha(Rx)| dx \\ &= R^n \int_{\varepsilon \le |x_j| \le \frac{1}{2}} \sum_{k \in \mathbb{Z}^n} |K^\alpha(Rx + k)| dx \\ &\le c_{n,\alpha} R^{\frac{n-1}{2} - \alpha} \int_{\varepsilon \le |x_j| \le \frac{1}{2}} \sum_{k \in \mathbb{Z}^n} \frac{1}{|x + k|^{n + \alpha - \frac{n-1}{2}}} dx.\end{aligned}$$

Note that the series above converges uniformly, implying

$$\int_{\varepsilon \le |x_j| \le \frac{1}{2}} |R^n \delta^R(L^\alpha)| dx \to 0 \text{ as } R \to \infty.$$

□

As you can see in the proof above, when $\alpha > \frac{n-1}{2}$, the Bochner-Riesz means can be written as a convolution operator in which the convolution kernel is an approximate identity. As you have no doubt noticed, our study of the Bochner-Riesz means have been limited to functions on the torus. Naturally, the Bochner-Riesz means of $f \in L^p(\mathbb{R}^n)$ are defined as

$$B_R^\alpha(f)(x) = \int_{|\xi| < R} \left(1 - \left(\frac{|\xi|}{R}\right)^2\right)^\alpha \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where the summation is replaced by an integral. Grafakos[1] and Stein[2] go on to show the Bochner-Reisz means of functions defined on Euclidean space have the same results as those on the torus thanks to a transference result. Given it's good behavior and flexibility, the Bochner-Riesz means are a wonderful tool when studying the convergence of Fouier series and integrals. For more on Bochner-Riesz means please refer to Grafakos[1] and Stein[2].

REFERENCES

[1] L. Grafakos, *Classical Fourier Analysis*, 2nd edition, Graduate Texts in Math 249, Springer-Verlag (2008).

[2] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis On Euclidean Spaces*, 2nd edition, Princeton Mathematical Series 32, Princeton University Press (1975).

[3] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd edition, Cambridge University Press (2004).

[4] S. Bochner, *Summation of Multiple Fourier Series By Spherical Means*, Trans. Amer. Math. Soc. 40 (1936), 175-207.

[5] G. Weiss, *Harmonic Analysis*, M.A.A. Studies in Mathematics 3 (1965), 124-178.

[6] W.O. Bray, *Growth and Integrability of Fourier Transforms on Euclidean Space*, (2013).

[7] G.N. Watson, A Treatise on the Theory of Bessel Functions, 2nd edition, Cambridge University Press, 1980.

[8] W.O.Bray and M.A.Pinsky, Growth Properties of Fourier Transforms via Moduli of Continuity, Jour. Func. Anal., 255, p2265-2285 (2009).

[9] S.S. Platonov, The Fourier transform of functions satisfying the Lipschitz condition on rank one symmetric spaces, Siberian Math. Jour. 45, no. 6, p1108-1118 (2005).