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# A Geometric Approach To Ramanujan's Taxi Cab Problem And Other Diophantine Dilemmas

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# A GEOMETRIC APPROACH TO RAMANUJAN'S TAXI CAB PROBLEM AND OTHER DIOPHANTINE DILEMMAS

A Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

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Master of Science, Mathematics

By

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## ABSTRACT

In 1917, the British mathematician G.H. Hardy visited the Indian mathematical genius Ramanujan in the hospital. The number of the taxicab Hardy arrived in was 1729. Ramanujan immediately recognized this as the smallest positive integer that can be expressed as the sum of two cubes in two essentially different ways. In this thesis, we use properties of conics and elliptic curves to investigate this problem, its generalization to fourth powers, and a Diophantine equation involving the distance of a point from the vertices of a regular tetrahedron (the latter extends work of Christina Bisges).

**KEYWORDS**: Diophantine, Taxi Cab Problem, tetrahedron, elliptic curve, conic section

This abstract is approved as to form and content

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Chairperson, Advisory Committee
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# 1 INTRODUCTION

We apply a geometric approach in investigating solutions for several (well-known in the case of one of our problems) Diophantine equations. In the first section, we outline facts from the theory of Diophantine Geometry, and in general, algebraic geometry, that we will be of use in this work, or that we simply find of interest. Specifically, we introduce the group law for elliptic curves and show that, given a point on a conic, we can find infinitely many more points on the conic by creating a parameterized family of lines that pass through our known point on the conic. In the second section, we presents our work on the famous Taxi Cab problem, that is, the problem of looking at numbers that can be represented as the sum of two cubes in two different ways. We use geometric methods to find solutions to our problem instead of purely algebraic methods. After considering the Taxi Cab problem, we "upgrade" the problem and look at the sum of two fourth-powers that can be written in two different ways. Finally, in the last chapter, we extend the work done by Christina Bisges in her Master's Thesis [1]. Instead of considering an equilateral triangle as she did, we investigate the tetrahedron and look to see if there exists a tetrahedron of integer side lengths with a point inside it that is integer side length from each of the vertices. We employ three different techniques for finding solutions to this problem, all of which make heavy use of properties from conics.

## 2 THEORETICAL FOUNDATIONS

### Brief Facts on Conics

Conic sections (conics for short) are curves obtained from slicing a cone in three-dimensional space with a plane. The most familiar examples of conics are the parabola, the ellipse, and the hyperbola. In general, conics are quadratics with the following equation:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \tag{2.1}$$

where each of the coefficients is taken to be from a field, and the variables $x$ and $y$ are indeterminates. An important property about conics is that given a point on the conic, one can create a parameterized family of solutions for the conic. In this way, through the use of a single known point, all other points can be found. To see this, let $C$ be a conic, and let $P = (x_0, y_0)$ be a point on $C$. Then, we can pass the following line through $P$:

$$y - y_0 = t(x - x_0).$$

But, a line will cross a conic in two places, thus producing a new point for $C$. The slope $t$ in the above line is called a parameter for our line, since, allowing it to vary produces new lines, and new points on $C$. Finally, if $t$ is rational, then the resulting new point for $C$ found from the second intersection point of the line and $C$ forces this new point to have rational coordinates.

An example of such a parameterization is the following:

EXAMPLE 2.1: Consider the conic $x^2 + y^2 = 1$ (a circle of radius 1). One quickly sees that $P = (0, 1)$ is a point on this conic. The line $y = tx + 1$ passes through $P$.

This gives us the following system of equations

$$\begin{cases} x^2 + y^2 = 1 \\ y = tx + 1 \end{cases} \tag{2.2}$$

Inserting the second equation into the first gives

$$x^2 + (tx + 1)^2 = 1.$$

The above equation gives us $x((t^2 + 1)x + 2t) = 0$ implying that $x = 0$ or $x = \frac{2t}{1-t^2}$. From $P$ we already knew $x = 0$ and so we use the second solution for $x$. This gives us parameterized solutions for $C$:

$$\left(\frac{2t}{1-t^2}, \frac{1-t^2}{1+t^2}\right),$$

provided that $t^2 \neq 1$.

**Elliptic Curves and the Group Law**

The study of elliptic curves, and topics related to them, is a key part of the theory of algebraic geometry. As we will see in this section, one can use elliptic curves to aid in studying solutions to Diophantine equations, that is, integer solutions to certain multi-variable equations. One of the main tools applied to these curves is the fact that a specific type of addition can be placed on the points of the curve to create what is known as a group structure on those points. We begin this section then by first defining what is meant by a group, and then investigating how to put this structure on the points of an elliptic curve.

Before we can discuss a group, we must first state what is meant by a binary operation.

DEFINITION 2.2: A binary operation $*$ on a set $G$ is a function $* : G \times G \to G$

such for $a, b \in G$, $a * b \in G$.

Using (2.2) on a set $G$, we can now define what we mean by a group.

DEFINITION 2.3: A group is an ordered pair $(G, *)$ such that $G$ is a set and $*$ is a binary operation. The binary operation $*$ on $G$ satisfies the following properties:

1. For $a, b, c \in G$, $(a * b) * c = a * (b * c)$. This property is known as associativity.
2. There exists an element $e$ in $G$ such that for all $a \in G$, $e * a = a * e = a$. Such an element $e$ is called the identity of $G$.
3. For every $a$ in $G$, there exists an element $a^{-1}$ of $G$ such that $a * a^{-1} = a^{-1} * a = e$. This element $a^{-1}$ is called an inverse of $a$.

If, in addition to the above properties on $G$, $a * b = b * a$ for all $a$ and $b$ in $G$, then $G$ is called an abelian group.

From the properties above, one can show that the identity element of a group is unique and that the inverse of any element is unique. Some basic examples of groups are the familiar sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ where the binary operation $*$ is taken to be ordinary addition, which is denoted as $+$ instead of $*$.

As mentioned, our main goal in this section will be to form the set of points on an elliptic curve into a group. To do this, we first begin with a general cubic equation. Such a general equation is:

$$ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy + j = 0, \qquad (2.3)$$

where the coefficients are taken to be from some field.

We remind the reader what a field is in the following definition:

DEFINITION 2.4: A field is a set $k$ with two binary operations, $+$ and $\times$, (called addition and multiplication respectively) such that the following properties are satisfied under these operations:

1. $(k, +)$ is an abelian group.

2. $(k - \{0\}, \times)$ is an abelian group.

3. Multiplication distributes across addition.

Some common examples of fields are: $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. We note that $\mathbb{Z}$ is not a field since the element 2 is in $\mathbb{Z}$ but $2 \times \frac{1}{2} = 1$ but $\frac{1}{2}$ is not in $\mathbb{Z}$.

If the field for (2.3) is $\mathbb{Q}$, then we say that the cubic equation is rational. Let $F(x, y) = 0$ be a cubic equation. A point $P = (a, b)$ on $F$ is said to be simple if $F_x(P) \neq 0$ or $F_y(P) \neq 0$, that is, there exists a tangent line at $P$. The case when $P$ is not simple motivates the following definition:

DEFINITION 2.5: Let $F$ be a cubic equation. A point $P$ on $F$ such that all partial derivatives at $P$ on $F$ vanish, and $F(P) = 0$, is called a singular point.

The topic of singular points is very important to us. Later, we will use them to help find solutions to certain equations known as Diophantine equations. In addition, they set the stage for our next definition:

DEFINITION 2.6: Let $F$ be a cubic curve. $F$ is called an elliptic curve if $F$ has no singular points. We sometimes refer to a curve that has no singular points as being smooth or non-singular.

EXAMPLE 2.7: An example of a elliptic curve is the curve given by the equation $y^2 = x^3 + x + 2$. To see this, consider the equation $x^3 + x + 2 - y^2 = 0$. Taking partial derivatives we have

$$3x^2 + 1 = 0$$

and,

$$-2y = 0$$

There does not exist any point on the curve that makes the above equations simultaneously 0, and so $y^2 = x^3 + x + 2$ contains no singular points and thus is an elliptic curve by Definition 2.6.

REMARK 2.8: The equation in (2.7) takes on a special form. Any elliptic curve that has the form $y^2 = x^3 + ax + b$ where $a$ and $b$ are constants, is called the Weierstrass form of an elliptic curve. As it turns out, through an appropriate change of variables, any elliptic curve (whose field characteristic is not 2 or 3) can be written in Weierstrass form.

We now introduce some more notation and definitions that will be needed shortly.

An important concept from algebraic geometry is the distinction between a homogeneous polynomial and a non-homogeneous polynomial.

DEFINITION 2.9: A polynomial of degree $n$ is said to be homogeneous if all the degrees of the non-zero terms of the polynomial are equal to $n$ as well.

While any polynomial can be rewritten so that all degrees are the same (through a process called homogenization), we will be primarily interested in "de-homogenizing" a homogeneous polynomial, that is to say, removing the property that all the degrees of the terms of the polynomial are the same. A common technique for de-homogenizing a homogeneous polynomial is to set one of variables equal to 1. For example, consider the polynomial $p(x, y) = x^3 + xy^2$. We see that $p$ is homogeneous of degree 3. If we wanted to de-homogenize $p$, we could fix $y$ to be 1, changing $p(x, y)$ to $p(x) = x^3 + x$, and this resulting polynomial is no longer homogeneous.

The concept of homogeneous polynomials leads naturally into the discussion of projective curves. A projective curve, also called a projective plane curve, is a curve that exists in projective space. Projective space is an $n + 1$ dimensional space built of points, other than the origin, that satisfy the following equivalence relation

$$(x, y, z) \sim (\lambda x, \lambda y, \lambda z) \tag{2.4}$$

for $\lambda \neq 0$. A projective plane curve is a homogeneous equation in three variables that is equal to 0. We denote projective space by $\mathbb{P}^2$.

While the definition and use of projective curves and projective space is far more technical than what has been presented here, we do not make our definition more rigorous since what we have discussed is sufficient for the next several definitions and theorems.

DEFINITION 2.10: Let $F$ and $G$ be projective plane curves of degree $m$ and $n$ respectively. Then, the intersection cycle of $F$ and $G$ is given by:

$$F \cdot G = \sum_{P \in \mathbb{P}^2} I(P, F \cap G)P$$

The expression $I(P, F \cap G)$ in Definition 2.10 is formally known as the intersection number. Intuitively, it can be thought of the number of times two given curves intersect each other at a given point $P$, counting multiplicities. For example, the $x$ and $y$ axis in $\mathbb{R}^2$ intersect at the origin with intersection number equal to 1. It must be noted that our definition of an intersection number is very informal and not complete, however, we will not need to make significant use of it, and so will not go into the details that would be necessary to formulate a complete definition. Instead, we recommend [3] for a rigorous take on what is meant by intersection number.

LEMMA 2.11: [3] Let $F$, $G$, and $H$ be plane curves with $P \in F \cap G$. If all the points of $F \cap G$ are simple points of $F$, and $H \cdot F \geq G \cdot F$, then there is a curve $C$ such that $C \cdot F = H \cdot F - G \cdot F$.

PROPOSITION 2.12: Let $C$ be an irreducible cubic and $C,' C''$ cubics. Suppose $C' \cdot C = \sum_{i=1}^{9} P_i$ where each point $P_i$ on $C$ is simple (but not necessarily distinct). Suppose further that $C'' \cdot C = \sum_{i=1}^{8} P_i + Q$. Then, $P_9 = Q$.

*Proof.* Let $L$ be a line through $P_9$ that does not pass through $Q$. So, $L \cdot C = P_9 + R + S$ for some points $R, S \in C$. Then, $LC'' \cdot C = C' \cdot C + Q + R + S$, so, by the above fact, there is a line $L^{'}$ such that $L^{'} \cdot C = Q + R + S$. But then,

$$L^{'} = L \text{ implying that } P_9 = Q.$$

□

We are now ready to introduce what is meant by addition of points on an elliptic curve.

DEFINITION 2.13: Let $C$ be an elliptic curve. For any two points $P, Q \in C$, there exists a unique line $L$ such that $L \cdot C = P + Q + R$ for some additional point $R \in C$. Let $\varphi : C \times C \to C$ defined by $\varphi(P, Q) = R$.

REMARK 2.14: If $P = Q$, then $L$ is the tangent line at $P$.

The definition for $\varphi$ is somewhat abstract, but as we shall soon see, it can be made to hold all the familiar properties of "normal" addition. We can see this by realizing that $\varphi$ is just an algebraic way of saying that, geometrically speaking, we are drawing a line between the points $P$ and $Q$ on the curve $C$ ($P$ and $Q$ need not be distinct), and then stating that $P$ "plus" $Q$ is equal to the third point of intersection of the line through $C$, which we have labeled as $R$. This process makes sense, since it is a fact from geometry that a line will interest a cubic at most three times.

We know that "normal" addition is commutative. Addition given by $\varphi$ is also commutative. To see this, let $\varphi(P, Q) = R$. We know that, from above, this is just the line passing through the points $P$, $Q$, and $R$. However, any line passing through $P$ and $Q$ is the same as the line passing through $Q$ and $P$, therefore, we have: $\varphi(P, Q) = \varphi(Q, P) = R$

This addition on the points of $C$, while intuitive when related to the concept of lines, has one serious draw back as we see in the following remark:

REMARK 2.15: Let $C$ be an elliptic curve and let $L$ and $N$ be lines such that

$L \cdot C = P + Q + R$ and $N \cdot C = Q + T + S$. Then,

$$\varphi(P, \varphi(Q, T)) = \varphi(P, S)$$

and,

$$\varphi(\varphi(P, Q), T) = \varphi(R, T)$$

The line passing through $R$ and $T$ must be different than that of the line passing through $P$ and $S$. Therefore, we have $\varphi$ is not associative!

To remedy the lack of an associative property for $\varphi$ , we introduce a new point, $\mathcal{O}$, and define it to be the identity point on $C$ (this new point is sometimes called the "point at infinity," and labeled as $\infty$, but it is not the same idea of infinity when talking about the cardinality of sets). While possibly contrived in feeling, $\mathcal{O}$ can geometrically be thought of as living at the "top" and "bottom" of the coordinate plane.

With this notion of $\mathcal{O}$, we can enhance our earlier definition of addition.

DEFINITION 2.16: Let $\varphi$ be as above. Let addition on $C$ be given by: $P \oplus Q = \varphi(\mathcal{O}, \varphi(P, Q))$

Definition 2.16 gives us all the usual properties of addition, thus creating a group structure on $C$. As we shall see, most of the normal properties of addition are easy to prove; the only property that is hard is the associative property which we relay as a theorem.

THEOREM 2.17: The operation $\oplus$ on the elliptic curve $C$ is associative.

*Proof.* Suppose $P, Q, R \in C$. Let $L_1 \cdot C = P + Q + S'$, $N_1 \cdot C = \mathcal{O} + S' + S$, $L_2 \cdot C = S + R + T'$, $N_2 \cdot C = Q + R + U'$, $L_3 \cdot C = \mathcal{O} + U' + U$, and $N_3 \cdot C = P + U + T''$

We have that $L_i$ and $N_j$, $i, j \in \{1, 2, 3\}$ are simply lines through $C$. Now,

$$(P \oplus Q) \oplus R = \varphi(\mathcal{O}, S') \oplus R = S \oplus R = \varphi(\mathcal{O}, T')$$

and,

$$P \oplus (Q \oplus R) = P \oplus \varphi(\mathcal{O}, U^{'}) = P \oplus U = \varphi(\mathcal{O}, T^{''})$$

Our goal is to show that $T^{'} = T^{''}$.

Let $C^{'} = L_1L_2L_3$ and $C^{''} = N_1N_2N_3$. Then,

$$C^{'} \cdot C = P + Q + S^{'} + S + R + \mathcal{O} + U^{'} + U + T^{'}$$

and,

$$C^{''} \cdot C = P + Q + S^{'} + S + R + \mathcal{O} + U^{'} + U + T^{''}$$

It follows that $T^{'} = T^{''}$ by Proposition 2.12. □

Theorem 2.17 shows us that $\oplus$ is an associative operation on any elliptic curve. However, this binary operation gives more than just this important property. Consider $P \oplus \mathcal{O}$. We know that

$$P \oplus \mathcal{O} = \mathcal{O} \oplus P$$

by commutativity above. But,

$$P \oplus \mathcal{O} = \varphi(\mathcal{O}, \varphi(\mathcal{O}, P)) = P$$

by $\mathcal{O}$ being defined as the identity. Further, if we define $-P$ to be the point reflected across the $x-$ axis, then the line through $P$ and $-P$ must be vertical, and so it intersects the curve at $\mathcal{O}$. From this we have

$$P \oplus -P = \varphi(\mathcal{O}, \varphi(P, -P)) = \varphi(\mathcal{O}, \mathcal{O}) = \mathcal{O}$$

Thus, every point on the curve has an additive inverse. It is clear that every condition of Definition 2.3 has been met for $\oplus$ on the curve $C$ and so we have

$(C, \oplus)$ forms a group. In fact, even more can be said about $\oplus$. By our discussion above, $(C, \oplus)$ is not only a group, but an abelian group as well!

REMARK 2.18: We sometimes refer to $\oplus$ on $C$ as the group law on $C$, or simply as the group law.

The primary interest of this thesis will be to investigate solutions to elliptic curves that are either rational or integral. In light of this, we have the following definition:

DEFINITION 2.19: Let $C$ be an elliptic curve. Define

$$C(\mathbb{Q}) = \{(x, y) \,|\, x, y \in \mathbb{Q},\ (x, y) \text{ on } C\} \cup \{\mathcal{O}\}$$

That is, $C(\mathbb{Q})$ is the set of all rational solutions to the curve $C$.

Under the operation $\oplus$ above, $C(\mathbb{Q})$ is a subgroup of $(C, \oplus)$. That is, $C(\mathbb{Q})$ is a subset of $C$, and is a group under $\oplus$. The fact that $C(\mathbb{Q})$ is a subgroup of $C$ is very helpful for our purposes, however, $C(\mathbb{Q})$ has even more structure to it than its parent set as we now show:

THEOREM 2.20: (Mordell) The set $C(\mathbb{Q})$ is finitely generated. That is, there exists some finite set of points $P_1, P_2, \ldots, P_n$ of $C(\mathbb{Q})$ such that for any point $Q \in C(\mathbb{Q})$, $Q = r_1P_1 + \cdots + r_nP_n$ for constants $r_k$, $k = 1, \ldots, n$.

What this theorem implies is that given any rational point on the elliptic curve, we are guaranteed to produce another rational point on the curve using the addition $\oplus$. This is a very powerful theorem indeed.

Some clarification is needed for Theorem 2.20 before we can move on, centered on what is meant by $r_iP_i$ for $i = 1, \ldots, n$ in the above theorem. Clearly $r_iP_i$ is an integer multiple of the point $P_i$ on the curve. To understand what is meant by taking an integer multiple of a point on an elliptic curve, we note the following definition

DEFINITION 2.21: Let $n$ be an integer and let $P$ be a point on an elliptic curve $C$. Then, if $n > 0$, define $nP = P \oplus P \oplus \cdots \oplus P$, $n$ times. Likewise, for $-n$, we have $-nP = -P \oplus (-P) \oplus \cdots \oplus (-P)$, $|-n|$ times. For the case where $n = 0$, define $nP = 0P$ to be the identity, $\mathcal{O}$.

Remark 2.14 above gives a hint on how to calculate $nP$. We know that to add points on elliptic curve using $\oplus$, we must pass lines through them. Finding these lines is fairly easy when presented with distinct points $P$ and $Q$ on our elliptic curve since, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then the slope of the line connecting them is given by the equation $m = \frac{y_2 - y_1}{x_2 - x_1}$. Building the appropriate line between $P$ and $Q$ then reduces to making use of the standard formula for a line, $y = mx + b$. However, the situation changes when we do not have distinct points. Say we wish to calculate $2P$, where $P = (x_1, y_1)$ is a point on our elliptic curve. From Definition 2.16, we know that $2P = P \oplus P$, but if we try and make use of the standard equation for slope, we see that $m = \frac{0}{0}$, clearly a result that we don't want. However, as Remark 2.14 suggests, if we consider the tangent line at $P$, we now have a means at finding a line through $P$.

EXAMPLE 2.22: With the above in mind, we present an example of how to compute $2P$ for some point $P$ on an elliptic curve. Consider the elliptic curve

$$y^2 = x^3 + 3 \tag{2.5}$$

and the point $P = (1, 2)$ on it. The tangent line at $P$ can be found by taking the implicit derivative of (2.5) and evaluating it at $P$. Doing so gives us $m = \frac{3}{4}$ as the slope of the tangent line. This implies that our tangent line is

$$y = \frac{3}{4}x + \frac{5}{4}.$$

Substituting this equation in for $y$ in (2.5) and simplifying gives us $x^3 - \frac{9}{16}x^2 -$

$\frac{15}{8}x + \frac{23}{16} = 0$. However, we know that $x = 1$ must be a double root from $P$ and the fact that we are considering the tangent line at $P$. We have $x^3 - \frac{9}{16}x^2 - \frac{15}{8}x + \frac{23}{16} = 0$ factors into $(x-1)^2(x + \frac{23}{16}) = 0$ giving us $Q = (-\frac{23}{16}, \frac{11}{64})$ as the third point of intersection of our tangent line. Now, this point $Q$ is not $2P$. We know from our discussion earlier that we must pass another line through $Q$ and the point at infinity $\mathcal{O}$ to find $2P$. It turns out (after some technical considerations) that $2P$ will be the reflection of $Q$ across the $x$-axis. Therefore, $2P = (-\frac{23}{16}, -\frac{11}{64})$. For more information on calculating $2P$ and the mentioned technical issues involving the line through $Q$ and $\mathcal{O}$, please see [5].

Another result that is useful in the discussion on elliptic curves is the following theorem

THEOREM 2.23: Let $C$ be an elliptic curve. Then $C(\mathbb{Q}) \cong \mathbb{Z}^r \times C(\mathbb{Q})_{tor}$ where $r$ is a positive integer called the rank of the elliptic curve, and $C(\mathbb{Q})_{tor}$ is the set of all points on $C$ with finite order, i.e, $nP = \mathcal{O}$ for some integer $n < \infty$.

While Theorem 2.23 is interesting, the following theorem which uses it is very important to the study of elliptic curves:

THEOREM 2.24: (Mazur) Let $C$ be an elliptic curve in Weierstrass form. Then, $C(\mathbb{Q})_{tor} \cong \mathbb{Z}/n\mathbb{Z}$ for $1 \leq n \leq 10$, and $n = 12$, or $C(\mathbb{Q})_{tor} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ for $1 \leq n \leq 4$.

**Diophantine Equations**

One of our primary goals will be to determine integer solutions for certain equations. Equations where one is only interested in integer solutions are known as Diophantine equations. We make this notion formal in the following definition:

DEFINITION 2.25: Let $p(x_1, x_2, \ldots, x_n) = 0$ be a polynomial equation with integer coefficients. If we restrict the solutions $x_i$ for $i = 1, \ldots, n$, of $p$ to only those that are integers, then $p = 0$ is called a Diophantine equation.

**Genus of a Curve**

Roughly speaking, by "genus," we simply mean how many holes there are in a surface. For our purposes, we define the (arithmetic) genus for a curve as follows:

DEFINITION 2.26: Let $C$ be an irreducible, smooth plane curve with degree $d$. Then the genus of $C$ is given by the genus-degree formula:

$$g = \frac{(d-1)(d-2)}{2}$$

If $C$ is not smooth, and contains only $n$ ordinary singular points, then the genus of $C$ is given by:

$$g = \frac{(d-1)(d-2)}{2} - \frac{1}{2}\sum_{i=1}^{n} r_i(r_i - 1)$$

where each $r_i$ is the multiplicity of the given singular point.

EXAMPLE 2.27: Let $C$ be a cubic with one singular point of multiplicity 2. Then $C$ has genus

$$g = \frac{(3-1)(3-2)}{2} - \frac{1}{2}(2)(2-1) = 0$$

If $C$ is an elliptic curve, then $C$ has genus

$$g = \frac{(3-1)(3-2)}{2} = 1$$

We see that all elliptic curves have genus of 1.

# 3 TAXI CAB PROBLEM AND SUM OF CUBES

## The Number 1729

Upon first glance, the number 1729 may appear to be a rather dull number. As with many other numbers, one may quickly assume that 1729 does not possess any interesting or awe inspiring properties, outside the well known factoring properties of number theory. The famous mathematician of the early 20th century, G. H. Hardy, certainly thought so when he was visiting his friend and colleague, the equally famous Srinivasa Ramanujan, in the hospital. To Hardy's surprise, the assumption of the blandness of 1729 was incorrect as is recorded in the following tale, taken from [6]:

> "Once, in the taxi from London, Hardy noticed its number, 1729. He must have thought about it a little because he entered the room where Ramanujan lay in bed and, with scarcely a hello, blurted out his disappointment with it. It was, he declared, 'rather a dull number,' adding that he hoped that wasn't a bad omen. 'No, Hardy,' said Ramanujan, 'it is a very interesting number. It is the smallest number expressible as the sum of two [positive] cubes in two different ways'"

Ramanujan was famous for his many elegant but highly complex appearing solutions for mathematical questions; his obscure observation on the above property of 1729 was not out of place for him. Also not out of place, was the fact that he was correct. To see how, let's consider the equation:

$$x^3 + y^3 = 1729. \tag{3.1}$$

Our goal is to show that there exists positive integer solutions $(x_1, y_1)$ and

$(x_2, y_2)$ such that

$$x_1^3 + y_1^3 = x_2^3 + y_2^3 = 1729 \tag{3.2}$$

The process is easier than one might first assume since the expression, $x^3+y^3$ is factorable. We have:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2). \tag{3.3}$$

Setting the right hand side of (3.3) equal to 1729 gives:

$$1729 = (x + y)(x^2 - xy + y^2). \tag{3.4}$$

We now let $1729 = AB$ for $A, B \in \mathbb{Z}$ and set

$$A = x + y \tag{3.5}$$

and,

$$B = x^2 - xy + y^2. \tag{3.6}$$

By (3.5), we have $y = A - x$. Inserting $A - x$ for $y$ in (3.6), we have

$$B = x^2 - x(A - x) + (A - x)^2. \tag{3.7}$$

We know that $1729 = 7 \cdot 13 \cdot 19$, and so we can apply this fact to the factorization $1729 = AB$, to find positive integer solutions. We do this by taking combinations of the three factors of 1729 and inserting them into (3.6). For example:

EXAMPLE 3.1: Take $A = 19$ and $B = 7 \cdot 13 = 91$. Then by (3.6), we have

$$91 = x^2 - x(19 - x) + (19 - x)^2$$

Solving using the quadratic formula, we see that $x = 9$ and $x = 10$. Let's assume that $x = 9$. Then $y = 10$ and so $(9, 10)$ must be a solution to (3.1).

Indeed, $9^3 + 10^3 = 1729$, and so we have found at least one integer solution to (3.1).

REMARK 3.2: Example 3.1 really gives us to two solutions to (3.1) since $9^3 + 10^3 = 10^3 + 9^3$ and so both $(9, 10)$ and $(10, 9)$ satisfy (3.1). However, we are interested in unique solutions to (3.1) and so count $(9, 10)$ and $(10, 9)$ as being the same solution.

So far, we have only produced one integer solution to $x^3 + y^3 = 1729$. Ramanujan claimed there was two and no more; we show that he was correct in the following combinations of $A$ and $B$ values, created using the method above:

$$A = 91 \qquad B = 19$$

$$A = 7 \qquad B = 247$$

$$A = 247 \qquad B = 7$$

$$A = 13 \qquad B = 133$$

$$A = 133 \qquad B = 13$$

We see that only $A = 13$ and $B = 133$ produce integer solutions when applied to (3.6). The solution is: $(1, 12)$.

Thus,

$$9^3 + 10^3 = 1^3 + 12^3 = 1729.$$

And by the factorization of $x^3+y^3$, we know that the above solutions are the only possible integer solutions.

REMARK 3.3: The restriction to positive integers is important. If we instead allow for the possibility for any integer solution, we would have that the number 91 is the smallest number expressible as the sum of two cubes since

$$91 = 6^3 + (-5)^3 = 4^3 + 3^3.$$

As it turns out, while 1729 is the smallest number that can be represented as the sum of two cubed positive integer, it is not the only number with this property. There exists many such numbers, which we represent with the letter $r$, that also hold this property. The above discussion is based largely on the arguments found in [4] and more details on the matter can be found there.

What we have talked about thus far inspires the following definition:

DEFINITION 3.4: The equation $\alpha^3 + \beta^3 = \gamma^3 + \delta^3 = r$ for some integer $r$ is called a Ramanujan equation if there exists integer pairs $(\alpha_0, \beta_0)$ and $(\gamma_0, \delta_0)$ that satisfy it. The integer $r$ in this equation is called a Ramanujan number.

The method used earlier to find the solutions to Ramanujan's problem involved a factorization argument, and was thus purely algebraic. There is almost always more than one way to solve a mathematical problem, and another way to solve Ramanujan's problem is to apply techniques from geometry to it.

**The Equation** $x^3 + y^3 - z^3 - 1 = 0$

In this section we assume that we do not know the integer $r$ that satisfies Definition 3.4. We develop methods to uncover $r$, even showing another way at finding the case when $r = 1729$.

To begin, we consider the equation

$$\alpha^3 + \beta^3 = \gamma^3 + \delta^3 \tag{3.8}$$

where each of the variables are taken to be positive integers.

From this we have

$$\alpha^3 + \beta^3 - \gamma^3 - \delta^3 = 0. \tag{3.9}$$

Equation (3.9) is a homogeneous equation in four variables. We can dehomogenize (3.9) by dividing both sides of the equation by $\delta^3$, i.e.,

$$\frac{\alpha^3}{\delta^3} + \frac{\beta^3}{\delta^3} - \frac{\gamma^3}{\delta^3} - 1 = 0. \tag{3.10}$$

Setting $x = \frac{\alpha}{\delta}$, $y = \frac{\beta}{\delta}$, and $z = \frac{\gamma}{\delta}$, the equation in (3.10) becomes:

$$x^3 + y^3 - z^3 - 1 = 0. \tag{3.11}$$

Since $\alpha$, $\beta$, $\gamma$, and $\delta$ were assumed to be positive integers, it follows that $x$, $y$, and $z$ are positive rational numbers.

A quick glance at (3.11) shows us that $(t, 1, t)$ is a trivial solution. We know that on our surface there must be a a line containing these trivial solutions (Figure 1), and so we take a plane through this line. Slicing our surface in this way will give us a cubic curve, however, since this line must be a component of the cubic curve, our end result will be the union of the line and a conic. We can then utilize this conic to find more solutions to (3.11).

We start by considering a general plane in three-space

$$px + qy + rz = s. \tag{3.12}$$

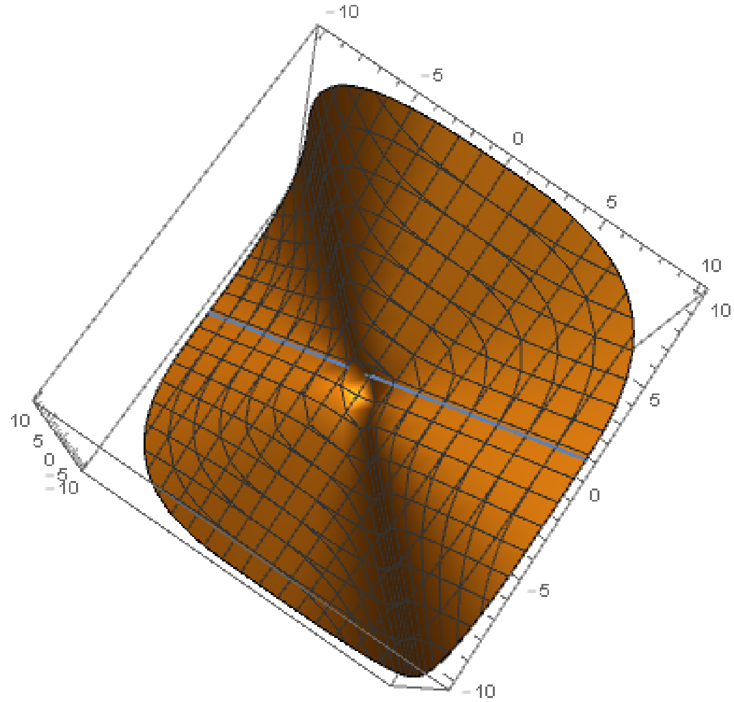


Figure 1: The Surface in (3.11) with Line $(t, 1, t)$

To determine what the the coefficients are in (3.12), we slice our cubic in (3.11) with the plane along the line that crosses through a line of trivial solutions on our cubic. Doing so gives

$$pt + q + rt = s. \tag{3.13}$$

Comparing coefficients across the equation in (3.13) shows

$$q = s.$$

Our plane above becomes

$$px + qy + rz = q,$$

and dividing both sides of the equation by $q$ gives

$$\frac{p}{q}x + y + \frac{r}{q}z = 1.$$

We now have our plane written in a form that easily gets us $y$.

$$y = -\frac{p}{q}x - \frac{r}{q}z + 1.$$

We also see from equation (3.13) that $p = -r$. Therefore, letting $a = \frac{p}{q}$ gives us

$$y = ax - az + 1. \tag{3.14}$$

From our work above, we now have a system of equations

$$\begin{cases} x^3 + y^3 - z^3 - 1 = 0, \\ y = ax - az + 1 \end{cases}$$

Replacing $y$ from the first equation with that found in the second equation gives

$$-1 + x^3 - z^3 + (1 + ax - az)^3 = 0.$$

Factoring the left hand side of the equals sign becomes

$$(x - z)(3a + 3a^2x + x^2 + a^3x^2 - 3a^2z + xz - 2a^3xz + z^2 + a^3z^2) = 0.$$

As expected with have the line $x - z = 0$ and a conic. We now consider the conic in the above equation. From our work earlier in this thesis, we know that we can create a parameterization for our conic, and thus create as many solutions as we want for (3.11).

So, again, let us consider the above conic

$$3a + 3a^2x + x^2 + a^3x^2 - 3a^2z + xz - 2a^3xz + z^2 + a^3z^2 = 0. \tag{3.15}$$

Setting $z = x$ in (3.15) we get $3a + 3x^2$. We want to determine which value for $a$ will make $3a + 3x^2 = 0$ since this will be the point where our line crosses the cubic. We see that $x = L$ and $a = -L^2$ gives us a solution.

Since $(L, L)$ is on our curve, we can pass a line though it using the standard point-slope equation of a line

$$z - L = m(x - L),$$

where $m$ is the slope of the line. Inserting $z = m(x - L) + L$ into the above and factoring, we have $-(L - x)(2L + 3L^4 + L^7 + 2Lm - 3L^4m - 2L^7m - Lm^2 + L^7m^2 + x - L^6x + mx + 2L^6mx + m^2x - L^6m^2x) = 0$.

The expression at the end of the previous paragraph looks messy, but it is linear in $x$, so solving for $x$ gives

$$x = \frac{2L + 3L^4 + L^7 + 2Lm - 3L^4m - 2L^7m - Lm^2 + L^7m^2}{-1 + L^6 - m - 2L^6m - m^2 + L^6m^2}.$$

Replacing $x$ in the linear equation from earlier and simplifying gives

$$z = \frac{-L + L^7 + 2Lm + 3L^4m - 2L^7m + 2Lm^2 - 3L^4m^2 + L^7m^2}{-1 + L^6 - m - 2L^6m - m^2 + L^6m^2}.$$

At this stage we have almost everything needed to find solutions for (3.11). All that is required is a solution for $y$. To uncover $y$, we insert the above values for $x$ and $z$ into the plane $y = ax - az + 1$ and simplify.

We have,

$$y = \frac{-1 - 3L^3 - 2L^2 - m + 4L^6m - m^2 + 3L^3m^2 - 2L^2m^2}{-1 + L^6 - m - 2L^6m - m^2 + L^6m^2}.$$

If we allow $x_0$, $y_0$, and $z_0$ to represent the solutions for $x$, $y$, and $z$ listed above, we see that

$$x_0^3 + y_0^3 - z_0^3 - 1 = 0$$

And thus, the triple $(x_0, y_0, z_0)$ is a solution to $x^3 + y^3 - z^3 - 1 = 0$.

The fact that $(x_0, y_0, z_0)$ is a solution to our cubic is great news, since we now have a means to find many numerical solutions to the cubic. Finding these solutions is possible because $x_0$, $y_0$, and $z_0$ are expressed in terms of $L$ and $m$, which can be viewed as parameters similar to how $t$ was a parameter in the section on conics.

We now find some numerical values for $x_0$, $y_0$, and $z_0$. To do this, we first fix $L$ and allow $m$ to vary. Our $L$ value can be almost anything, but for sake of example, we let it equal 2. Then our triple $(x_0, y_0, z_0)$ becomes

$$\left(\frac{180 - 300m + 126m^2}{63 - 129m + 63m^2}, \frac{-153 + 255m - 105m^2}{63 - 129m + 63m^2}, \frac{126 - 204m + 84m^2}{63 - 129m + 63m^2}\right). \tag{3.16}$$

With the above solution triple, we are close to having solutions to (3.11). We recall from earlier that $x$, $y$, and $z$ in (3.11) must be positive rational numbers. To prevent having too many subscripts, let $x_0$, $y_0$, and $z_0$ now take on the respective coordinates in (3.16). Since (3.11) is cubic, we know that $x_0$, $y_0$, and $z_0$ can all be positive or $z_0$ can be negative with $x_0$ and $y_0$ having opposite signs. To see this, consider

$$(-x_0)^3 + y_0^3 = (-z_0)^3 + 1.$$

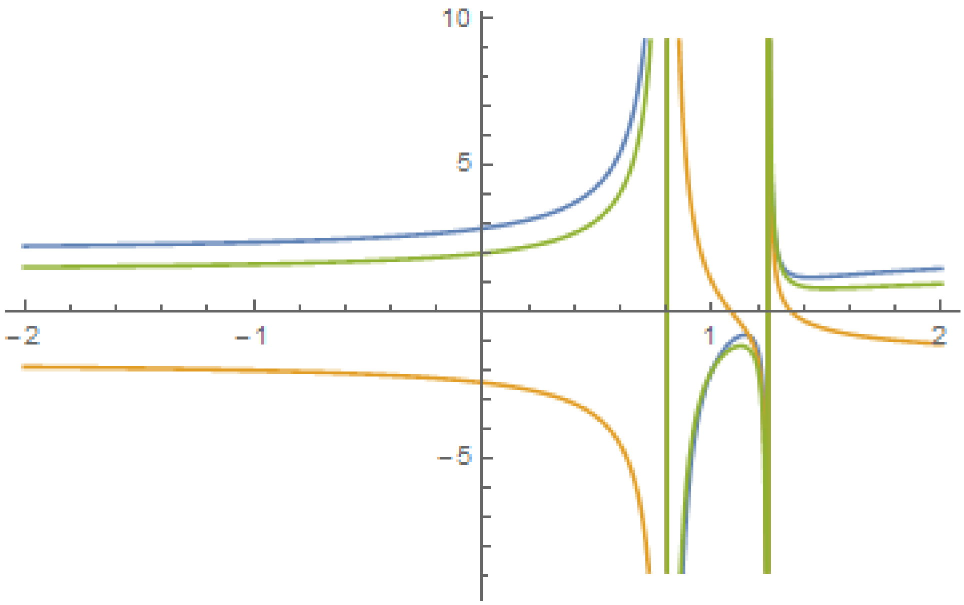


Figure 2: Graphs of $x_0$, $y_0$, and $z_0$

This equation can be re-written as

$$z_0^3 + y_0^3 = x_0^3 + 1$$

giving all positive solutions for (3.11).

With this in mind, we plot $x_0$, $y_0$, and $z_0$ as in Figure 2 above to help us see when the above situations occur. The blue curve in Figure 2 models the $x_0$ coordinate, the yellow curve models $y_0$, and the green curve is the $z_0$ coordinate. We want to determine values for $m$ that are rational and give us the sign values for $x_0$, $y_0$, and $z_0$ that we discussed above. From the graph, along with some equation solving, we see that the interval $(.804297, 1.08258)$ will be the location of $m$ that gives us $y_0$ as positive and the other coordinates as negative. The interval $(1.24332, 1.34599)$ is where $m$ gives us positive coordinate values. In the second interval above, we see that an $m = \frac{13}{10}$ is one rational value for $m$. This $m$ value gives us the solution triple $(\frac{98}{59}, \frac{35}{59}, \frac{92}{59})$. This implies

$$\left(\frac{98}{59}\right)^3 + \left(\frac{35}{59}\right)^3 = \left(\frac{92}{59}\right)^3 + 1.$$

We find the solutions for the Ramanujan equation by multiplying the above equation by its least common denominator, that is, $59^3$. Doing so gives us

$$98^3 + 35^3 = 92^3 + 59^3. \tag{3.17}$$

Calculating $98^3 + 35^3$ gives us 984067. This means that 984067 is a Ramanujan number, and it can be written as the sum of two positive integer cubes as seen in (3.17). In a similar way, an $m$ value of $\frac{9}{10}$ gives a solution triple $(-\frac{134}{23}, \frac{95}{23}, -\frac{116}{23})$ therefore,

$$116^3 + 95^3 = 134^3 + 23^3 \tag{3.18}$$

by our discussion on negative numbers and cubes from earlier. We have that $134^3 + 23^3 = 2418271$, giving us 2418271 as a Ramanujan number.

As a grand finale, if we let $m = \frac{5}{4}$, we get the solution triple

$$(10, 9, 12)$$

Which are the solutions from Ramanujan's story since, as we know, $10^3 + 9^3 = 12^3 + 1^3$, and of course, these cubes add to 1729.

Thus, we have presented a purely geometric way at finding $r$ values that correspond to those conditions found in Definition 3.4.

## 4 SUMS OF FOURTH POWERS IN TWO DIFFERENT WAYS

The story discussed in the beginning of the previous section is fairly well-known. However, it is not complete. According to Hardy, after Ramanujan saved 1729 from a mundane life by stating that it had some interesting properties to it, Hardy asked him if there was any positive integer that could be written as the sum of two fourth-powers, in two different ways. Sadly for Hardy, Ramanujan did not know the answer. In this section, we answer Hardy's question.

We begin in a similar way to the last chapter, but instead of dealing with a third-degree equation, we have the following fourth-degree equation:

$$\alpha^4 + \beta^4 = \gamma^4 + \delta^4. \tag{4.1}$$

As before, we can divide both sides of the equation by $\delta^4$ and make appropriate substitutions using $x$, $y$, and $z$ as new variables to turn (4.1) into

$$x^4 + y^4 = z^4 + 1.$$

Which implies

$$x^4 + y^4 - z^4 - 1 = 0. \tag{4.2}$$

As in the previous section, $(t, 1, t)$ is a line of trivial solutions on our surface (Figure 3). The difference now is, when we slice the surface above with a plane through the line, instead of getting a cubic we get a fourth degree equation. Since the line is a component of this resulting surface, we known that we will have the union of a cubic and a line. This cubic will be an elliptic curve, and so we can use the ideas of elliptic curves to find more solutions for (4.2).

We thus can build the following plane in a similar fashion as in the last sec-

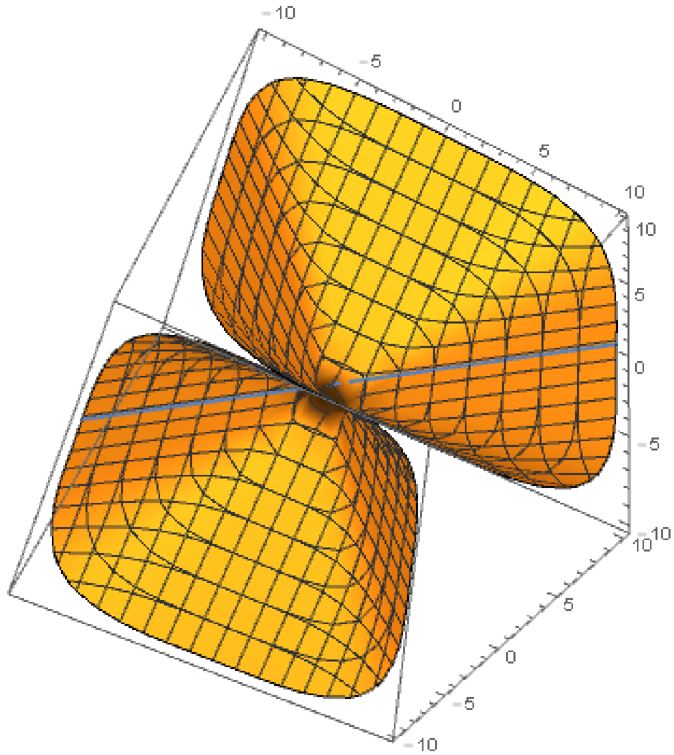


Figure 3: Graph of (4.2) with Line $(t, 1, t)$

tion:

$$y = ax - az + 1. \tag{4.3}$$

Inserting (4.3) into (4.2) and factoring gives $(x-z)(4a + 6a^2x + 4a^3x^2 + x^3 + a^4x^3 - 6a^2z - 8a^3xz + x^2z - 3a^4x^2z + 4a^3z^2 + xz^2 + 3a^4xz^2 + z^3 - a^4z^3 = 0$. We see that we have recovered our line $(x - z) = 0$ and the expected cubic. We consider only the cubic, i.e.,

$$\begin{aligned} 4a + 6a^2x + 4a^3x^2 + x^3 + a^4x^3 - 6a^2z - 8a^3xz + x^2z - \\ 3a^4x^2z + 4a^3z^2 + xz^2 + 3a^4xz^2 + z^3 - a^4z^3 = 0. \end{aligned} \tag{4.4}$$

We have that (4.4) is an elliptic curve. To see this, let $F(x, z)$ be the left hand side of (4.4). We take partial derivatives on $F$ and arrive at the following set of equations:

$$\frac{\partial F}{\partial x} = 6a^2 + 8a^3x + 3x^2 + 3a^4x^2 - 8a^3z + 2xz - 6a^4xz + z^2 + 3a^4z^2 = 0$$

and,

$$\frac{\partial F}{\partial z} = -6a^2 - 8a^3x + x^2 - 3a^4x^2 + 8a^3z + 2xz + 6a^4xz + 3z^2 - 3a^4z^2 = 0.$$

Attempting to find solutions for $x$ and $z$ that satisfy $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial z} = 0$, and $F(x, z) = 0$ gives no results, and so $F$ satisfies Definition 2.5.

**Finding Solutions**

In developing a means of finding solutions to (4.2), we tried two different approaches. Both approaches are similar in appearance, however, as will be shown later, only the second approach grants non-trivial solutions.

**First Method for Solving (4.2)**

The methods used in the previous section for finding solutions to $\alpha^3 + \beta^3 = \gamma^3 + \delta^3$ worked very well. Thus, we now attempt to employ similar techniques for (4.2).

From above, we saw that slicing $x^4 + y^4 - z^4 - 1 = 0$ with a general plane through the line $(t, 1, t)$ and constructing the plane $y = ax - az + 1$, we were able to produce the elliptic curve (4.4)

We now set $P = (t, t)$ and consider the new point $2P$, which we hope will give us a new, non-trivial point. To find $2P$ we must first determine the tangent line that passes through $P$.

Taking the implicit derivative of (4.4) we have

$$\frac{dz}{dx} = \frac{6a^2 + 3x^2 + 8a^3(x - z) + 3a^4(x - z)^2 + 2xz + z^2}{6a^2 - x^2 + 8a^3(x - z) + 3a^4(x - z) - 2xz - 3z^2}. \tag{4.5}$$

Inserting the point $P = (t, t)$ into (4.5) gives us

$$\frac{a^2 + t^2}{a^2 - t^2}.$$

Letting $m = \frac{a^2+t^2}{a^2-t^2}$ we form the line

$$z = m(x - t) + t.$$

Replacing $z$ in (4.4) with the line above gives us, after factoring and solving, the following three solutions for $x$

$$x = -a^{\frac{1}{3}},$$

$$x = -a^{\frac{1}{3}}$$

and,

$$x = -\frac{1}{a}. \tag{4.6}$$

The first two solutions above are obvious. The last solution is new and so we study it more. We see that plugging (4.6) into the line above, we find $z = \frac{1}{a}$. Therefore, through similar methods used in the first section, we see that $2P$ is $(-\frac{1}{a}, \frac{1}{a})$. Finally, inserting the $x$ and $z$ values from the respective coordinates of $2P$ into the plane $y = ax - az + 1$ gives $y = -1$.

While the triple $(-\frac{1}{a}, -1, \frac{1}{a})$ is a solution to our equation $x^4 + y^4 - z^4 - 1 = 0$, multiplying both sides of the equation by $a^4$ gives us

$$(-1, -a, 1, a)$$

which is a solution to (4.1), but this a trivial solution.

We have demonstrated that the methods in this section do not grant any new or interesting points for the equation (4.1), and thus we sought another method for finding solutions.

**The Second Method for Solving (4.2)**

Consider, again, the line $(t, 1, t)$. Since our surface is three-dimensional, as we have already seen, our line must run along our surface. In the previous method, we could pass infinitely many planes through the line $(t, 1, t)$ which then gave us an elliptic curve to use. Instead, we now consider a new trivial point for (4.2). There are many choices for trivial points, but for the sake of example, we use the point $T = (-1, t, t)$. We then pass a plane through the line $(t, 1, t)$ with the condition that it also contain $T$. This reduces the number of planes that we can work with from infinitely many planes to just one unique plane. By this relation, we see that our choice for a plane depends on the value for $a$ from the equation $y = ax - az + 1$, or $a$ depends on our value of $t$.

We consider the case when $t$ depends on $a$. We set $Q = (-1, t)$ and evaluate (4.4) at this point. From this, we find that $t = \frac{1-a}{1+a}$.

Our unique plane outlined above gives rise to a unique elliptic curve that we use to calculate what we hope will be a new point. We now find the tangent line to $Q$ and then consider the third point of intersection of this tangent line on our elliptic curve. Through similar methods as before, we find the slope of the tangent line at $Q$ is:

$$\frac{1+2a+6a^2-2a^2+a^4}{(-1+a)^2(1+a)}$$

Letting $m = \frac{1+2a+6a^2-2a^2+a^4}{(-1+a)(1+a)}$, we can construct the tangent line

$$z = m(x+1)+t$$

Replacing $z$ in (4.4) with the line above gives us, after factoring and solving, the following three solutions for $x$

$$x = -1$$

$$x = -1$$

and,

$$x = \frac{-1+9a+8a^2-6a^3+23a^4-3a^5+2a^6}{-2+3a-23a^2+6a^3-8a^4-9a^5+a^6} \tag{4.7}$$

From $Q$, we already know $x = -1$, but the third solution is new. Letting $x$ be as in (4.7) and inserting this $x$ into the line above shows us that

$$z = \frac{(-1+a)(-1-9a+8a^2+6a^2+23a^4+3a^5+2a^6}{(1+a)(-2+3a-23a^2+6a^3-8a^4-9a^5+a^6}$$

Setting $x$ and $z$ from above into our original plane $y = ax - az + 1$, we find that $y$ must be

$$y = \frac{2+a+20a^2-17a^3+2a^4-17a^5+8a^6+a^7}{2-a+20a^2+17a^3+2a^4+17a^5+8a^6-a^7}$$

Indeed, letting $(x_0, y_0, z_0)$ take on the above values for $x$, $y$, and $z$ we see that that $x_0^4 + y_0^4 - z_0^4 - 1 = 0$ as desired.

We now have all the information necessary to arrive at various, new solutions to $x^4 + y^4 - z^4 - 1 = 0$.

For example, if we let $a = -\frac{1}{2}$, we have the triple $(\frac{76}{653}), \frac{1203}{653}, \frac{1176}{653})$ and so

$$(-67)^4 + 133^4 = (-59)^4 + 158^4$$

REMARK 4.1: We note that, unlike in the case of (3.11), we do not have to concern ourselves with the sign values of our solutions since the degree of each of the terms in our equation is 4.

Continuing the above process, we see that a value of $a = \frac{1}{3}$ gives $(\frac{-67}{79}, \frac{133}{158}, \frac{-59}{158})$ and so

$$76^4 + 1203^4 = 1176^4 + 653^4$$

Finally, a value of $a = 2$ gives

$$(-1203)^4 + 76^4 = (-653)^4 + 1176^4$$

Thus, one can pick various $a$ values and find new solutions for $\alpha^4 + \beta^4 = \gamma^4 + \delta^4$ as we have done above.

## 5 THE TETRAHEDRON PROBLEM

There is a classic problem that asks for the side lengths of an equilateral triangle if the distance from the vertices of the triangle are 3, 4, and 5. The answer turns out to be $\sqrt{25 + 12\sqrt{12}}$. This raises the question: does there exist an equilateral triangle with a side of integer length having a point inside it that is integer distance from each of the triangles vertices? Christina Bisges, in her master's thesis [1], discussed this problem and it's solution.

We investigate an analogous problem in three dimensions by considering a tetrahedron. We wish to know if there exists a tetrahedron of integer side length with a point inside that is integer distance from each of the vertices of the tetrahedron.

By a tetrahedron in the previous paragraph, we mean a solid existing in three space that is formed by taking four vertices and taking their convex hull. The type of tetrahedron we will be considering is formulated in the following definition:

DEFINITION 5.1: We say a tetrahedron is regular if its four triangular faces are composed of congruent triangles. For brevity, we will refer to regular tetrahedrons as simply tetrahedrons for the remainder of this thesis.

### Main Problem

To create some usable structure to help solve our problem, we place our tetrahedron inside a cube. To do this, we consider a cube whose side lengths are all equal to $u$. We then place the tetrahedron inside the cube by aligning the vertices of the tetrahedron with that of the cube, as in Figure 4 below. However, if $s$ is the length of the sides of our tetrahedron, then $s = \sqrt{2}u$ implying that $u = \frac{s}{\sqrt{2}}$.

We see that we can apply the distance formula to the line segments in Fig-

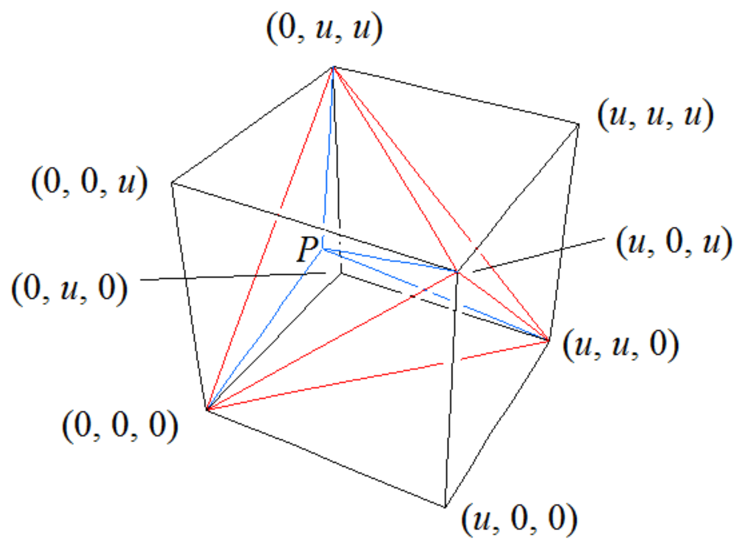


Figure 4: Tetrahedron Inside a Cube

ure 4 giving:

$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x^2 + (y - \frac{s}{\sqrt{2}})^2 + (z - \frac{s}{\sqrt{2}})^2 = b^2 \\ (x - \frac{s}{\sqrt{2}})^2 + y^2 + (z - \frac{s}{\sqrt{2}})^2 = c^2 \\ (x - \frac{s}{\sqrt{2}})^2 + (y - \frac{s}{\sqrt{2}})^2 + z^2 = d^2 \end{cases} \tag{5.1}$$

Our objective is to eliminate $x$, $y$, and $z$ to obtain an expression involving only $a$, $b$, $c$, $d$, and $s$. To start, we solve for $y$ by subtracting the first equation from the last. Doing so gives us

$$-2\frac{s}{\sqrt{2}}y = d^2 - a^2 - s^2 + 2\frac{s}{\sqrt{2}}z. \tag{5.2}$$

If we subtract the third equation from the second, we get

$$-2\frac{s}{\sqrt{2}}y + 2\frac{s}{\sqrt{2}}z = b^2 - c^2.$$

This shows us that

$$z = \frac{\sqrt{2}}{2s}(b^2 - c^2 + 2\frac{s}{\sqrt{2}}y). \tag{5.3}$$

If we insert (5.3) into (5.2), we can find a solution for $y$. We have

$$-2\frac{s}{\sqrt{2}}y = d^2 - a^2 - s^2 + b^2 - c^2 + 2\frac{s}{\sqrt{2}}y,$$

which implies,

$$y = \frac{\sqrt{2}}{-4s}(-a^2 - s^2 + b^2 - c^2 + d^2). \tag{5.4}$$

Inserting the equation for $y$ in (5.4) into (5.2) and solving for $z$ gives us

$$z = \frac{\sqrt{2}}{4s}(a^2 + b^2 - c^2 - d^2 + s^2)$$

To find $x$, we consider the equation created by $d^2 - c^2$. Taking the equation found from $d^2 - c^2$, inserting the above values for $y$ and $z$ and then solving for $x$ gives

$$x = \frac{\sqrt{2}}{4s}(a^2 - b^2 - c^2 + d^2 + s^2)$$

From our work above, we now have $x$, $y$, and $z$ expressed in terms of the side lengths as desired. If we let $x_0$, $y_0$, and $z_0$, we see that the equation $x_0^2 + y_0^2 + z_0^2 - a^2 = 0$. If we simplify this equation (keeping in mind what $x_0$, $y_0$, and $z_0$ are equivalent to) gives

$$\begin{aligned} &3(a^4 + b^4 + c^4 + d^4 + s^4) \\ &= 2(c^2d^2 + c^2s^2 + d^2s^2 + b^2c^2 + b^2d^2 + b^2s^2 + a^2b^2 + a^2c^2 + a^2d^2 + a^2s^2) \end{aligned} \tag{5.5}$$

Much like in the previous section when we dealt with the Taxi Cab problem, our method for finding solutions to the above equation involves de-homegenizng it. We de-homogenized by dividing the equation by $s^4$. Doing so yields:

$$\begin{aligned} & 3\left( (\frac{a^4}{s^4}) + (\frac{b^4}{s^4}) + (\frac{c^4}{s^4}) + (\frac{d^4}{s^4}) + 1 \right) \\ = 2 & \left( \frac{c^2d^2}{s^4} + \frac{c^2s^2}{s^4} + \frac{d^2s^2}{s^4} + \frac{b^2c^2}{s^4} + \frac{b^2d^2}{s^4} + \frac{b^2s^2}{s^4} + \frac{a^2b^2}{s^4} + \frac{a^2c^2}{s^4} + \frac{a^2d^2}{s^4} + \frac{a^2s^2}{s^4} \right) \end{aligned} \tag{5.6}$$

If we set $X = \frac{a}{s}$, $Y = \frac{b}{s}$, $Z = \frac{c}{s}$, and $W = \frac{d}{s}$, we see that this change of variables turns the above equation into

$$\begin{aligned} & 3(X^4 + Y^4 + Z^4 + W^4 + 1) \\ = 2 & (Z^2W^2 + Z^2 + W^2 + Y^2Z^2 + Y^2W^2 + Y^2 + X^2Y^2 + X^2Z^2 + X^2W^2 + X^2) \end{aligned} \tag{5.7}$$

We will be making significant use of the above equation, and so for the rest of this section, instead of using the capital letters found above, we rename our variables and reuse lower case $x$, $y$, $z$, and $w$ for equation (5.8), i.e.,

$$\begin{aligned} & 3\left(w^4 + x^4 + y^4 + z^4 + 1\right) \\ = 2 & \left(w^2x^2 + w^2y^2 + w^2z^2 + w^2 + x^2y^2 + x^2z^2 + x^2 + y^2z^2 + y^2 + z^2\right) \end{aligned} \tag{5.8}$$

We now discuss three different methods for finding integer solutions for (5.8), but before we can do so, we must talk about the existence of some singular points.

### The Existence of Singular Points

Our equation in (5.8) is a four dimensional object and so it is impossible to graph as is, however, there is hope of visualizing this hyper-surface. If we slice the hyper-surface created from equation (5.5) with a hyper-plane, we can gain a glimpse at its appearance. For example, if we slice our equation with $x+y+z+w = 3$ we get Figure 5.

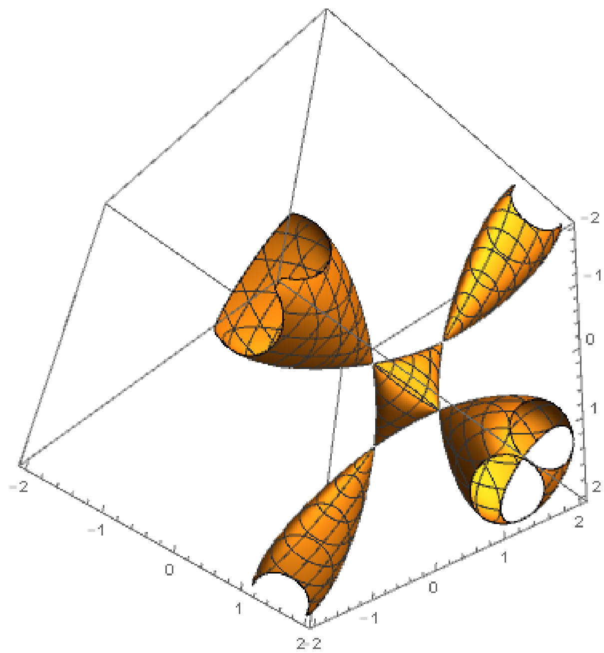

Figure 5: Slice Using $w = 3 - x - y - z$

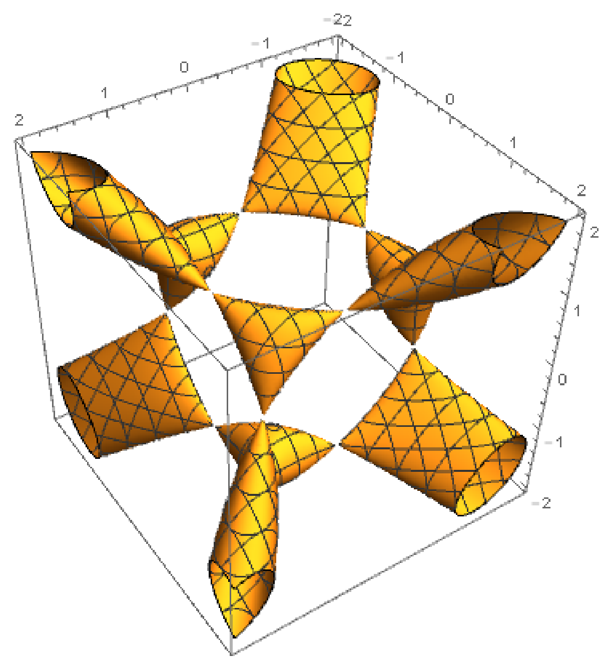

Figure 6: Slice Using $w = 1 - x - y - z$

Or, if we slice with $x + y + z + w = 1$, we get the above Figure 6.

In both Figure 5 and 6 we see that the graph is not completely connected together, but rather has several areas where the surface appears to pinch together. The fact that our graphs are not nice and smoothly connected illustrates the possible existence of singular points for (5.5).

With the thought of singular points in mind, we let $F(x, y, z, w) = 3(x^4 + y^4 + z^4 + w^4) - 2(x^2y^2 + x^2z^2 + x^2w^2 + y^2z^2 + y^2w^2 + z^2w^2 + x^2 + y^2 + z^2 + w^2)$ and consider partial derivatives on $F$. We see,

$$\frac{\partial F}{\partial x} = 12x^3 - 2(2xy^2 + 2xz^2 + 2xw^2 + 2x)$$

$$\frac{\partial F}{\partial y} = 12y^3 - 2(2yx^2 + 2yz^2 + 2yw^2 + 2y)$$

$$\frac{\partial F}{\partial z} = 12z^3 - 2(2zx^2 + 2zy^2 + 2zw^2 + 2z)$$

$$\frac{\partial F}{\partial w} = 12w^3 - 2(2wx^2 + 2wy^2 + 2wz^2 + 2w)$$

If we set each of the above equations equal to 0 and solve for $x$, $y$, $z$, and $w$, we obtain a whole collection singular point solutions. If we run through this process we, would see that all the singular points for $F$ fit the following form: $(\pm 1, \pm 1, \pm 1, 0)$ where the 0 permutes through the four coordinates. So, for example, $(-1, -1, 1, 0)$ is a singular point, as well as, $(0, -1, -1, 1)$. From this we see that there are $(4)(2)(2)(2) = 32$ singular points for $F$.

REMARK 5.2: The above discussion does not not technically give all the singular points for $F$. There exists some at the point of infinity, but since we will not need to make use of them, we do do not discuss such points further.

**First Solving Method: Using Given Solutions Along with Singular Points**

The strategy for our first method is to consider what we shall refer to as boundary points satisfying equation (5.8), and use them along with some properly

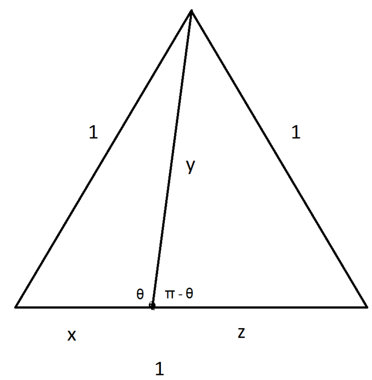


Figure 7: A Triangular Face

chosen singular points to find additional points.

**Boundary Points**

We shall refer to points found along the edges of our tetrahedron as boundary points. These boundary points are nice in the sense that it is fairly easy to find such points satisfying (5.8).

One property of the boundary points that makes them so helpful is the fact that we can find parameterizations for them. To see this, consider the situation presented in Figure 7. Since, for equation (5.5), we de-homogenized by removing $s$, which is equivalent to $s = 1$, we can rewrite $z$ as $1 - x$, further simplifying our problem.

From Figure 7, we see that finding solutions for $(x, y, z)$ reduces to a law of cosines problem. From above, we see that we have two triangles to consider, and using the law of cosines on both of them, and then combining them together, will grant us values for $y$ and $x$ (and thus $z$). For the first triangle above, we have

$$1 = x^2 + y^2 - 2xy\cos\theta. \tag{5.9}$$

For the second triangle, the law of cosines gives

$$1 = (1-x)^2 + y^2 - 2y(1-x)\cos(\pi - \theta). \tag{5.10}$$

However, since $\cos(\pi - \theta) = -\cos\theta$, (5.10) becomes

$$1 = (1-x)^2 + y^2 + 2y(1-x)\cos\theta. \tag{5.11}$$

From (5.9),

$$\cos\theta = \frac{x^2 + y^2 - 1}{2xy}. \tag{5.12}$$

Inserting (5.12) into (5.11), and simplifying, we get

$$0 = -2x + x^2 + y^2 + 2y(1-x)(\frac{x^2 + y^2 - 1}{2xy}).$$

Expanding the above and carrying the $x$ in the denominator of $\frac{x^2+y^2-1}{2xy}$ throughout, we have

$$0 = x^2 - x - y^2 + 1. \tag{5.13}$$

The equation in (5.13) is a hyperbola, but more importantly, it's a conic and so we can create parameterized solutions for it. We mimic the process found in the section on conics to find the parametrizations we require. Letting $x = 0$ in (5.13) gives us a trivial solution of $T = (0, 1)$. We can use $T$ to construct a line that passes through the hyperbola and thus giving us more solutions. Such a line that passes through the conic is $y = tx + 1$, and inserting this line in for $y$ in (5.13) and solving for $x$ gives us

$$x = \frac{2t+1}{1-t^2},\ t^2 \neq 1. \tag{5.14}$$

We can place this above $x$ value into the line $y = tx + 1$ from earlier to get us

$$y = \frac{t^2+t+1}{1-t^2},\ t^2 \neq 1. \tag{5.15}$$

Since $z = 1 - x$, we have the triple $(\frac{2t+1}{1-t^2}, \frac{t^2+t+1}{1-t^2}, \frac{-t^2-2t}{1-t^2})$ will give us any point on the boundary of our triangle; we simply need to let $t$ vary to uncover these new boundary points.

The triple that we just found does more than satisfy points on the boundary, it also gives us a means to find solutions to (5.8). Since each of the triangular faces on the tetrahedron have the same dimensions, we can determine $w$ in (5.8) by noting that, in the case of a boundary point, $w$ must equal $y$.

Therefore, we set $w = \frac{t^2+t+1}{1-t^2}$, $t^2 \neq 1$ and thus, now have a full parameterized family of degenerate solutions.

**Using Degenerate Solutions with Singular Points**

We are now prepared to find non-trivial solutions for (5.8). To do so, consider a singular point. Selecting a $t$ value for the parameterizations in the previous section generates another point, and this point must be rational. We can use these points to help us find other solutions. One way to determine new points is to construct the line through one of singular points on (5.8) and a boundary point and then see where else our line intersects the hyper-surface defined by our equation.

Since the degree of a line is 1 and (5.8) has degree of 4, by the Fundamental Theorem of Algebra, we know that a line should intersect $1 \times 4 = 4$ times.

It can be shown that singular points have a multiplicity of 2, and that any arbitrary non-singular point will have multiplicity 1, therefore a line passing through a (5.8) must intersect four times, counting multiplicities. Thus far, the line we have been working with intersects three times (counting multiplicities); our line therefore must cross (5.8) one more time, and this additional point of intersection must be rational.

For demonstration purposes, we use for our first singular point, the point $(-1, 1, -1, 0)$. For our boundary point, we select $t = -\frac{1}{3}$ for the parameterizations in (5.14) and (5.15). This gives the boundary point $(\frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{7}{8})$ and the parametric equation of the line through these points is given by:

$$x = -1 + \frac{5}{8}p$$

$$y = 1 - \frac{3}{8}p$$

$$z = -1 + \frac{15}{8}p$$

$$w = \frac{7}{8}p$$

Inserting these values for $x$, $y$, $z$, and $w$ above into (5.8) and factoring gives

$$\frac{1}{16}(-1 + p)p^2(-409 + 469p) = 0$$

Values of $p = 1$ and $p = 0$ are trivial, however, $p = \frac{409}{469}$ is not trivial and so we make use of it. If we place $p = \frac{409}{469}$ into the above equations we have $x = -\frac{1707}{3752}$, $y = \frac{2525}{3752}$, $z = \frac{2383}{3752}$, and $w = \frac{409}{536}$. At this point, we have accomplished our goal for finding solutions to (5.8), since placing our new-found solutions into said equation grants a value of 0. However, we decided not to stop. There is nothing inherently special about the boundary points from earlier, save that they are easy to find. We can use other known points that satisfy (5.8), along with new singular points to find new solutions.

Let us take as our second singular point, the point $(1, -1, 1, 0)$ (we note that we must use a different singular point each time, otherwise will get back the original point we started with). Following the method above, we obtain another set of parameterized solutions. They are:

$$x = -1 - \frac{5459}{3752}m$$

$$y = -1 + \frac{6277}{3752}m$$

$$z = 1 - \frac{1369}{3752}m$$

$$w = \frac{409}{536}m$$

Passing these values above into (5.8) and factoring gives us

$$\frac{57(-1+m)m^2(-791274757 + 549514857m)}{1650587344} = 0$$

Again, we don't really care about $m = 1$ and $m = 0$. However, $m = \frac{791274757}{549514857}$ is non-trivial and so we use it to generate other non-trivial solutions. Placing $m = \frac{791274757}{549514857}$ into the equations above gives us the solution tuple:

$$\left(-\frac{4814049371}{4396118856}, \frac{6194140525}{4396118856}, \frac{2086406399}{4396118856}, \frac{4830319039}{4396118856}\right) \tag{5.16}$$

We can repeat the above process multiple times over. However, doing so quickly generates rather large solutions. For example, using the solution tuple above, in conjunction with the singular point $(0, 1, -1, 1)$, gives us

$$x = -\frac{5136282575359797718437668 6449}{23499071295923767674102523224}$$

$$y = \frac{42682811754085433189783602535}{23499071295923767674102523224}$$

$$z = \frac{4566532430035548249452555 8621}{234990712959237676741025 23224}$$

and

$$w = \frac{281317093565045042072023 99901}{234990712959237676741025 23224}$$

The numerator and denominators of those fractions contain some huge numbers!

We have thus answered our question. We can uncover $a$, $b$, $c$, $d$, and $s$ from the fractions above by clearing denominators. For example, for the solution

$$(-\frac{1707}{3752}, \frac{2525}{3752}, \frac{2383}{3752}, \frac{409}{536}),$$

found towards the beginning of this current section, we see that upon clearly denominators, $a = 1707$, $b = 2525$, $c = 2383$, $d = 2863$, and $s = 3752$. We can preform a similar process for the other solutions found above.

**Second Solving Method: Singular Fourth Degree Curves Intersected with Conics**

In the previous section, we passed a line through one singular point and a rational point, and then used this line to find more solutions for (5.8). Now, we use two singular points to find solutions. We slice two hyperplanes through the surface given by (5.8) in such a way that the hyperplanes pass through both singular points and a rational point on the surface. This results in a quartic curve with two singular points and an additional rational point. We wish to intersect this quartic with a conic in a very special way. This conic must be a member the following set:

$$\text{Conics} = \{ax^2 + bxy + cy^2 + dx + ey + f = 0, \ (a, b, c, d, e, f) \in \mathbb{Q}\} \tag{5.17}$$

Since we are taking our conic to be over $\mathbb{Q}$ we can divide by a constant and rewrite the above set as

$$\text{Conics} = \{Ax^2 + Bxy + Cy^2 + Dx + Ey - 1 = 0,\ (A, B, C, D, E) \in \mathbb{Q}\} \quad (5.18)$$

Through dimension counting, any constraint imposed on a conic will lower its dimension by 1. Thus for the conic that passes through the two singular points and the rational point, we see that since 3 constraints have been applied, the conic has dimension 2.

To lower this number even further, we consider both the tangent line and the second derivative at the rational point, and ask that they agree for the quartic and the conic, providing two more restrictions to the conic. These additional conditions lowers the dimension to 0 implying a unique conic.

The fact that our conic is unique is powerful since, passing it through two singular points and a rational point, as well as considering both the tangent line and second derivative at the rational point means that our conic intersects the surface given by (5.8) 7 times. However, the following theorem tells us more about the number of times our conic should intersect (5.8).

THEOREM 5.3: (Bezout's Theorem) [5] Let $C_1$ and $C_2$ be projective curves with no common components. Then,

$$\sum_{P \in C_1 \cap C_2} I(P,\, C_1 \cap C_2) = (\deg C_1)(\deg C_2)$$

where $I(C_1 \cap C_2,\, P)$ is the intersection multiplicity of $P$ defined in the first section.

Thus, we see that the conic must pass through the surface at an additional point, and as it turns out, this point will be rational.

In order to establish the the two hyperplans necessary for the scheme outlined above, we first consider the equation for a general hyperplane in four-space. We have:

$$ax + by + cz + dw - 1 = 0 \tag{5.19}$$

We now determine what conditions must be satisfied in order for two hyperplanes to pass through $F$ at two given singular points, and a known rational, non-singular point. To do this, we evaluate 5.19 using said points. We take, for the sake of example, the singular points $(1, -1, 1, 0)$ and $(1, 1, 0, 1)$ and for our rational point, $R = (\frac{7}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8})$.

Evaluating (5.19) with our selected points gives us the following system of equations:

$$\begin{cases} a - b + c = 1 \\ a + b + d = 1 \\ 7a + 5b + 3c + 7d = 8 \end{cases} \tag{5.20}$$

Solving the above system for $d$ gives us: $a = \frac{3-d}{4}$, $b = \frac{1-3d}{4}$, and $c = \frac{1-d}{2}$. These values allow us to write the coefficents of (5.20) in terms of $d$, and so our plane becomes:

$$\frac{3-d}{4}x + \frac{1-3d}{4}y + \frac{1-d}{2}z + dw - 1 = 0. \tag{5.21}$$

If we let $d = 1$, we can express $w$ in terms of $x$ and $y$. Likewise, with $d = 0$, we can express $z$ in terms of $x$ and $y$. Letting $d$ take on these respective values gives us the following equations:

$$w = \frac{2 - x + y}{2} \tag{5.22}$$

and,

$$z = \frac{4 - 3x - y}{2}. \tag{5.23}$$

Placing (5.22) and (5.23) into the equation $F(x, y, z, x) = 0$, gives us the curve

$$\begin{aligned} &3(x^4 + (-\frac{x}{2} + \frac{y}{2} + 1)^4 + (-\frac{3x}{2} - \frac{y}{2} + 2)^4 + y^4 + 1) - \\ &\quad 2((x^2((-\frac{x}{2} + \frac{y}{2} + 1)^2 + (-\frac{3x}{2} - \frac{y}{2} + 2)^2 + y^2) + x^2 + \\ &\quad y^2((-\frac{x}{2} + \frac{y}{2} + 1)^2 + (-\frac{3x}{2} - \frac{y}{2} + 2)^2) + (-\frac{3x}{2} \\ &\quad - \frac{y}{2} + 2)^2(-\frac{x}{2} + \frac{y}{2} + 1)^2 + (-\frac{x}{2} + \frac{y}{2} + 1)^2 + (-\frac{3x}{2} - \frac{y}{2} + 2))^2 + y^2). \end{aligned} \tag{5.24}$$

We now consider the following conic,

$$px^2 + qxy + ry^2 + sx + ty = 1. \tag{5.25}$$

In order to find when this conic passes through the same points that our hyperplanes do, we must evaluate it at the same singular points and rational point from above. The following system of equations result

$$\begin{cases} p+q+r+s+t=1 \\ p-q+r+s-t=1 \\ \frac{49p}{64}+\frac{35q}{64}+\frac{25r}{64}+\frac{7s}{8}+\frac{5t}{8}=1 \end{cases} \tag{5.26}$$

As mentioned earlier, we want to consider the tangent line at our point $R$. However, since $R$ simultaneously exists on both our conic and $F$, we are forced to make our tangent line in conjunction with both equations.

For $F$, using implicit differentiation $\frac{dy}{dx}$ and then evaluating the resulting derivative at the point $(\frac{7}{8}, \frac{5}{8})$ gives us a value of 3.

Repeating the same process above, only now on the conic grants us, $\frac{-\frac{7p}{4}-\frac{5q}{8}-s}{\frac{7q}{8}+\frac{5r}{4}+t}$. We need this rational expression to equal the slope above, and so we subtract it by 3, giving

$$\frac{-\frac{7p}{4}-\frac{5q}{8}-s}{\frac{7q}{8}+\frac{5r}{4}+t}-3. \tag{5.27}$$

The above becomes:

$$-\frac{2(7p+13q+15r+4s+12t)}{7q+10r+8t}. \tag{5.28}$$

Subtracting 3 in (5.27) forces (5.28) to be 0. Therefore, the rational expression in (5.28) equals 0, which implies that it's numerator is 0. We thus have the equation

$$-2(7p+13q+15r+4s+12t)=0. \tag{5.29}$$

We now find the second derivative at $(\frac{7}{8}, \frac{5}{8})$, which can be found using the usual formula.

We have, noting that the value for $\frac{dy}{dx}$ must be 3 from our work above, that:

$$8 - \frac{15y^{''}}{8} = 0$$

Therefore, solving for $y^{''}$ gives

$$y^{''} = \frac{64}{15} \tag{5.30}$$

Applying the same process to our conic, we have

$$2p + 3q\left(\frac{7q}{8} + \frac{5r}{4} + t\right)y^{''} + 3(q + 6r) = 0$$

From (5.30), we know the value for $y^{''}$. Substituting for $y^{''}$ gives

$$2p + \frac{64}{15}\left(\frac{7q}{8} + \frac{5r}{4} + t\right) + 3(q + 6r) + 3q = 0 \tag{5.31}$$

We now have everything needed to find solutions for $F$.

We arrange all of the above formulas into a system of equations as follows:

$$\begin{cases} p + q + r + s + t = 1 \\ p - q + r + s - t = 1 \\ \frac{49p}{64} + \frac{35q}{64} + \frac{25r}{64} + \frac{7s}{8} + \frac{5t}{8} = 1 \\ -2(7p + 13q + 15r + 4s + 12t) = 0 \\ 2p + \frac{64}{15}\left(\frac{7q}{8} + \frac{5r}{4} + t\right) + 3(q + 6r) + 3q = 0 \end{cases} \tag{5.32}$$

The system of equations in (5.32) contains five equations, and five unknowns.

This means that we can solve for the respective unknowns; doing so, and then applying found values for $p$, $q$, $r$, $s$, and $t$, we get the following, unique conic

$$-\frac{13x^2}{8}+\frac{5xy}{12}+\frac{31x}{12}+\frac{y^2}{24}-\frac{5y}{12}-1=0 \tag{5.33}$$

We can now solve for $y$ by eliminating $x$ between the conic in (5.33) and $F$. Doing so gives: $356352y^8-1024000y^7+372096y^6+1544000y^5-1726373y^4-16000y^3+911050y^2-504000y+86875=0$. Factoring gives us

$$(y-1)^2(y+1)^2(8y-5)^3(696y-695)=0$$

We see that we are successful! We have generated a new, non-trivial point for $F$ that is rational. Plugging $y=\frac{695}{696}$ into the conic in (5.33), gives

$$-\frac{13x^2}{8}+\frac{25051x}{8352}-\frac{15980159}{11625984}=0 \tag{5.34}$$

We find that $x=\frac{589}{696}$ and $x=\frac{2087}{2088}$. However, if we plug these $x$ values, along with our solution for $y$ above, into $F$, we see that only $x=\frac{589}{696}$ will work for us. Placing $x=\frac{589}{696}$ and $y=\frac{695}{696}$ into (5.22) and (5.23) from earlier gives us $w=\frac{749}{696}$ and $z=\frac{161}{696}$. Placing these new found values for $x$, $y$, $z$, and $w$ into $F$ gives us 0, proving that we have found solutions for our equation.

Thus, we have again answered our question posed in the beginning of this section, this time we will have an tetrahedron of integer side length $s=696$ provided $a=589$, $b=695$, $c=161$, and $d=749$.

**Third Solving Method: Tangent Cone**

For each one of the singular points of (5.8), we can form a tangent cone from that singular point. We can then use tangent lines to the singular point to find more solutions.

We choose the singular point $S = (1, 1, 1, 0)$, and consider what happens arbitrarily close to $S$. In other words, we let $x = 1 + r$, $y = 1 + s$, $z = 1 + t$, and $w = u$, where $r$, $s$, $t$, and $u$ are taken to be as close as we like to their respective coordinates of $T$ (without being equal to them). Doing this gives:

$$\begin{aligned} &3r^4 + 12r^3 - 2r^2s^2 - 4r^2s - 2r^2t^2 - 4r^2t - 2r^2u^2 + \\ &\quad 12r^2 - 4rs^2 - 8rs - 4rt^2 - 8rt - 4ru^2 + 3s^4 + 12s^3 - 2s^2t^2 - 4s^2t - 2s^2u^2 + \\ &12s^2 - 4st^2 - 8st - 4su^2 + 3t^4 + 12t^3 - 2t^2u^2 + 12t^2 - 4tu^2 + 3u^4 - 8u^2 = 0. \end{aligned} \tag{5.35}$$

To find the tangent cone, we need only consider the quadratic terms in (5.35). Collecting the quadratic terms above and simplifying gives

$$3r^2 - 2rs - 2rt + 3s^2 - 2st + 3t^2 - 2u^2 = 0. \tag{5.36}$$

Our singular point exists at the bottom of our cone, and so there exists infinitely many tangent lines to that point. In addition, each tangent line intersects with multiplicity 3, giving us another point of intersection. Since a tangent cone is a quadratic surface, it behaves in a similar way to a quadratic curve, i.e., a conic. What this means is that once we have a line lying our the tangent cone, we can obtain infinitely many other lines through similar techniques as preformed with conics in earlier sections using paremeterization arguments. With this in mind, we can form a tangent line to the singular point with the following parameterization:

$x = 1 + s\alpha$, $y = 1 + r\alpha$, $z = 1 + t\alpha$, $w = u\alpha$.

Now, from inspection of (5.36), we see that $(1, 1, 2, 2)$ is a solution. We have the above parameterization become: $x = 1 + \alpha$, $y = 1 + \alpha$, $z = 1 + 2\alpha$, and $w = 2\alpha$. Inserting these parameterizations into (5.5) gives us

$$3\left(1296\alpha^4 + (2\alpha+1)^4 + 2(5\alpha+1)^4 + 1\right) - \\ 2(36\alpha^2 + (5\alpha+1)^2\left(36\alpha^2+1\right) + (5\alpha+1)^2\left(36\alpha^2 + (5\alpha+1)^2 + 1\right) \\ + (2\alpha+1)^2\left(36\alpha^2 + 2(5\alpha+1)^2 + 1\right))$$

Factoring the above gives

$$36\alpha^4 = 0 \tag{5.37}$$

But, (5.37) implies $\alpha = 0$, which is trivial. We repeat the same process above and find $(2, 5, 5, 6)$ is another solution for (5.36). This point gives us $x = 1 + 2\alpha$, $y = 1 + 5\alpha$, $z = 1 + 5\alpha$, and $w = 6\alpha$. Inserting these values into (5.5) gives, after factoring,

$$12\alpha^3(179\alpha - 16) = 0. \tag{5.38}$$

Thus, we see that $\alpha = \frac{16}{179}$. Inserting this value of $\alpha$ into our above parameterizations gives us the following solution

$$\left(\frac{211}{179}, \frac{259}{179}, \frac{259}{179}, \frac{96}{179}\right) \tag{5.39}$$

Indeed, inserting the above values for $x$, $y$, $z$, and $w$ into (5.5) gives us 0 as desired. We have that a tetrahedron must have side lengths of $s = 179$ to have a point inside it of integer distance away from the vertices of the tetrahedron, with said integer distance being given by $a = 211$, $b = 259$, $c = 259$, and $d = 96$.

## 6 CONCLUSION

We have developed geometric methods for solving several major Diophantine equations. These include the famous Taxi Cab Problem of the mathematician Ramanujan and an analogous problem to it involving fourth powers. As we saw, we could use parameterized solutions to a particular conic to give us solutions to the Taxi Cab problem, while properties of elliptic curves gave us solutions for the fourth degree problem. In addition to these, we considered a Diophantine equation that arose from a problem involving a tetrahedron and gave three solving methods for finding solutions. Our Diophantine equation turned out to be homogeneous, but to give us a way at finding solutions for it, we de-homogenized it. We then saw that the first method for solving relied on lines passing through certain points on the de-homogenized equation, while the second dealt with a particular conic and was required to simultaneous meet certain conditions that affected both it, and the equation. The last method considered a tangent cone that formed from one of the singular points to the de-homogenized equation, and used tangent lines to the singular point to find solutions. In essence, we saw that for each of our problems, once one solution was known, we could then find many more solutions (in fact, for most, infinitely many more solutions), thus showing how powerful the power of geometry truly can be.

## REFERENCES


[1] Christina Bisges, Geometric Techniques for Solving a Certain Diophantine Equation, Missouri State University, Springfield MO, 2012.

[2] Robert Bix, Conics and Cubics: A Concrete Introduction to Algebraic Curves (Undergraduate Texts in Mathematics), Springer-Verlag New York, 1998.

[3] William Fulton, Algebraic Curves: An Introduction to Algebraic Geometry, 2008, http://www.math.lsa.umich.edu/ wfulton/CurveBook.pdf

[4] Joseph H. Silverman, Taxicabs and Sums of Two Cubes, MAA, 1993.

[5] Joseph H. Silverman, John Tate, Rational Points on Elliptic Curves, Springer-Verlag New York, 1992.

[6] Eric W. Weisstein, 1729, From MathWorld–A Wolfram Web Resource, http://mathworld.wolfram.com/1729.html.