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TRANSITIONING STUDENTS FROM THE AREA MODEL TO THE NUMBER LINE MODEL WHEN DEVELOPING FRACTION COMPARISON STRATEGIES

A Master's Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science in Education, Elementary Education

By

Joann Barnett

July 2016

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TRANSITIONING STUDENTS FROM THE AREA MODEL TO THE NUMBER LINE MODEL WHEN DEVELOPING FRACTION COMPARISON STRATEGIES

Childhood Education and Family Studies

Missouri State University, July 2016

Masters of Science in Education

Joann Barnett

ABSTRACT

In this study, 21 participants (control and treatment) were examined based on differences between their abilities to use a variety of strategies to compare fractions after considering the fractions' approximate positions on a number line. A pre-post test and interviews were conducted in Fall 2015 and Spring 2016. Results showed greater gains for the treatment group than the control group in their ability to correctly estimate a fractions' position on the number line and to count fractional intervals on the number line. Even after students in the treatment group developed the ability to correctly place fractions on a number line, these students did not see the number line as a tool to use for comparing fractions, and preferred to use other strategies instead, even faulty strategies.

KEYWORDS: area models, common numerator and denominator, comparing fraction strategies, fraction concepts, iterating, gap reasoning, linear models, number lines, partitioning, residual reasoning, visual models.

This abstract is approved as to form and content

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Diana Piccolo, PhD
Chairperson, Advisory Committee
Missouri State University

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Approved:

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My sweet husband, Steve Barnett, who kept the house clean, my family who said I could do this, and my great friends who kept calling and texting to make sure I was writing.

I dedicate this thesis to the seventh grade students with whom we worked who welcomed us into their classroom and made us feel like we were part of the whole.

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INTRODUCTION

In a graduate-level research class, 18 adults were asked to compare the fractions $4/9$ and $6/11$. Of the 18 adults, 11 compared these two fractions by rewriting each fraction as an equivalent fraction with 99 as the common denominator.

4/9 = 44/99 6/11 = 54/99 54/99 > 44/99, therefore 6/11 $>$ 4/9

Of the 18 adults, five compared the two fractions by rewriting the decimal equivalents.

4/9 = $0.\overline{4}$ 6/11 = $0.\overline{54}$ $0.\overline{54} > 0.\overline{4}$, therefore 6/11 $>$ 4/9

Only two of the adults compared the two fractions by using the benchmark $1/2$.

4/9 is less than 1/2 6/11 is greater than 1/2 6/11 $>$ 4/9

Rewriting two fractions as equivalent fractions with the same denominator or rewriting the two fractions using their decimal equivalents are procedural ways to compare the two fractions. The results of research from Charlambous and Pitta-Pantazi (2007) show more students experience success with mathematics if the introduction of procedures is delayed for the purpose of developing conceptual knowledge of a fraction's magnitude. Of the 18 adults in this graduate level class, 16 of them immediately resorted to a procedure to compare these two fractions. Only two of the adults compared the two fractions conceptually by comparing each to the benchmark ½ in order to determine their magnitude (the size of a fraction). The reason so many students choose procedural strategies over conceptual strategies might be related to American textbooks spending minimal time on developing fraction concepts and moving too quickly to fraction procedures (Alajmi, 2012).

Developing fraction comparison strategies, conceptually instead of procedurally, will require carefully sequenced instructional tasks providing an intentional focus of transitioning students from an area model (where comparisons can be made without attending to the number itself) to the use of the number line (where the focus is on the magnitude of the fraction). Elementary teachers, who launch children's primary formal experiences with fractions, need additional resources with instructional tasks including details about the sequence and implementation of the tasks so initial ideas about fractions are developed conceptually instead of procedurally.

Purpose of the Study

The purpose of this study was to design an instructional model transitioning students from representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude (size) of the fraction and to develop fraction comparison strategies. By placing a fraction on a number line, the fraction could be seen as a number, but when using an area model, some might have perceived the fraction as a shape instead of a number. This idea that a fraction is a number and not a shape is important because computations are performed with numbers, not with shapes (Wu, 2011).

Rationale for the Study

The rationale for this study was based on results from the research of Charlambous and Pitta-Pantazi (2007), who state more students are successful with present and future work in mathematics if the introduction of procedures is delayed,

while more instructional time is allowed for conceptualizing fractions. However, students continue to exit their school experiences with little conceptual knowledge of the magnitude of fractions. Teachers at all grade levels complain, “Why can’t my students do fractions?” (National Mathematics Advisory Panel [NMAP], 2008). A major reason students have difficulty learning algebra is directly related to their lack of fraction understanding (NMAP, 2008). Research indicates instruction focusing on developing strategies to compare magnitudes of fractions will help students make sense of the work they do with fractions. For instance, when asking children to calculate (2/3) × (5/6), there is no meaning attached to correct responses such as 10/18 or 5/9 if students lack an understanding of conceptualizing the magnitude of the fraction (Behr, Wachsmuth, Post & Lesh, 1984). Because students know so little about the magnitude of fractions, students are unable to determine if the solution to a fraction operation problem makes sense. Working with fractions is usually the first time in their schooling when students give up on making sense of a concept and resort to simply following the procedure demonstrated by the teacher (Wu, 2011).

On an assessment item in the 1980 National Assessment of Educational Progress (NAEP) 55% of the 13 year olds selected either 19 or 21 as the best estimate for 12/13 + 7/8 as cited by Carpenter, Corbitt, Kepner, Lindquist, and Reys (1980). In this situation, students were not attending to the magnitude of each fraction and instead allowed their knowledge of whole numbers to interfere with the quantitative notion of each fraction in the sum being slightly less than the whole number 1, which would result in a best estimate of 2 for the above addition problem (Carpenter et al., 1980, Behr et al., 1984).

In a developmental math class at a local community college, adult students were asked to draw a number line, partition and mark the number line with the whole numbers 0-3, and then place the fraction 2/3 where it belonged on the number line. Out of the 60 students who completed the task, only ten of the students correctly placed 2/3 on the number line. These adult students had spent years of schooling in elementary, middle, and high school math classes where fractions were part of the content and still 1/6 of the students were unable to place 2/3 correctly on the number line.

As it turns out, a child's difficulty in working with fractions may be related to teachers moving too quickly towards procedures so students can *do* fractions. Instead of rushing to procedures, students need the time and freedom to conceptualize, organize and assimilate the notion of a fraction into their already developed informal framework of fractions (Cooper, Wilkerson, Montgomery, Mechell, Artebury, & Moore, 2012). Promoting rote memorization and resorting to algorithms too quickly does not allow for the development of mental operations with fractions, which is essential in an authentic understanding of fractions (Cooper et al., 2012). Most teachers' instructional decisions are heavily influenced by the textbook from which they teach. Behr et al. (1984) point out textbook deficiencies which include experiences with the following: (a) conceptualizing unit fractions, (b) composing and decomposing fractions, (c) qualitative reasoning about fraction magnitude, and (d) comparing and ordering fractions. The typical textbook moves too quickly to computation with fractions, neglecting the idea of estimating before operating (Alajmi, 2012).

The significance of this study was the development of an instructional sequence to help students determine the magnitude of fractions and provided teachers with effective

supplemental material for their existing curriculum. The results of this study may be shared with teachers at state and national conferences. The efforts of this study resulted in an eventual acknowledgement of the importance of conceptualizing the magnitude of fractions so such ideas may be incorporated into future publications of textbooks. The greatest significance of this study may be in developing a curriculum resource containing the sequence of instructional tasks developed from this study to help students exit their schooling with the ability to *do fractions*.

Research Questions

The following questions were used to guide the research study:

1. What types of strategies were students using to compare fractions?

2. What difficulties did students have in naming a fraction when represented with a variety of visual models?

3. What misconceptions with area models confounded students' use of the number line model?

4. What needs to be included in the design of an instructional model to support students' ability to conceptualize a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies?

Assumptions

For the purpose of this study, the following assumptions were made:

1. It was assumed students in this study were going to have difficulty describing the strategy they used to compare fractions.

2. It was assumed students in this study would be reluctant to try new strategies, even if their present strategies were more difficult or successful only part of the time.

3. It was assumed students in this study may have simply wanted a teacher to tell them what to do procedurally instead of relying on their own thinking.

4. It was assumed these were typical students in seventh grade in a mid-size middle school in southwest Missouri in fall and spring 2015-2016.

5. It was assumed the data is valid and reliable.

Limitations

For the purpose of this study, the following limitations were identified:

1. The study was limited because the teacher of the control group was not restricted from teaching conceptual lessons related to the magnitude of fractions.

2. The study was limited to working with the treatment group 45 minutes one day a week from September 2015 through March 2016.

3. The study was limited to two intact pre-algebra classes of students, some who had high absentee rates and some who were from transient families.

4. The study was limited to data collected from a self-constructed pre- and post-test, student pre- and post-interviews, informal assessments, and a collection of journal entries.

Definition of Terms

For the purpose of this study, the following terms were defined as:

1. Magnitude of a number: the size of a number, or an order or ranking of numbers. For example, the magnitude of 1/4 is less than the magnitude of 3/4.

2. Conceptualizing mathematics: to form an idea of something in the mind related to mathematics.

3. Procedural thinking in mathematics: the idea relating work in mathematics as simply a set of procedures to memorize, and understanding is unnecessary.

4. Unit fraction: One piece of the whole. For example, 1/2, 1/3, 1/4 and 1/5 are unit fractions.

5. Iterate a unit fraction: to repeat the length of a unit fraction.

6. Whole number reasoning to compare fractions: Whole number reasoning is unreliable but is often used by students to compare fractions by comparing just the numerators or just the denominators. For example, students are using whole number reasoning if they say 6/11 is greater than 4/9 because 6 is greater than 4, and 11 is greater than 9.

7.Gap reasoning to compare fractions: Gap reasoning is unreliable and is a form of whole number thinking, where the student is not considering the size of the denominator and therefore the size of the relevant parts (or the ratio of numerator to denominator), but merely the absolute difference between numerator and denominator (Clarke & Roche, 2009). For example, students are using gap reasoning when they say 4/9 and 6/11 are equivalent because 4 is 5 away from 9, and 6 is 5 away from 11.

8. Residual reasoning to compare fractions: residual reasoning considers not only the need to fill in another piece to complete the whole, but also takes into consideration the size of the missing piece (Clarke & Roche, 1984). For example, students are using residual reasoning when they say 7/8 is greater than 5/6, because they are both one piece away from the whole, and since eighths are smaller pieces than sixths, 7/8 is closer to the whole than 5/6, so 7/8 is greater, as seen in Figure 1.

Residual Reasoning

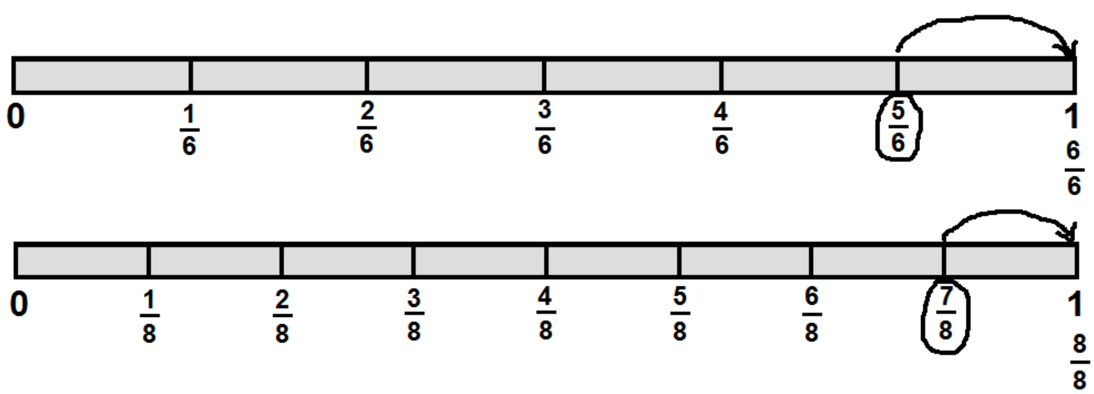

Figure 1. Using Residual Reasoning

LITERATURE REVIEW

The purpose of this study was to design an instructional model transitioning students from representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. Presented in this section of the review of the related literature will be: (a) the obstacles students encounter when comparing fractions, (b) the importance of students developing fraction magnitude knowledge, (c) the importance of using a variety of visual aids with special attention paid to the number line to help students understand the magnitude of fractions, and (d) summary.

Importance of Students Developing Fraction Magnitude Knowledge

Fractions comprise what is undeniably the most challenging number system of both elementary and middle school curriculum (Mathematics Learning Study Committee, 2001). Besides being a challenging topic for students, how students become proficient with fractions is less understood than how students become proficient with whole numbers. The most critical issues for success with fractions are a student's ability to understand what a fraction is and how different fractions relate to each other. (Behr et al., 1984; Clarke & Roche, 2009; Wu, 2011). After students have worked with equal sharing, recording the quantity of each share leads students into part-whole concepts and examining the structure of fractions. The complexity of student thinking when recording a quantity in fraction form requires a student to identify the whole, partition the whole

into equal size pieces, identify the number of pieces under consideration for the given problem, and then relate the number of pieces under consideration to the pieces in the partitioned whole (Cooper et al., 2012). Because understanding part-whole concepts is critical to understanding rational number concepts, such as ratios and quotients, much of the curriculum in lower grades focuses on equal partitioning before formal instruction of fractions begins (Charlambous & Pitta-Pantazi, 2007). The ability to perceive how the numerator and denominator coordinate as a conceptual unit (rather than two distinct numbers) in the part-whole construct has been found to be an indicator for successful performance with rational numbers (Behr et al., 1984). Other researchers argue the part-whole and the measurement interpretation of fractions are the most relevant topics in fraction conceptualization (Fuchs et al., 2013).

Just as magnitude knowledge is vital to whole number understanding, it is equally vital when developing fraction concepts and operating with fractions (Schneider & Siegler, 2010). Students' ability to position a fraction correctly on a number line and then to compare and order fractions have a high correlation to students successes with fraction operations and other mathematical achievement (Schneider & Siegler, 2010).

Understanding the magnitude of a fraction helps determine the reasonableness of an answer when operating with fractions, thus deterring students from employing flawed procedures, such as adding unlike denominators together when adding fractions. Students are motivated to persevere in solving problems when the answer makes sense to them, thus providing opportunities for greater success in higher level math classes (Siegler & Pyke, 2013). The relationship between understanding fraction magnitude and math

achievement scores is stronger in 8th grade than in 6th grade, due in part to the fraction work required in algebra and pre-algebra.

Fractions are used across the curriculum and across grade levels in a myriad of topics besides algebra, such as probability, data analysis, ratio and proportions, percents, and measurement (Barnett-Clarke, 2010). Even though the relationship was not as strong for 6th grade, the relationship was still there, suggesting fraction magnitude knowledge facilitates learning of less advanced mathematics also (Siegler & Pyke, 2013). Students who were able to demonstrate fraction magnitude knowledge by placing the fraction correctly on the number line experienced greater overall mathematical achievement than those who placed fractions on the number line by attending to only the numerator or only the denominator (Siegler & Pyke, 2013). More students are successful with present and future work in mathematics if the introduction of procedures is delayed, while more instructional time is allowed for conceptualizing fractions (Charlambous & Pitta-Pantazi, 2007). Working with rational numbers (like fractions) is considered to be the most complex mathematics students will encounter in their presecondary schooling (Mack, 1993).

Obstacles Students Encounter when Comparing Fractions

Students' success with whole numbers is often a major contributor of an inability to work with fractions. When working with whole numbers, students can count a set of objects, knowing the last number counted names the number of items in the set. Whole numbers represent a specific amount, but a fraction represents a portion of the amount. Numbers represented with fraction notation can also be interpreted in other ways such as

ratios or division, leading to more confusion (Cooper et al., 2012). Since fractions are not part of the counting sequence, students often reject fractions as numbers, which leads students to conceptualize fractions as two distinct whole numbers, resulting in computational errors such as (1/2) + (1/4) = 2/6 (Charlambous & Pitta-Pantazi, 2007). Some of the other whole-number properties students have generalized are: (a) each number can be represented with a unique numeral; (b) each number has a unique number preceding it and another unique number following it; and (c) a whole number is never decreased with multiplication or increased with division. None of these generalizations are true about fractions, which leads to a whole number bias when working with fractions, causing misconceptions to thrive (Siegler & Pyke, 2013). Even high school students will often claim there are no fractions between 5/7 and 6/7 since 5 and 6 are consecutive whole numbers (Seigler & Pyke, 2013).

The one property all numbers possess is they have magnitude, which means they have a specific spot on a number line (Schneider & Siegler, 2010). This one property creates obstacles for students because of the unique difficulties related to the structure of the number line and determining the placement of a fraction on the number line. When working with placing fractions on number lines, students typically have difficulty determining the whole, partitioning the whole, counting the partition marks instead of the intervals, and often using the wrong unit (Charlambous & Pitta-Pantazi, 2007).

Another difficulty in conceptualizing fractions is students' inability to coordinate the numerator and denominator as a single number (Cramer, Behr, Post, & Lesh, 2009). Children do not see a number like 2/3 as a single entity, and instead treat the 2 and 3 as two separate whole numbers. In the 1980 NAEP assessment, 55% of the 13 year olds

selected either 19 or 21 as the best estimate for (12/13) + (7/8) (Carpenter et al., 1980; Behr et al., 1984) . In this situation, the 55% who chose 19 or 21 allowed what they know about whole numbers to interfere with the quantitative notion each fraction in the sum is slightly less than the whole number 1, which would result in a best estimate of 2 for the above problem.

Another whole number concept that surfaces during work with fractions results in a prevalent erroneous comparison strategy where students focus on the gap (gap thinking) between the numerator and the denominator instead of using residual thinking. When using gap thinking, students believe the fractions 5/6 and 7/8 are equivalent, justifying this conclusion because each fraction requires only *one more piece* to complete the whole. This focus on *one more piece* without consideration of the size of the missing piece is rooted in students' prior work with whole numbers. Residual thinking, as opposed to gap thinking, considers not only the need to fill in another piece to complete the whole, but also takes into consideration the size of the missing piece (Clarke & Roche, 2009).

Language itself can also interfere with students' sense-making of fractions as the phrases *more* and *greater* or *less* and *fewer* cause confusion when describing the size of the pieces as compared to the number of pieces (Clarke & Roche, 2009). Students must be able to internalize when two fractions have the same denominator, the fraction with the greatest numerator has the greatest value, but if two fractions have the same numerator, the fraction with the least number in the denominator has the greater value (Pantziara & Phillipou, 2012).

An often-overlooked obstacle when examining student difficulties with fractions is the instructional model itself, teaching students through procedures and memorization instead of allowing students the time and the freedom to conceptualize, organize and assimilate the notion of a fraction into their already developed informal framework of fractions (Cooper et al., 2012). Promoting rote memorization and resorting to algorithms too early does not build on a foundation students already have of equal sharing, nor does it allow for the development of mental operations with fractions, which is essential in an authentic understanding of fractions (Cooper et al., 2012). But most teachers' instructional decisions are heavily influenced by the textbook from which they teach. Behr and colleagues (1984) point out textbook deficiencies, which include experiences with the following: conceptualizing unit fractions, composing and decomposing fractions, qualitative reasoning about fraction magnitude, and comparing and ordering fractions. The typical textbook moves too quickly to computation with fractions, neglecting the idea of estimating before operating (Alajmi, 2012).

Importance of Using a Variety of Visual Aids

Physical models play a major role in the development of mathematical concepts, as the learner moves from concrete to abstract (Behr et al., 1984). Research specifically supports the use of concrete representations as students develop comprehension of fractions (Van de Walle, Karp, & Bay-Williams, 2007). The two types of concrete models are continuous (area and linear) and discrete (sets), but researchers do not agree which visual models should be used first. The use of manipulatives is vital because it addresses the development levels from Piaget's work and also meets the needs of the

learner to have physical knowledge of a concept before procedural knowledge (Cooper et al. 2012). As important as visual tools are, children are often *model-poor* and may have only the circle model to use for their concrete model (Clarke & Roche, 2009).

In another study, Alajmi (2012) compared three textbooks from three different countries--the United States, Japan, and Kuwait. The Japanese textbook used few physical models when developing fractional concepts, but the number line (a linear or measurement model) was used throughout, focusing on measurement at each grade level. It has been argued the number line is a powerful model for developing fraction knowledge but it is highly underused in American textbooks.

Summary

In Clarke and Roche's study (2009), the two strategies used most often by students who were successful in comparing fractions were residual reasoning and benchmarking. Ironically, little time is devoted to these topics in traditional American textbooks (Alajmi, 2012). This suggests more lessons need to be designed for children to develop the conceptual ideas of a fraction and its magnitude. It appears there should be greater opportunities for students to estimate the values of fractions and approximate their position on a number line. Data from Clarke and Roche's interviews clearly showed students had little sense in determining the size of a fraction, and most students either focused on the numerator or denominator, without considering the relationship between the two.

Helping students develop the essential understandings related to comparing fractions is challenging. Teachers need the skills to carefully select tasks where the focus

and meaning lay the foundations to develop rational number concepts (Chval, Lannin & Jones, 2014). Mullis, Martin, Foy and Arora (2012) indicate fractions are more of a problem for American students than for students of any other country. Perhaps the difficulties the students have with fraction concepts are due to the difficulties American teachers have with fraction concepts. However, for the most part, teachers teach the way they learned. It is time to stop this cycle by making sure tomorrow's teachers have effective conceptually-based fraction instruction today.

METHODOLOGY

The purpose of this study was to design an instructional model transitioning students from representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. Presented in this section of the methodology will be: (a) instrumention and design, (b) subjects and site of the study, (c) procedures, and (d) analysis of data.

Instrumentation and Design

A mixed methods study using one quantitative and two qualitative instruments with a multi-stage evaluation design was conducted to determine if an instructional model could be developed to transition students from exclusively representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. To calculate a base line for both the treatment and the control group, a 13-item teacher-created pretest (see Appendix A) was administered in September 2015. The same pretest was administered as a posttest in March 2016. Mean raw scores were calculated for each group on the pre- and posttest. A weekly reflective journal (a qualitative instrument) was used to reflect on the outcomes of each lesson between the pretest and the posttest.

Clarke and Roche's research (2009) looked at the different strategies students used to compare fractions, but this study focused on the initial comparison strategies

students used, and then designed instruction to help students choose new comparison strategies based on experiences with the number line. To determine if the treatment group developed new strategies, video-taped, transcribed semi-structured pre-and post-interviews were administered in person to five students from both the treatment group and the control group to more fully understand the phenomena of student-thinking when comparing and ordering fractions. Each interview lasted an average of 10 minutes.

This study was designed to determine if improved student scores were the result of developing new conceptual comparison strategies or were the improved student scores the result of students simply developing better skills using procedures. To accomplish this, the interviewed students were asked to compare two fractions and describe the strategy used to compare the fractions. The types of comparison strategies students chose to use were analyzed for similarities and differences between the pre- and post-interviews. During the pre-interview, students were asked to compare eight pairs of fractions and then placed eight additional fractions on a number line (see Appendix B and C). In the post-interview, students were asked to compare at least 10 pairs of fractions and placed each pair of fractions they were comparing on the number line (see Appendix D).

A convenient sampling of 44 seventh graders students who struggled conceptually with fractions were chosen for this study. In the treatment group, two of the 19 students were proficient at working with fractions procedurally. However, students who were procedurally confident may still have needed support with making sense out of the magnitude of fractions.

The treatment group consisted of 19 seventh graders in one section of pre-algebra, while the control group contained 25 seventh graders from another section of pre-algebra in the same school. Results from using district MAP (Missouri Assessment Program) and IReady achievement data showed 75% of the students in both groups tested below grade level. The same teacher taught both classes and for the same length of time. However, once a week, 3-5 extra instructors from the university presented a lesson to the treatment group, often working in small groups with a teacher/student ratio of 1 to 4.

Subjects and Site of the Study

This study took place at an inner-city middle school in a Midwestern community of approximately 160,000. This was a Title 1 school with 89% of the students on free or reduced lunch. Often, the students at this site came from one-parent households where there was minimal educational support from home due to parents' unwillingness or inability to help.

There were 12 boys and 7 girls in the treatment group, with 14 Caucasians and 5 African Americans. In the treatment group, 17 of the 19 students tested multiple grade levels below their own grade level. Most parents of children in this district had a high school education or less.

Procedures

Permission to conduct the study was obtained from the building principal (see Appendix E). Prior approval for this project was obtained from the Missouri State University IRB (January 13, 2015; approval #15-0274).

The treatment and control groups were selected using convenience sampling because the classroom teacher and the principal were willing to allow the sequence of the regular curriculum to be interrupted and replaced with content placing a greater emphasis on fractions than the current curriculum. The class chosen as the treatment group had a timeframe allowing the greatest number of university instructors to attend. The treatment group received special instruction on conceptualizing fractions, while the control group received the typical instruction provided within the framework of the current curriculum.

In September 2015, a researcher-constructed test (see Appendix A) was developed to administer as a pre-and posttest to both the treatment group and the control group to determine student knowledge of representing fractions, comparing fractions, and placing fractions on a number line. During the pretest, five students were selected for pre-interviews (see Appendix B and Appendix C) from both the treatment and the control group. During March 2016, the posttest and post-interviews were administered to both the treatment group and the control group (see Appendix D).

Each week, fraction tasks were developed (see Appendix F) by the university instructors to move students in the treatment group towards conceptually understanding the value of a fraction by considering its location on a number line. Initial lessons used area models to help students understand fraction notation. Fraction strips were used to connect area models with linear models before the number line was introduced. Once the number line model was introduced, lessons were designed to help students develop strategies to locate fractions on the number line.

Analysis of Data

Quantitative data were collected from a researcher-constructed pretest and posttest to determine student knowledge of representing fractions, comparing fractions, and placing fractions on a number line. The raw mean test scores for both the treatment group and the control group were calculated and compared. Data were collected from two specific problems on both the pretest and the posttest to determine student gains on locating fractions on a number line and counting on the number line.

Qualitative data were collected from student pre- and post-interviews to determine which strategies students used to compare fractions. Pre and post-interview comparison strategy results compared the frequency and type of strategies used in the pre-interview to those strategies used during the post-interview. Data were analyzed by coding to identify common themes and were used to answer the overarching research questions. Assurances of trustworthiness including data triangulation were employed to uncover the salient themes. The findings of the study are reported in the Results.

RESULTS

The purpose of this study was to design an instructional model transitioning students from representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. Presented in this chapter of the results were: (a) data analysis, and (b) summary.

Data Analysis

A mixed methods study with a multi-stage evaluation design was conducted to determine if an instructional model could be developed to transition students from exclusively representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. Data were collected during the 2015-2016 school year. Data were analyzed to determine if a difference existed between the instructional gains for the treatment group compared to the control group. The data collected reflects only those students in each group who had taken both the pretest and the posttest.

Table 1 shows the means and standard deviations for the pre-and posttests for both the treatment group and the control group. The highest score possible on each test was 36 points.

Table 1. Means and Standard Deviations

Pre- and Posttest Scores	N	Means	S.D.
Treatment Group			
Pretest	11	9.36	3.14
Posttest	11	18.73	6.48
Control Group			
Pretest	10	10.41	4.74
Posttest	10	14.40	6.31

The quantitative results of the pre-test showed the mean score of the treatment group increased 9.33 points while the mean score of the control group improved 4.01 points. While both the treatment group and the control group improved their mean scores from the pretest to the posttest, the treatment group's mean score increase was 2.33 times greater than the control group's mean score increase. In both groups, the standard deviation for the posttests varied closely to the same amount.

A specific task on the pretest and the posttest asked students to place nine fractions at their approximate location on a number line already marked with 0, 1, and 2 (see Figure 2). In Table 2, the pretest results show the treatment group correctly placed 10 fewer fractions than the control group. The posttest results show the treatment group correctly placed 11 more fractions than the control group. The only two fractions the

Locating Fractions on the Number Line

On the number line, mark and label: $\frac{15}{16}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{5}{8}$, $\frac{4}{4}$, $\frac{6}{8}$, $\frac{6}{12}$, and $\frac{5}{4}$

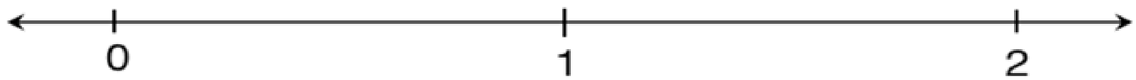

Figure 2. Locating fractions on the number line on pre- and posttest.

Table 2. Results of Students Placing Fractions Correctly on Number Line

Pre- and Posttest Scores	Percent of Fractions Placed Correctly
Treatment Group	
Pre-test	2%
Post-test	41%
Control Group	
Pre-test	13%
Post-test	33%

treatment group placed correctly on the pretest were 1/4 and 5/4. It appeared the two students who placed one of these fractions correctly did so using whole number reasoning by locating fractions in order of the size of the numerator, and coincidentally placed these

two fractions at the correct spot on the number line. On the posttest, the four students who did not place 5/4 correctly, placed it between 0 and 1. Only two of the four students who did not correctly place 4/4 placed it to the left of 1, while another student placed it halfway between 1 and 2, and the fourth student didn't place 4/4 on the number line at all.

Another task from the pre- and posttest asked students to count by fifths on a number line numbered with 0 and 2 (see Figure 3). Table 3 shows the results of the pre- and posttest for both the treatment and the control group.

Counting by Fifths

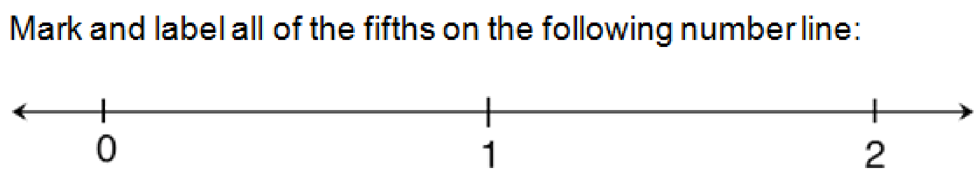

Figure 3. Counting by fifths on the number line.

On the pretest, 55% of the students in the treatment group either did not attempt this problem, or they marked the number line with multiples of 5. The other 45% placed fractions on the number line, but none of them were correctly placed. On the posttest, 64% of the students in the treatment group counted and placed fifths accurately on the number line, while the other 36% of the students counted by fifths, but had difficulty placing fractions correctly on the whole numbers. On the pretest, the control group placed 4% of the fractions correctly while seven out of ten students in the control group either used only whole numbers or did not attempt the problem at all.

Table 3. Results of Counting by Fifths on the Number Line

Pre- and Posttest Scores	Percent of Fractions Placed Correctly
Treatment Group	
Pre-test	0%
Post-test	89%
Control Group	
Pre-test	4%
Post-test	64%

On the posttest, the control group correctly placed 20% of the fractions with one of the students doing so by using mixed numbers instead of fractions. Another student in the control group placed 1/5, 2/5, 3/5, and 4/5 between each interval, but paid no attention to equal partitions for the fifths. Another student in the control group counted by fifths, but none of the fractions were correctly placed. A different student in the control group counted up to 5/5 correctly, but in the interval between 1 and 2, counted down by fifths. On this task of counting by fifths on a number line, the students in the treatment group out-performed the students in the control group. In the control group, six of the ten students either left the number line blank or only attempted to place tick marks, while in the treatment group, seven out of 11 students correctly completed this task with four of the 11 students misplacing only one or two of the fractions.

The qualitative results of the teacher journal and member checks with the other university professors showed three themes emerging from the work with the treatment group shown in Figure 4: (a) maintaining a focus on the whole unit, (b) naming a whole number with a fraction on the number line, and (c) having the prerequisite skills for each lesson.   Evidence of these three themes surfaced after a series of weekly reflections recorded in the researcher's journal.

Difficulties Developing Concepts

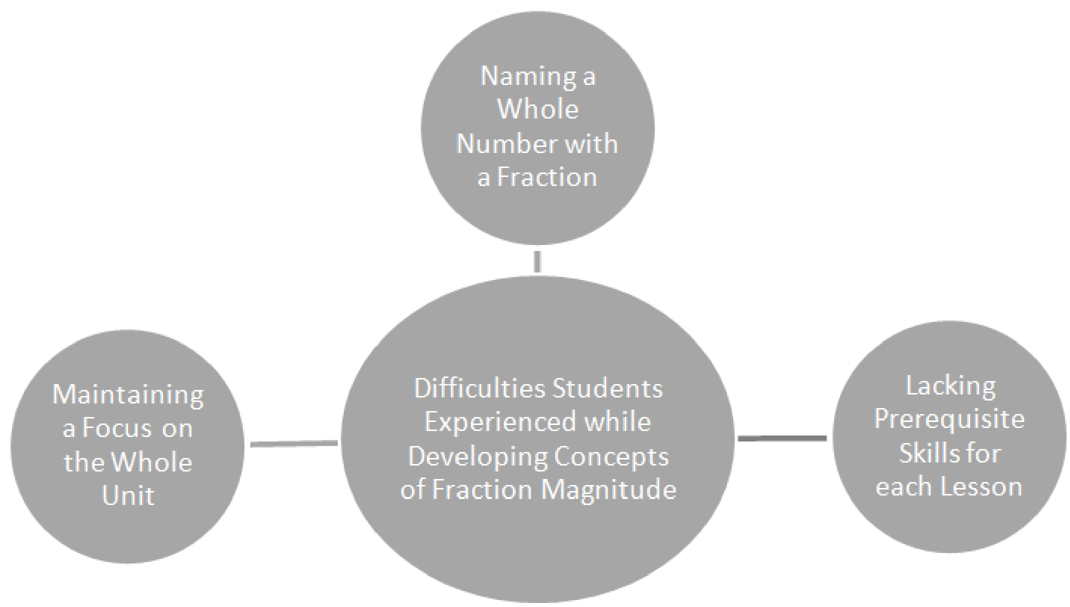

Figure 4. Themes emerging from researcher's journal.

Maintaining a Focus on the Whole Unit. The first lesson was simply counting sections of a circle. Students counted up to the whole successfully, but faltered when counting past the whole. It appeared students understood the concept of the whole and what a fraction looked like as the whole when representing the fraction with the circle. Students even articulated 6/6 was different than the other fractions recorded on the board

because the top number and bottom number were the same, justifying the notation by saying “Because you used six of the sixths to make the whole circle.” Following this initial lesson with the circles, instruction was designed to move students toward the use of a linear model (the number line) so fractions strips were folded and then the number line was introduced. During this transition to linear models, students either demonstrated an indifference to the whole unit or they identified whatever piece of the number line they could see as the whole unit. The teacher recorded in her journal “Today students had to count by eighths on a number line marked 0-2. I was totally surprised by the number of students who placed 8/8 on the number 2 or close to the number 2 without any consideration or acknowledgement of the location of the number 1 halfway between 0 and 2.”

Naming a Whole Number with a Fraction. Students had difficulties determining whether or not a fraction could be located at a whole number on the number line. When fraction strips were folded in the second lesson, students first demonstrated their difficulty in placing fractions at a whole number. Figure 5 shows both sides of the folded fraction strips.

Both Sides of a Fraction Strip

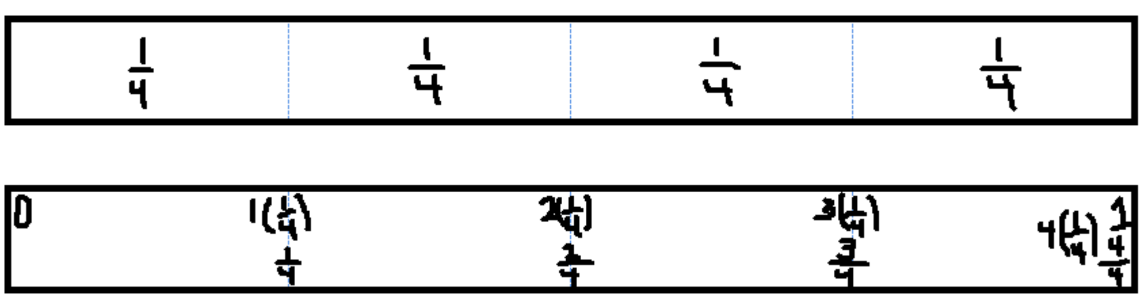

Figure 5. Fourths shown as an area and linear model.

On one side of each fraction strip, equal partitions were marked with the unit fraction, while on the other side of the fraction strip, students counted the partitions, marking the value on each fold. Each fold was first counted as an iteration of 1/4 [specifically 1(1/4), 2 (1/4)]. Underneath the iterations of groups of 1/4, students recorded their counting by fourths. Many students did not place 4/4 directly underneath the whole number 1. Initially, the cause for the incorrect placement of 4/4 appeared to be due to a lack of space to write directly underneath the whole number. However, students continued to avoid placement of fractions at whole number positions even on the posttest. At the April 2016 National Council of Teachers of Mathematics annual conference, recent research from the *Rational Number Project* showed evidence of the same phenomenon of students resisting placing a fraction directly where the whole units were located (T. Wyberg, personal communication, April 2016).

The breakthrough in this phenomenon of student resistance to marking a whole number with a fraction occurred six months into the study. As students practiced counting by fractional units on a series of number lines, one of the students raised her hand and said, "I just noticed something. If we are counting by thirds, I write 3/3 at 1. If I am counting by sixths, I write 6/6 at 1." Then she generalized, "So whatever I am counting by, that's how many of them are at 1." She then went on to explain how this idea helped her partition the whole. "So, if I am counting by eighths, I know I am going to write 8/8 at 1, and then I have to have seven tick marks for the other eighths."

On the pretest, 0% of the students in the treatment group placed 5/5 at the whole number 1, while 82% of the students placed 5/5 correctly on the whole number 1 on the posttest. In the control group, 1 out of the 10 students placed 5/5 correctly on the posttest,

but this was also the same student who, once reaching 5/5 at 1, counted back down to 0/5 to get to the whole number 2.

Having the Prerequisite Skills for Each Lesson. According to a series of journal entries, another common theme was the researchers' repeated realizations students were not ready to achieve success with a given lesson, and at times students were unable to enter a task. In an initial task where students were asked to represent a given fraction using pattern blocks, the reflections in the journal describe students as being unable to enter the task. "They (students in the treatment group) did not have meaning for the fraction notation. They did not have a way to talk about fractions." However, 5 months into the study, a lesson was designed to answer specific research questions about students and their ability to determine the magnitude of the fractions by positioning the fractions in their approximate location on the number line. The journal reflection on this lesson began "So that was a disaster." In the journal, the researcher recorded the students inattention to precision, and unwillingness to consider the placement of benchmark fractions such as 6/12 and12/12. During journal writing, the researcher noted that "They (students) didn't care about being exact (with partitioning or placement) and didn't seem interested in using the unit fraction for spacing unless prompted by the math coach." To help students develop the supporting skills needed to complete these tasks of partitioning carefully and iterating the unit fraction, a new task was implemented the following week. This new task asked students to first estimate the placement of a fraction count on the number line, and then use paper strips to make a unit fraction that could be used as an iterative tool to determine a more exact placement on a second number line. After this lesson, the teacher reflected in her journal "The very first

thing they (students) counted by was 1/2. A student I was working with placed 2/2 past the whole number 1. Another student counted1/2, 2/2, and then placed 3/1 underneath the whole number 1. I believe next week we are going to have to do even more simple counting on the number line."

This same task had to be continued the next week, after which the researcher recorded in her journal:

> On the top pair of number lines only 0 and 2 are marked. Student F estimated his first half where it belonged, and then all the other halves were very close together. On the second number line, he was supposed to check his work, but he needed direction about cutting a paper strip the length of the whole, and then folding and cutting in half to use as his 1/2 piece to iterate. Then he successfully iterated the 1/2 strip after a few erasures (see Figure 6).

Estimating and Counting by Halves

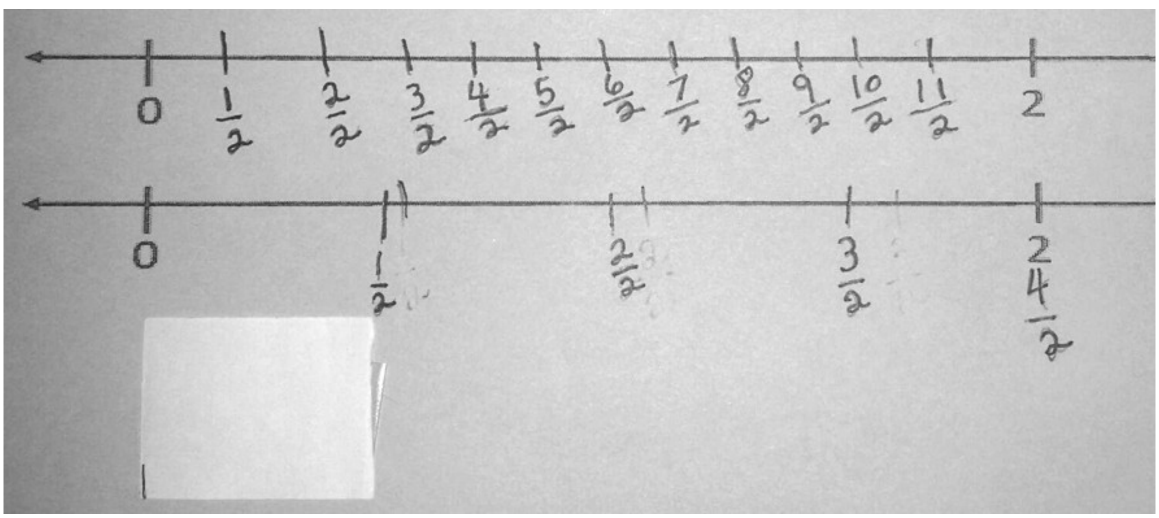

Figure 6. Estimating a count of 1/2, and then checking estimate by iterating 1/2.

The researcher continued to write:

> This same student (whose work is shown in Figure 6) continued to the second problem with a pair of number lines marked with 0 and both 1 1/2 and 3/2. He was much more careful about estimating a more uniform interval (space between tick marks), thus being much more successful when he checked his work on the second

> number line. However, students were not very engaged. Moved very slowly. We will never get through this. Sigh.

Because students appeared to still be having difficulties iterating a unit fraction on a number line, it was decided a game would be played requiring students to count fractions on a number line. The next week, before the game was played, students practiced counting on the number line, first counting by halves, then thirds, fourths, etc., each time counting from 0-2 with the numbers 0, 1 and 2 already marked on the number line. It was during this counting sequence students experienced the breakthrough about writing a fraction at the whole number. After playing this game, students were ready to go back to the original task from four weeks prior to this where they started placing fractions on the number line.

A comparison of the qualitative results of the student pre- and post-interviews showed students in the post-interview treatment group paid more attention to the whole unit and used less gap reasoning than in the pre-interview, but all the students still returned to gap thinking when all other reasoning failed. The results also showed students could partition a number line and iterate a unit fraction to estimate the placement of halves, thirds, fourths, fifths, sixths, eighths, and ninths, but 4 of the 5 students still demonstrated they did not know how to use the fraction's position on the number line to determine which number was greater. If the fractions' positions on the number line were contradictory to gap or whole number reasoning, the students opted to use the faulty gap reasoning instead of relying on which fraction was furthest to the right on the number line.

Table 4 shows how often whole number reasoning, gap reasoning, or some other type of strategy was used during the treatment group interviews to compare fractions on the pretest and the posttest.

Table 4. Fraction Comparison Strategies Treatment Group Used in Post-Interviews

Type of Reasoning	Pre-Interview	Post-Interview
Whole Number	27%	2 %
Gap	46%	49%
Improper Fraction	12%	9%
Number Relationships	6%	9%
Benchmarking	3%	17%
Common Numerator	3%	2%
Draw a Picture	3%	0%
Residual Reasoning	0%	4%
Equivalent Fractions	0%	2%
Unsure	0%	6%

The use of whole number reasoning was the strategy students used the most in the pre-interview, but was only used once in the post interview, and when it was used in the post-interview it was described in a context as though pieces of something were being

shared. The use of gap reasoning appeared to remain steady from the pre-interview to the post-interview. The students who were no longer using whole number reasoning in the post-interview were now incorrectly using gap reasoning while students who had been using gap reasoning in the pre-interview appeared to move towards more reliable reasoning during the post-interview, such as using residual reasoning, comparing fractions to benchmarks, and using number relationships.

In the pre-interview, about 75% of the strategies used to compare fractions were the unreliable whole number or gap reasoning strategies. Of the other strategies used in the pre-interview, two of the responses included a picture or a manipulative, with no consideration for the whole units to be the same size. Other responses showed evidence students knew the relationship between the numerator and denominator when fractions were equal to ½. Another student compared fractions using a benchmarking strategy by recognizing one fraction was greater than ½ and the other fraction was less than ½. All four students interviewed in the treatment group knew the term *improper fraction* but only two students used the term properly. However, these two students could only define improper fractions procedurally as “the top number is bigger than the bottom number”. Another student declared “you can’t compare 5/3 to 3/5 because 5/3 is not a fraction.”

The post-interview was presented differently than the pre-interview. In the post-interview, not only were students asked to compare two fractions, students were then asked to estimate the location of the two fractions on the number line. The number of times some form of gap reasoning was used in the post-interview was 23 out of 46 responses. The first student interviewed in the treatment group during the post-interview used gap reasoning to initially compare two fractions, but as soon as he was asked to

place the two fractions on the number line, he was able to correctly compare them. In the pre-interview, another student compared 7/8 and 5/6 by saying, “They’re equal because they are the same distance away from being a whole.” In the post-interview, this same student compared the same two fractions by saying, “They’re both one away, so I’m going to have to say 5/6 is greater.” Once he started thinking about placing the two fractions on the number line, he says, “No wait. No it’s 7/8… I was looking at it and thinking about where these pieces would probably be…And I looked and thought it would be right around here.” He then placed 7/8 closer to 1 than 5/6 was placed. The interviewer then asked, “Now at first you said 5/6 was greater and I think you were referring to sixths being greater than eighths.” The student responded, “I was thinking about the size of the pieces. Well sixths are greater than eighths. Yeah, but because it’s going to be closer. What I’ve realized is that the bigger the pieces the farther it’s going to be from that (points to 1) so the smaller the pieces, the closer it’s going to be.”

During the pre-interview, another student used whole number reasoning to compare fractions 87.5% of the time. For example, when comparing 5/8 and 3/7, this student correctly said 5/8 was greater, but his reasoning was, “Five is greater than three, and eight is greater than seven.” This whole number reasoning resulted in a correct response for this problem, but he used the same reasoning when comparing 7/14 and 2/4. Even though these two fractions are equal, this student chose 7/14 as being the greater fraction “because seven is greater than two and 14 is greater than four.” The only other reasoning this student used in the pre-interview was recognizing an improper fraction was greater than a fraction whose numerator was less than its denominator. In the post-interview, this student no longer used any whole number reasoning, but instead used a

combination of comparing to the benchmark 1/2 with additional gap reasoning when needed. For example, when comparing 2/3 and 4/10, this student reasoned that 2/3 was the greatest "because it's closer to 1 whole, but 4 is under 1/2." This student also correctly pointed to the positions of 2/3 and 4/10 on the number line. In the post-interview, once this student was asked to compare 1/4 and 2/5, the student used only gap reasoning for all the remaining pairs of fractions, even though he could still use benchmarking to 1/2 on some of the remaining problems. When this student was asked to compare 1/4 and 2/5 in the post-interview, he said "They are kind of equal because you do need 3 to get to the whole (points to 2/5) and you need 3 for this one (points to 1/4)." When the interviewer wanted to know more about this, the student still believed the two fractions were equal and said "Even though they both had different wholes but they need the same number to get to the whole." On the number line he pointed to where he thought 1/4 would be. Then he iterated what he thought was a length of 1/4. When the fourth iteration was not on the whole number 1, he adjusted the length of 1/4, iterated that length again, with the fourth iteration being closer to the whole number 1. He was satisfied, pointed to where 1/4 would be on the number line, and the interviewer then placed her finger at the same spot to keep track of 1/4's position. When attempting to place 2/5 on the number line, he said it would be close to where 1/4 was. He iterated what he thought was 1/5, but the fifth iteration was less than the whole number 1, so he started with a larger approximation for 1/5 just slightly less than 1/4, and then approximated that 2/5 was where 1/5 would be. The interviewer asked the student to count by fifths, stopped him when he got to 2/5, and reminded him the fraction 2/5 was the number he was supposed to be locating on the number line. The student placed his

finger on a point to the right of 1/4 and slightly to the left of 1/2 and said “2/5 goes here.” The interviewer still had her finger at 1/4 while the student had his finger at 2/5. The interviewer asked which fraction was greater and the student now said 1/4 was greater “because it’s taking less to get to the whole, but 2/5 is a higher goal for it, but yet they still need the same numbers to get to the whole.” In a later interview, the student explained fourths were bigger pieces than fifths, so it took fewer fourths to reach the goal which is the whole number 1. So, using gap reasoning, this student first thought 1/4 and 2/5 were equal, but when he correctly iterated to locate correct positions of 1/4 and 2/5, he decided 1/4 was greater because fourths were bigger than fifths.

There was no evidence in the pretest or pre-interview of students in either the treatment or control group acknowledging the whole unit as an important idea when working with fractions. Even when using gap reasoning, students appeared to use reasoning as filling in a whole, but there was no evidence that students’ thought-processes included the idea whole unit should be the same size in order to compare. When drawing pictures on the pretest to compare fractions, 85% of the drawings had different sized whole units. On the pretest, one student used Cuisenaire rods to compare 3/4 and 7/9. This student used the purple rod for the whole unit when partitioning into fourths, and he used the blue rod for the whole unit when partitioning into ninths, and he did not seem at all bothered the whole units were two different sizes as he compared the fractions. In the post-interview, when asked to compare 3/4 and 7/9, this same student paused for 20 seconds, and then said:

> It’s gotta’ be7/9. Six is 2/3 of this (points to the denominator 9), so 7 (of the ninths) is going to be greater than 2/3. Three-fourths is right about here on the number line (estimates correctly), and I think 7/9 is going to be right about here

(points slightly to the right of 3/4). Is there any way you can be totally confident about that… no.

Summary

At the beginning of this study, the verbal fraction comparison strategies seventh grade students in the treatment group used most often were whole number reasoning and gap reasoning. If students had to record their reasoning on a test, they most often simply drew a rectangular or circular area model to explain their reasoning. At the end of the study, students used faulty gap reasoning as their strategy in 48.9% of the interview questions. Even though at the end of the study, students used gap reasoning in half of the comparison situations, only one student used whole number reasoning. However, all students interviewed in the treatment group had developed and started using a greater variety of more reliable strategies, such as comparing to the benchmark 1/2, using a number line, and considering the multiplicative relationship between the numerator and the denominator. When prompted to explain reasoning on a written test, students at the end of the study still showed a preference for using an area model to explain their reasoning, but the preferred shape referenced was now a rectangle. During the post-interview, 6.5% of the reasoning strategies began with the correct use of the number line, while another 6.5% of the time, students made corrections using a number line after using gap reasoning. In 15.2% of the responses during the post-interview, students used gap reasoning to incorrectly compare fractions even after their correct placement of the fractions on the number line contradicted the gap reasoning results.

DISCUSSIONS

The purpose of this study was to design an instructional model transitioning students from representing fractions using area models towards conceptualizing a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies. Presented in this section will be: (a) discussions, (b) conclusions, and (c) recommendations for future study.

Discussions

A mixed-methods research study with a multi-stage evaluation design was conducted to design an instructional path to transition students from representing fractions with an area model to conceptualizing a fraction's placement on the number line in order to develop strategies to compare and order fractions. A pre- and posttest, a pre- and post-interview, the researcher's journal, and various informal assessments were the instruments used to evaluate the effectiveness of the instructional model. While the researchers were encouraged by the greater increase of the treatment group's mean score from the pre- to the posttest when compared with the increase of the control group's mean score, the more surprising results of the study were revealed during the post-interviews when students could correctly estimate a fraction's position on the number line by partitioning and iterating but were unable to use the number line as a tool to compare these fractions, and instead resorted to faulty gap reasoning to compare.

While both the treatment group and the control group improved their mean scores from the pretest to the posttest, the increase of the treatment group's mean score was 2.33

times greater than the increase of the control group's mean score. The posttest mean score for the treatment group was 18.73 (maximum score 36) while the posttest mean score for the control group was 14.40. The greater mean score for the treatment group was likely a result of the students in the treatment group completing a series of tasks where students estimated the position of a unit fraction on the number line, iterated that fraction, partitioned whole units, and then counted fractional intervals on the number line. This instruction with the number lines was not part of the work of the control group. This was most evident on the task where students counted by fifths on a number line labeled 0, 1, and 2. The treatment group had 0% of the fractions correct on the pretest and 89% of the fractions correct on the posttest. The control group placed 4% of the fractions correctly on the pretest, and then 64% of the fractions were placed correctly on the posttest. Part of the treatment group's success could also be due to three to six mathematics teachers in the classroom with the students during one math class each week. Such a small student/teacher ratio likely played a part in some of the successes the students experienced. The increased variance of mean scores for both the treatment group and the control group is often due to the result of extreme differences in student mastery of new concepts, which unfortunately widens the learning gap as students progress through each successive grade level.

The post interviews used the same small sample of students from the pre-interviews. During the post-interviews, the four students in the treatment group revealed they were able to estimate a unit fraction, iterate that unit fraction for accuracy, adjust the original estimate, iterate, and repeat until they were satisfied with their estimate. Using this iterating process, students could accurately estimate the position of two fractions on

the number line. However, three of the four students interviewed from the treatment group were not able to use the number line to correctly compare the fractions. It appeared just teaching students how to locate numbers on the number line was not enough. The lesson needed to be designed with more explicit instruction on how the fraction's position on the number line also represented the distance the fraction was from zero. The lesson should have directed students to locate fractions on the number line and then employed the use of some other tool to represent the actual distance from zero. To represent the distance from zero, students could have either drawn the length, or cut and glued some other linear model such as a strip of paper, string, waxed string, or pipe cleaner. The value in this task was to make a specific connection between the position of the fraction on the number line and the fraction's distance from zero.

At some point during the interview, all four of the students from the treatment group experienced some disequilibrium or disconnect when the correct position of the fractions on the number line conflicted with the gap reasoning the student used to compare the fractions. Only one of the four students was able to see gap reasoning had resulted in a faulty comparison after seeing the position of the numbers on the number line. The other three students continued to use gap reasoning and were not motivated to change their answer even after correctly placing the two fractions on the number line. A student, after placing 7/8 and 5/6 correctly on the number line said, "Might be 7/8 (the greatest), well it's kinda' equals so they both need 1(more piece) to reach 1(whole). Yeah, I say it's equal, but not on the number line because they are different."

Gap reasoning is powerful and often prevails over any reasoning related to comparing fractions based on their positions on the number line. A student in the pre-

interview was asked to compare 2/7 and 85/95, and he started with the correct answer, but talked himself out of a correct answer after resorting to gap reasoning.

> (The greater fraction is) 85/95, but I can't really draw that one (haha). (The fraction) 85/95 is greater because it has more pieces. Now in this case, it's ten away from being a whole and this (points to 3/7) is four away from being a whole, but in a sense this (85/95) is bigger which kind of makes it closer to being this (points to 95) because of how big it is. It makes it feel (moves arms to look like he's holding something big to show that it's more of a feeling than a fact) like it's closer to 95 being a whole… but this (3/7) is the greater fraction because there is a difference of four. Okay, it's ten… ten is greater, four is less which means this (3/7) is closer to being a whole and this (3/7) is greater.

Since gap reasoning is an unreliable strategy to compare fractions, developing residual reasoning is important. Residual reasoning is the concept of how many more pieces a fraction needs to complete the whole, and if two fractions both need the same number of pieces, then the two fractions can easily be compared once students attend to the size of the pieces needed to complete the whole. Unfortunately, instruction about residual reasoning easily perpetuates students' use of risky gap reasoning because residual reasoning requires consideration of the gap. In addition to considering the gap, students using residual reasoning to compare fractions must also recognize the comparison can only be made when each fraction needs an equal number of pieces to fill the gap, and then students have to consider the size of the pieces. Unfortunately, attention to the same number of missing pieces to complete the whole and consideration of the size of the pieces is easily lost on the students. Instead, students turn reliable residual reasoning into unreliable gap reasoning when comparing fractions. Gap reasoning only looks at the difference between the numerator and denominator. So when teaching about residual reasoning, students appear to walk away from the lesson thinking fractions can be compared by only considering the difference between the numerator and denominator.

Unfortunately, gap reasoning works more often than it does not because of the denominators students are often limited to both in textbooks and on state standards.

For instance, a fourth grader can compare the fraction 4/5 with any of the fractions less than one whole whose denominators are thirds, fourths, fifths, sixths, eighths, tenths, and twelfths, and be correct about 75% of the time (see Figure 7). Because gap reasoning often results in a correct answer, each time a student compares two fractions correctly using gap reasoning, the use of gap reasoning is maintained until it appears to become the most often used strategy for many students, including practicing teachers. In a professional development workshop on ratios and proportions, some teachers thought a 3 to 4 ratio of juice to water would be the same as a 2 to 3 ratio of juice to water since both numbers in each ratio had a difference of 1. This is another example of gap reasoning leading people to wrong conclusions.

Comparing Four-Fifths Using Gap Reasoning

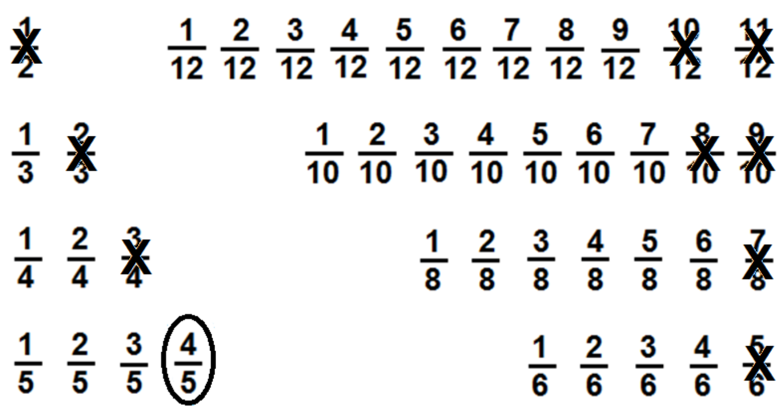

Figure 7. Gap reasoning correctly compares 4/5 to fractions not marked with X.

When asking students to compare two fractions on the posttest and the post-interview, students did not naturally seek out the number line. If students were writing out their reasoning, most students chose to draw a rectangular model to compare the two fractions. On the post-interview, the students were able to compare some of the fraction pairs using benchmarking strategies and number relationship strategies (such as knowing 2/4 and 7/14 were equivalent because the numerators were half the denominators), but when the comparisons became difficult, most students in both the treatment and control group returned to gap reasoning. Whole number reasoning was used only one time during the post-interview with the treatment group, so the results of the study showed success in moving students away from whole number reasoning, but unfortunately increased the reliance on gap reasoning when comparing fractions.

Conclusions

To the extent the subjects were representative of typical seventh graders experiencing difficulties in mathematics and based on the finding of this study, conclusions emerged from the research questions. These conclusions are described in the following paragraphs.

What types of strategies were students using to compare fractions? Initially, the comparison strategies students used to compare fractions when writing a response was to draw a picture of rectangles or circles. When verbalizing their reasoning, most students used whole number or gap reasoning. Drawing a picture, whole number reasoning, and gap reasoning are unreliable strategies to use when comparing fractions. Both whole number reasoning and gap reasoning don’t always lead to a correct answer. Drawing a

picture often leads to an incorrect result if students represent each fraction using a different size whole or if students draw unequal partitions within the shape. When students compared fractions on the pre- and posttest, drawing a shape was used 90% of the time. After the study, whole number reasoning to compare fractions was only used in 2% of the post-interview responses (see Table 4).Unfortunately, the students who were no longer using whole number reasoning started relying on gap reasoning.

After the study, what difficulties did students have in naming a fraction when represented with a variety of visual models? When naming a fraction, students had difficulties identifying the whole, creating equal partitions, and interpreting the number of equal parts it would take to make the whole if all equal portions were not delineated in a shape. Even when a correct numerator and denominator were named, students had difficulty coordinating these two numbers, often exchanging their positions or using whole number language when reading out loud a fraction like 3/5 as “three-five.”

What misconceptions with area models confounded students’ use of the number line model? With area models, the whole shape was typically perceived as the whole unit, but when students transferred this same idea to the number line, students viewed whatever part of the number line that could be seen as one whole, even if the number line went from 0-2 instead of 0-1. Also, when trying to represent a fractional amount with the area model, students would draw a shape, partition it, and if there were not enough pieces, students would incorrectly add more pieces to the whole. This incorrect partitioning carried over to the number line, but instead of adding more length to the number line, students just kept marking the intervals in the whole unit smaller and smaller, with no consideration of the importance of equal intervals when iterating. For

many students, there was no connection between partitioning the whole and iterating a unit fraction.

What needs to be included in the design of an instructional model to support students' ability to conceptualize a fraction's approximate location on a number line in order to determine the magnitude of the fraction and to develop fraction comparison strategies? In order to locate a fraction on the number line, students must be able to identify the whole, understand the importance of the whole, understand the purpose of the numerator and the denominator, determine the unit fraction, iterate that unit fraction, and know the point where they placed the fraction was measuring the fraction's distance from 0. Eventually, students could progress to a more procedural solution when representing a fraction on a number line by equally partitioning the whole, and then counting each interval using fraction notation. Explicit lessons need to be developed to help students articulate why whole number reasoning and gap reasoning are unreliable strategies to use when comparing fractions.

Recommendations for Future Study

Based on the finding of this study, recommendations for future study were made. These recommendations are described in the following paragraphs.

Pre-interview tasks have students not only compare two fractions, but also estimate each fraction's position on the number line. This will make the tasks in the pre-interview align with the tasks in the post-interview.

More students should participate in the pre- and post-interviews. It was during these interviews where research was most informed about student usage of

correct/incorrect fraction comparison strategies and where the greatest details of student-thinking were provided.

Tasks should be developed to help students iterate a unit fraction before students are partitioning the number line. A sample task might ask students to estimate the position of a unit fraction, measure the length of that unit fraction, and then iterate the estimated unit fraction to determine accuracy and to develop the notion of the number of iterations required to complete the whole. Tasks requiring the iteration of a unit fraction to complete the whole should be done prior to tasks where students partition the whole unit on a number line and then count the partitions.

Future instruction needs to move at a slower pace to allow for the conceptual development of estimating a fraction's position on the number line. Even though the efforts of the study were to delay procedural work with fractions, locating a fraction on the number line too quickly became a procedure of partitioning and counting. More time needed to be spent on estimating a unit fraction and iterating the unit fraction to complete the whole.

Tasks need to be developed with the singular purpose of helping students determine the unreliability of whole number reasoning and gap reasoning as fraction comparison strategies. This should help students avoid the use of unreliable strategies and instead consider using dependable fraction comparison strategies such as comparing to the benchmark fraction 1/2 or looking at equal number of residual intervals from a benchmark.

Future instruction should be more explicit on how to use the number line to compare fractions. Even though students were able to accurately estimate the position of

two fractions on the same number line, 3/4 of the students were unable to use the number line as a tool to compare the two fractions.

Future work with helping students use the number line to compare fractions should begin in 3-5th grade. The seventh graders from this study had already spent four years developing misunderstandings of fraction concepts, and relying on procedures with no meaning attached.

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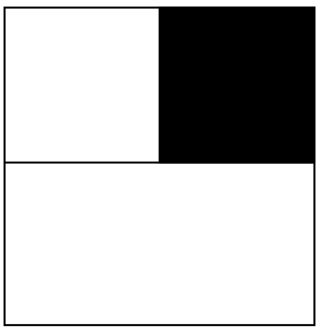
APPENDICES

Appendix A

Fraction Pre-Assessment Name______________________

1. Which of these fractions tells the part of this shape that is **shaded**?



a. 1/4
b. 1/3
c. 1/2
d. 3/4
e. I am not sure

2. For each pair of fractions, circle which of the fractions is greater? Explain your answer in words and show a **picture**.

A. $\frac{4}{5}$ or $\frac{5}{6}$

B. $\frac{2}{9}$ or $\frac{4}{9}$

C. $\frac{2}{8}$ or $\frac{2}{9}$

D. $\frac{4}{7}$ or $\frac{2}{5}$

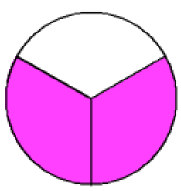
3.Compute:$\frac{2}{9} + \frac{5}{9}$

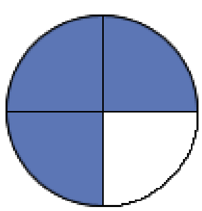
4. What number should replace the box to make this number sentence true?

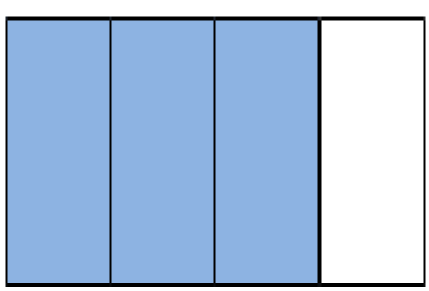
$$\frac{1}{3} = \frac{\square}{6}$$

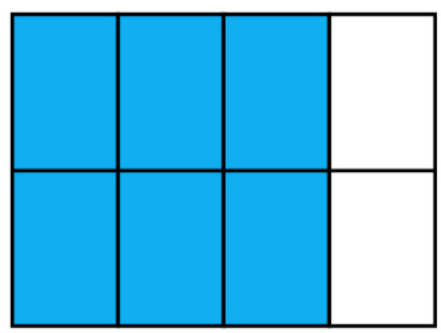
a. 1
b. 2
c. 3
d. 4

5. Circle which figure(s) shows 3/4 (there may be more than one correct answer)?

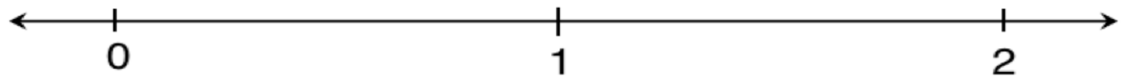
a. 

b. 

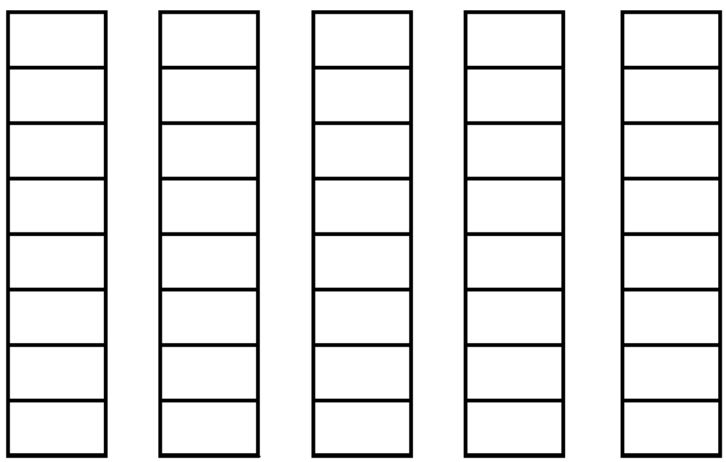
c. 

d. 

6. On the number line, mark and label: $\frac{15}{16}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, $\frac{5}{8}$, $\frac{4}{4}$, $\frac{6}{8}$, $\frac{6}{12}$, and $\frac{5}{4}$

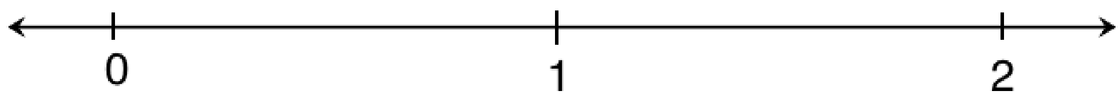


7. Shade these bars to represent $3\frac{3}{4}$



8. Which is greater, $\frac{3}{8}$ or $\frac{2}{5}$? WHY? Explain everything you can about these two fractions, and how you know which one is greater.

9. Mark and label all of the fifths on the following number line:



10. The zookeeper has 4 cups of frog food. His frog eats 1/3 cup of food each day. How long can he feed the frogs before the food runs out? (Explain your answer in words and show a picture.)

11. 8 people want to share 5 pizzas so that each person gets the same amount. How much would each person get? (Explain your answer in words and show a picture.)

12. Compute: 2 eighths + 5 eighths

13. Compute: $4 \div \frac{1}{3}$

Appendix B

Teacher Recording Sheet for Fraction Interviews

Teacher Recording Sheet Interview Protocol-Comparing Fractions	
Problem 1: Provide student with **The Fraction Pairs** worksheet. Cover up each pair and show only one fraction pair at a time. For each of the pairs of fractions, say the following: Point to the first pair of fractions and say, ***"Say these fractions out loud."*** After student names both fractions, say, ***"Circle the fraction that has the greatest value or place an equal sign between them if they name the same amount."*** After the student circles the greater fraction, ask, ***"How did you know that the fraction had the greatest value?" (or, "How did you know they named the same amount?")*** Go through this procedure for each pair of fractions on the Fraction Pairs worksheet. Students may draw pictures and use manipulatives to explain their reasoning.	
Each pair of fractions is used with a specific strategy in mind. Students may use different strategies than these. $\frac{2}{8}$ and $\frac{2}{5}$: Common numerator strategy $\frac{4}{9}$ and $\frac{2}{9}$: Common denominator	

strategy $\frac{4}{6}$ and $\frac{3}{8}$: Comparing to benchmark of $\frac{1}{2}$ $\frac{7}{8}$ and $\frac{9}{10}$: Each fraction is one unit fraction from a benchmark $\frac{4}{4}$ and 1: Students must understand these are equivalent.	

Appendix C

Fraction Cards for Number Line Pre-Interview

Use the cards with 0 and 1 to mark those locations on a number line. Then place the other fractions where they belong on the number line.

$\frac{3}{6}$	$\frac{7}{9}$	$\frac{2}{3}$	$\frac{1}{5}$
$\frac{12}{10}$	$\frac{9}{11}$	0	1
$\frac{6}{14}$	$\frac{62}{72}$	$\frac{12}{80}$	$\frac{8}{15}$

Appendix D

Post-Interview Questions

For each pair of fractions, students were first asked to determine which fraction was greater and then describe the strategy used to compare the fractions. Then students were asked to approximate the location of the two fractions on a given number line marked with 0 and 1.

$\frac{5}{9}$	$\frac{7}{9}$
$\frac{7}{14}$	$\frac{2}{4}$
$\frac{5}{8}$	$\frac{1}{2}$
$\frac{3}{8}$	$\frac{1}{2}$
$\frac{2}{3}$	$\frac{4}{10}$
$\frac{3}{5}$	$\frac{5}{3}$
$\frac{1}{4}$	$\frac{2}{5}$
$\frac{2}{8}$	$\frac{2}{5}$
$\frac{3}{7}$	$\frac{12}{90}$
$\frac{3}{7}$	$\frac{5}{8}$
$\frac{3}{7}$	$\frac{85}{95}$
$\frac{7}{8}$	$\frac{5}{6}$
$\frac{3}{4}$	$\frac{7}{9}$

Appendix E

Principal's Permission Letter

December 29, 2014

To Whom It May Concern:

On behalf of Reed Academy's students, faculty, and staff, let me state our overwhelming support of the work being done by the Missouri State Math Department with our most at-risk students. We are indeed fortunate to have such energetic life-long learners volunteering with our students on a weekly basis.

During the past semester our students participating in the project have been highly engaged, learning on a high level, and actually looking forward to math class!

I support the MSU Math Project without reservation. If I can be of further assistance, please let me know.

Sincerely,

Debbie Grega

Dr. Debbie Grega
Principal – Reed Academy

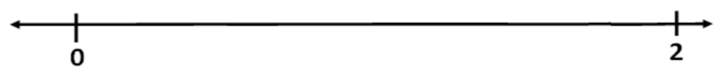
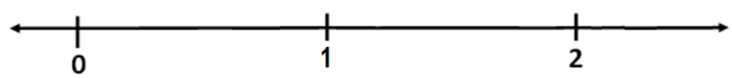
Appendix F

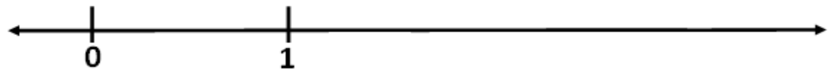
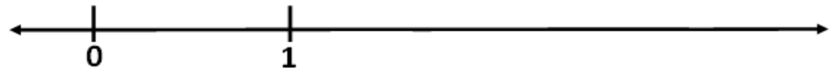
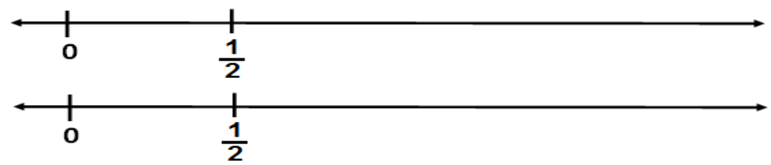
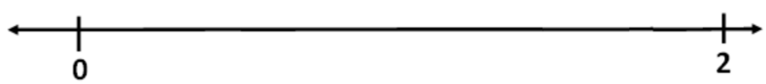
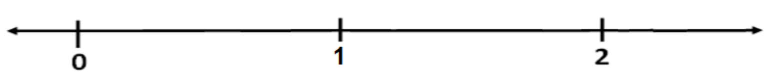
Overview of Lessons Developed by Instructors

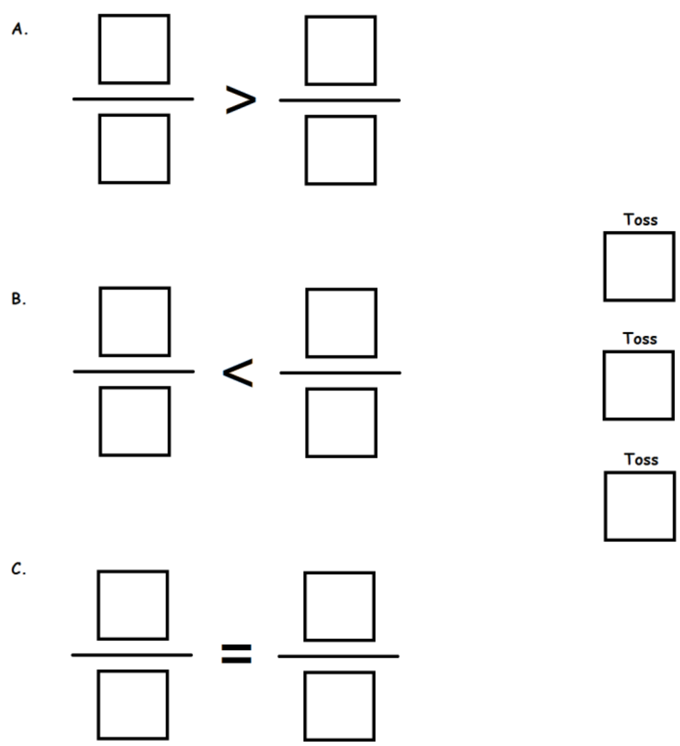
Sequence of Lessons for Reed 2015-2016

Date	Explanation of lesson
9/3	Pre-Test/ Interview/Pattern Block Activity
9/10	Paper Plate Lesson
9/17	Finish paper plate lesson
9/24	Folding Fraction Strips This time, when we folded fraction strips, we first just labeled using the area model, for instance 1/3, 1/3, 1/3
10/1	Finish Folding fraction strips
10/9	
10/16	Sharing Problems with Patrick
10/23	No class
10/30	Transitioning from Number Lines to Making Copies of Unit Fractions Engager: These Are and These Aren't to develop a way to describe "Unit Fraction". Started with 1/3 (These are) and 4/8 (These aren't). Students thought a unit fraction couldn't be equal to ½, so then I wrote ½ (these are) and 3/6 (these aren't). Student thought unit fractions had to have a numerator that was 3 or greater. So then I wrote 1/6 (these are) and 2/3 (these aren't). Students described unit fractions as a fraction that has a 1 in the numerator….that was all I wanted because I wanted to use that description as the attention signal. Lesson Part 1: Putting Fraction Cards in order from least to greatest. The idea of doing this first is to actually give students a reason for counting on the number lines and then going back to use the number lines to check to see if they got the fractions in order.We read the following prompt, and then students worked on getting fraction cards in order from least to greatest while math coach recorded notes about the strategies students were using. When students were finished ordering their cards, they wrote the fractions in order on the handout, and placed their fraction cards back in the Ziploc bags. During gym class, Mr. Miles had the students run the length of the playground. Mr. Miles planned to see how many seconds it took the students to travel the length of the playground.

	Mr. Miles blew the whistle for 9 students to begin running. Mr. Miles soon realized he had forgotten to start the stopwatch, so he blew the whistle and the 9 students stopped where they were. The fraction cards tell how much of the total distance each of the students had already traveled when Mr. Miles blew the whistle the second time. Place the fraction cards in order from least to greatest. After students had sufficient time to complete the fraction card sort, we brought them back together as a whole group and gave instructions about drawing and labeling numbers on the number line using only whole numbers. The idea was to introduce the term “Copies” of an interval and to write each whole number as a copy of the unit interval. For instance, underneath the number 4, we wrote 4 copies of 1 but agreed on using the notation of 4(1).
11/6	Finished counting on the back side of the number lines. Even though we wrote 5(1/8) and then wrote 5/8 on the fold, when students looked at the fold, some were calling the number 1/8 instead of 5/8.
11/9	Mrs. Russell worked with students on Monday on the task of categorizing fractions from their fraction strips as being less than 1/2, greater than 1/2, and equal to 1/2.
11/13	Place numbers on the number line for the Mr. Miller problem from two weeks ago. This time a number line is included for students to place fractions in order from least to greatest.  Who is Winning? During gym class, Mr. Miles had the students run the length of the playground. Mr. Miles planned to see how many seconds it took the students to travel the length of the playground. Mr. Miles blew the whistle for 9 students to begin running. Mr. Miles

	soon realized he had forgotten to start the stopwatch, so he blew the whistle and the 9 students stopped where they were. The fraction cards tell how much of the total distance each of the students had already traveled when Mr. Miles blew the whistle the second time. Place the fraction cards in order from least to greatest.
1/15	Students started the clothesline activity with twelfths only, and it was a disaster. Students just put them in numerical order with no attention to equal intervals unless prompted by the teacher.
1/22	Name______________________ Side A 1. Count by eighths on the number line below. Estimate where each tick mark belongs, being as precise as possible.  2. Place these numbers where they belong on the number line below: $\frac{5}{8}, \frac{5}{4}, \frac{2}{3}, \frac{1}{5}, \frac{9}{10}, \frac{6}{3}$  Students were unsuccessful with this task… must go back and practice counting on number line.
1/29	Short class period where we unsuccessfully tried to count on the number line Problems looked like this: **Counting by $\frac{1}{2}$ on the Number Lines**

	1.   2. 
2/5	Come on Six…. Practice counting by different unit fractions on the number line
2/12	Returned to the clothesline…. Only had students put some eighths on the number line Compared fractions with common numerator
2/26	Then worked with fractions that were less than half and greater than half Played war
3/1	Post test/Post Interview/Side B **Side B** 1. Count by eighths on the number line below. Estimate where each tick mark belongs, being as precise as possible.  3. Place these numbers where they belong on the number line below: $\frac{5}{8}, \frac{5}{4}, \frac{2}{3}, \frac{1}{5}, \frac{9}{10}, \frac{6}{3}$ 

3/4	Played games Dare to Compare and Rolling Something Close **Dare to Compare!** Your teacher will roll the number polyhedron 15 times. Each time a number is rolled, place the number in one of the squares below. You must place the number in a square before your teacher rolls the next number. Your task is to place numbers in the squares that will make each statement true. The Toss squares are for a rolled number that you cannot use. 
3/25	Dr. Sullivan prepared a sharing problem and related their work to all four operations.
4/1	A long time ago in a galaxy far far away Students used Cuisenaire rods as spaceships that could be partitioned equally into smaller parts. They searched for rod lengths that would partition equally several different ways….
4/8	Organized their work from last week and recorded sketches that showed the equal partitions for each rod.