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Solving Boundary Value Problems On Various Domains

Ibraheem Otuf

SOLVING BOUNDARY VALUE PROBLEMS ON VARIOUS DOMAINS

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

Domain-sensitivity is a hallmark in the realm of solving boundary value problems in partial differential equations. For example, the method used in solving a boundary value problem on an finite cylindrical domain is very different from one that arises from a rectangular domain. The difference is also reflected in the types of functions employed in the processes of solving these boundary value problems, as are the mathematical tools utilized in deriving an analytic solution. In this thesis, we solve an important class of partial differential equations with boundary conditions coming from various domains, such as the n dimensional cube, circles, and finite and infinite rectangles. We first enlist the functions and assemble the mathematical tools needed for the various domains. We then take the strategy of "divide-and-conquer" to solve the boundary value problems in a successive fashion. The main goal of solving these problems is to determine quantitatively how heat flow at any given time.

KEYWORDS: Fourier series, Fourier transform, boundary value problem, partial differential equation, Neumann problem, Laplace equation.

This abstract is approved as to form and content

Dr. Xingping Sun
Chairperson, Advisory Committee
Missouri State University

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Approved:

Dr. Sun Xingping

Dr. Shouchuan Hu

Dr. Matthew E. Wright

Dr. Julie J. Masterson, Graduate College Dean

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1. BASIC PROPERTIES OF ORTHOGONAL FUNCTIONS

Orthogonal special functions are efficient tools for solving boundary value problems on various domains. For example, Bessel functions naturally arise from solving differential equations on cylindrical domains. In Chapter 1, we introduce some orthogonal functions (Bessel functions and trigonometric functions) and summarize some of their basic properties. They will be used in later chapters.

1.1. Bessel's Equation

Bessel Functions arise as solutions of the differential equation,

$$x^2y^{''} + xy^{'} + (x^2 - \nu^2)y = 0, \quad \text{where } \nu \text{ is a constant.} \tag{1.1}$$

In this thesis, we are mostly concerned with the case in which ν is an integer. This equation was first studied by Daniel Bernoulli, and then generalized by Friedrich Bessel [7]. A fundamental solution set of Bessel functions consists of two functions whose Wronskian does not vanish. The first such function is the well-known Bessel function of the first kind of order ν.

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}, \; J_{-\nu} = J_\nu(x) \;\; \nu \geq 0.$$

Here:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

For a natural number n, we have $\Gamma(n) = (n-1)!$. The other function can be obtained first for non-integers ν:

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

For an integer order, Y_l is defined as in the following limit:

$$Y_l(x) = \lim_{\nu\to l} Y_\nu(x), \quad x \in (0,\infty).$$

In the mathematical literature the function $Y_\nu(x)$ are called Bessel function of the 2nd kind of order ν. If ν is a natural order, $J_\nu(0) = 0$ while $Y_\nu(0)$ is undefined. Indeed, $\lim_{x\to 0^+} Y_\nu(x) = -\infty$.

1.2. Some Properties of Bessel Functions

Here are some interesting properties of Bessel Functions, which all can be found in [4].

1. $J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x)$.
2. $J'_{\nu+1}(x) = \frac{1}{2}[J_{\nu-1}(x) - J_{\nu+1}(x)]$.
3. $J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x)$.
4. $J'_\nu(x) = J_{\nu-1}(x) - \frac{\nu}{x} J_\nu(x)$.

If we replace $J_\nu(x)$ by $Y_\nu(x)$, in the above identities, then the same identities still hold true. Namely, we have

1. $Y_{\nu+1}(x) = \frac{2\nu}{x} Y_\nu(x) - Y_{\nu-1}(x)$.
2. $Y'_{\nu+1}(x) = \frac{1}{2}[Y_{\nu-1}(x) - Y_{\nu+1}(x)]$.

3. $Y_\nu^{'}(x) = \frac{\nu}{x}Y_\nu(x) - Y_{\nu+1}(x)$.

4. $Y_\nu^{'}(x) = Y_{\nu-1}(x) - \frac{\nu}{x}Y_\nu(x)$.

1.3. Modified Bessel Functions

Modified Bessel functions are fundamental solutions of the differential equation:

$$x^2y^{''} + xy^{'} - (x^2 + \nu^2)y = 0.$$

Not surprisingly, this is referred to as the modified Bessel equation. To solve the above differential equation, we write it in the following form:

$$x^2y^{''} + xy^{'} + ((ix)^2 - \nu^2)y = 0,$$

in which i indicates the imaginary unit i, satisfying $i^2 = -1$. Bessel's equations and their solutions are valid for complex arguments, which allows us to carry out a simple change of variable, from x to ix, to solve the modified Bessel equation. We may formally write the solutions of the modified Bessel equation as

$$y(x) = c_1J_\nu(ix) + c_2Y_\nu(ix),$$

or

$$y(x) = c_3I_\nu(x) + c_4K_\nu(x).$$

Here $I_\nu(x)$, $K_\nu(x)$ are called, respectively, the modified Bessel functions of the first kind and the second kind of order v. By inspecting the series expansion for J_ν, we see that

$$I_\nu(0) = \begin{cases} 1 & \text{if} \quad v = 0, \\ 0 & \text{if} \quad v > 0. \end{cases}$$

However, the values of $K_\nu(0)$ are singular.

1.4. Some Properties of Modified Bessel Functions of the First Kind

In this thesis, we only use modified Bessel functions of the first kind. We list some of their useful properties as follows. These formulas can all be found in [4].

1. $I_{\nu+1}(x) = I_{\nu-1}(x) - \frac{2\nu}{x} I_\nu(x)$.
2. $I_\nu^{'}(x) = \frac{1}{2}[I_{\nu-1}(x) + I_{\nu+1}(x)]$.
3. $I_\nu^{'}(x) = \frac{\nu}{x} I_\nu(x) + I_{\nu+1}(x)$.
4. $I_\nu^{'}(x) = I_{\nu-1}(x) - \frac{\nu}{x} I_\nu(x)$.

1.5. Orthogonality of Bessel Functions of the First Kind

In the Bessel equation, if we pass the variable from x to λx, where λ is a constant, then the equation becomes

$$x^2 y^{''} + x y^{'} + (\lambda^2 x^2 - n^2) y = 0. \tag{5.2}$$

It can be directly verified that $y(x) = J_n(\lambda x)$ is a solution of the above equation. Let λ and k be two different constants. Let $u = J_n(\lambda x)$ and $v = J_n(kx)$ be respectively, solutions of the following two equations:

$$x^2 u^{''} + x u^{'} + (k^2 x^2 - n^2) u = 0; \tag{5.3}$$

$$x^2 v^{''} + x v^{'} + (k^2 x^2 - n^2) v = 0. \tag{5.4}$$

Now multiply (5.3) by v and (5.4) by u. Then subtract them. The result is

$$x^2(vu'' - uv'') + x(vu' - uv') = (k^2 - \lambda^2)x^2uv.$$

After divided by x, we can write the above as

$$\frac{d}{dx}[x(vu' - uv')] = (k^2 - \lambda^2)xuv.$$

We integrate on both sides of the above equation ignoring the integration constant to get

$$(k^2 - \lambda^2)\int xuvdx = x(vu' - uv'),$$

which implies

$$\int_0^1 xJ_n(\lambda x)J_n(kx)dx = \frac{kJ_n(k)J_n'(\lambda) - \lambda J_n(\lambda)J_n'(k)}{k^2 - \lambda^2}. \tag{5.5}$$

Thus, if λ and k are two different roots of J_n, then the above integral is zero. Let $k \to \lambda$ in Eq. (5.5). Using L'Hospital's Rule on the right side of the above equation. We get

$$\begin{aligned}\int_0^1 xJ_n^2(\lambda x)dx &= \lim_{k\to\lambda}\frac{\lambda J_n'(\lambda)J_n'(k) - J_n(k)J_n'(k) - uJ_n(\lambda)J_n'(k)}{2k} \\ &= \frac{\lambda[J_n'(\lambda)]^2 - J_n(\lambda)J_n'(\lambda) - \lambda J_n(\lambda)J_n''(\lambda)}{2\lambda}.\end{aligned} \tag{5.6}$$

Recall that J_n satisfy the Bessel equation

$$\lambda^2J_n''(\lambda) + \lambda J_n'(\lambda) + (\lambda^2 - n^2)J_n(\lambda) = 0.$$

Simplifying the right hand side of Eq. (5.6), we get

$$\int_0^1 xJ_n^2(\lambda x)dx = \frac{1}{2}\left[\left(J_n^{'}(\lambda x)\right)^2 + \left(1 - \frac{n^2}{\lambda^2}\right)J_n^2(\lambda)\right].$$

Upon proper normalization, $\sqrt{x}J_n^2(\lambda_l x)$ forms an orthonormal system for $L^2[0,1]$, where λ_1, λ_2, λ_3, $\ldots$ are zeros of J_n in ascending order. It can be shown that J_n has infinitely many roots. We refer readers to [7] for a proof of this important fact.

1.6. Classical Fourier Series

For $f \in L^2(-L, L)$, we can expand $f(x)$ in it's Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

in which the convergence is in the Banach space $L^2(-L, L)$. The coefficients $a_n(n \geq 0)$ and $b_n(n \geq 1)$ are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

The point-wise convergence of Fourier series to the function from which the series is developed has been extensively studied in the area of harmonic analysis; see Fourier Series, and Boundary Value Problems by Brown, James Ward, and Churchill, Ruel V.(2012) in [3]. In this thesis, we need the following sufficient condition. We call a function $f(x)$ defined on $[-L, L]$ piecewise-smooth, if there is an $n \in \mathbb{N}$, and a partition $-L = a_0 < a_1 < \ldots < a_n = L$ of the interval $[-L, L]$, such that $f(x)$ is in $C^1(a_i, a_{i+1})$ and

1. the two one-sided limits $f(a_i) = \lim_{x \to a_i^-} f(x)$ and $f(a_i) = \lim_{x \to a_i^+} f(x)$ $(1 \leq i \leq n-1)$ exist and are finite.

2. $\lim\limits_{x \to -L^+} f(x)$ and $\lim\limits_{x \to L^-} f(x)$ exist and are equal to, respectively, $f(-L)$ and $f(L)$.

THEOREM 1: If $f(x)$ is piecewise smooth on $[-L, L]$, then it's Fourier series converges to $f^*(x)$, where

$$f^*(x) = \begin{cases} f(x), & \text{if } x \neq a_i, \quad 1 < i < n-1, \\ \frac{f(a_i^-)+f(a_i^+)}{2}, & \text{if } x = a_i, \quad 1 < i < n-1. \end{cases}$$

Furthermore the converge is uniform on any compact subset of the open set $\cup_{i=0}^{n-1}(a_i, a_{i+1})$.

2. BOUNDARY VALUE PROBLEMS IN BOUNDED DOMAINS

There are mainly two kinds of boundary value conditions associated with solving a partial differential equation. If boundary value conditions are posed on the (unknown) function, then the system (the PDE and the corresponding boundary value conditions) is referred to as a Dirichlet problem; if boundary value conditions are posed on some lower order partial derivatives (of the unknown function), then the system is referred to as a Neumann problem. The phrase "Dirichlet problem" is named after Peter Gustav Dirichlet, and "Neumann problem" is named after Carl Neumann. A Neumann boundary value condition is also called the second type boundary value condition in the literature. We will extensively use the phrases "Neumann boundary value conditions" and "Neumann problems." While we focus on Laplace equations, the method can also be used to solve several other types of partial differential equations, such as the heat equations and the wave equations. The chapter is divided into five sections, in which we address, in a successive fashion, Neumann problem on rectangles, 3-D cubes, n-D cubes, circles, and cylinders. There are all Lipschitz domains. As such, if functions defined on the boundaries are piecewise smooth, then there exists a unique solution (up to an arbitrary constant) to a Neumann problem thus given. We refer readers to [3] for a proof of the uniqueness. The main mathematical tool and technique we will be using are Fourier series and separations of variables. Throughout this chapter, we use Δ to denote the Laplace operator. That is

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2},$$

where n is the dimension of the ambient space.

2.1. Neumann Problem for Rectangle

We consider a boundary value problem (BVP) on the rectangle $[0,a]\times[0,b]$. We will use the short hand notation

$$u_x = \frac{\partial u}{\partial x};\; u_y = \frac{\partial u}{\partial y}.$$

$$\triangle u = 0,$$

$$\begin{aligned} &u_x(0,y) = g(y), \quad u_x(a,y) = k(y), \quad &0 \;\leq\; y \;\leq\; b,\\ &u_y(x,0) = h(x), \quad u_y(x,b) = f(x), \quad &0 \;\leq\; x \;\leq\; a. \end{aligned} \tag{1.1}$$

To solve a BVP, we need into decompose the problem to four separate problems. After we solve each one, we add them together. We write the four problems as follows.

$$\triangle u = 0,$$

$$\begin{aligned} &u_x(0,y) = 0, u_x(a,y) = 0, \quad &0 \;\leq\; y \;\leq\; b,\\ &u_y(x,0) = 0, u_y(x,b) = f(x), \quad &0 \;\leq\; x \;\leq\; a. \end{aligned} \tag{1.2}$$

$$\triangle u = 0,$$

$$\begin{aligned} &u_x(0,y) = 0, u_x(a,y) = k(y), \quad &0 \;\leq\; y \;\leq\; b,\\ &u_y(x,0) = 0, u_y(x,b) = 0, \quad &0 \;\leq\; x \;\leq\; a. \end{aligned} \tag{1.3}$$

$$\triangle u = 0,$$

$$u_x(0,y)=0, u_x(a,y)=0, \qquad 0 \le y \le b,$$

$$u_y(x,0)=h(x), u_y(x,b)=0, \qquad 0 \le x \le a. \tag{1.4}$$

$$\triangle u = 0,$$

$$u_x(0,y)=g(y), u_x(a,y)=0, \qquad 0 \le y \le b,$$

$$u_y(x,0)=0, u_y(x,b)=0, \qquad 0 \le x \le a. \tag{1.5}$$

To solve the BVP in (1.2), we use separation of variables. Let $u(x,y)=X(x)Y(y)$, then

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Note that the above equation holds true for all $0 \le x \le a$ and $0 \le y \le b$, and that x and y are independent variables. This shows that values on both sides must be constants. Let

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = \lambda.$$

We get two ordinary differential equations

$$X'' + \lambda X = 0, \tag{1.6}$$

$$Y'' - \lambda Y = 0. \tag{1.7}$$

A nontrivial solution of the above equation exist for some particular values of λ, which we call "eigenvalues" of the differential equation. The corresponding solutions are called eigenfunctions. See [1, 2] for more details about eigenvalues and eigenfunction. We will first solve the following ordinary differential equation: $X''+\lambda X=0$, with $X'(0)=0, X'(a)=0$. Let $\lambda=-u^2$. Then we get $X''-u^2X=0$, which has solutions of the form:

$$X(x) = c_1 \cosh(ux) + c_2 \sinh(ux).$$

Enforcing the initial value conditions, we have

$$X'(x) = c_1 u \sinh(ux) + u c_2 \cosh(ux),$$

$$X'(0) = 0 \Rightarrow \ uc_2 = 0 \ \Rightarrow \ c_2 = 0,$$

$$X'(a) = 0 \Rightarrow \ uc_1 \sinh(ux) = 0 \Rightarrow \ c_1 = 0.$$

This means that there is no negative eigenvalue. Now let $\lambda = 0$. Then, we have $X'' = 0$. It follows that $X(x) = cx + d$, which gives $X'(x) = c$, hence $X'(0) = 0$ implies $c = 0$. We conclude that $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfuction $X_0(x) = 1$. Finally, let $\lambda = u^2$. Then we have $X'' + u^2 X = 0$, which has solutions $X(x) = A_n \cos(ux) + B_n \sin(ux)$. It follows that $X'(x) = -A_n u \sin(ux) + B_n u \cos(ux)$. That $X'(0) = 0$ implies that $B_n u = 0$. Thus, $B_n = 0$. Furthermore, that $X'(a) = 0$ implies that $-au \sin(ua) = 0$. We conclude that $\sin(ua) = 0$ which implies $ua = n\pi$, and $u = \frac{n\pi}{a}$. Therefore

$$\lambda_n = {u_n}^2 = \left(\frac{n\pi}{a}\right)^2, \ X_n(x) = \cos(\frac{n\pi x}{a}).$$

Corresponding to the case $\lambda_0 = 0$, the ordinary differential equation on the variable y is:

$$Y^{''}(y) = 0 \quad , \quad Y'(0) = 0,$$

which has solutions

$$Y(y) = c_1 y + c_2$$

Since $Y'(0) = 0$, we have $c_1 = 0$. Thus, $Y_0 = 1$. Corresponding to the cases $n > 0$, we have the ordinary differential equation on the variable y :

$$Y^{''} - \lambda Y = 0,$$

which has solutions:

$$Y(y) = A_n \cosh(\frac{n\pi}{a}) + B_n \sinh(\frac{n\pi}{a}).$$

Therefore,

$$Y^{'}(y) = A_n \sinh(\frac{n\pi}{a}) + B_n \cosh(\frac{n\pi}{a}).$$

Applying the initial value conditions, we get

$$Y^{'}(0) = 0,$$

which gives $B_n = 0$. Thus we have

$$Y_n(y) = A_n \cosh(\frac{n\pi y}{a}).$$

Hence

$$u(x, y) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{a}) \cosh(\frac{n\pi y}{a}). \tag{1.8}$$

Since $u_y(x, b) = f(x)$, we have

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(\frac{n\pi x}{a}) \sinh(\frac{n\pi b}{a})(\frac{n\pi}{a}).$$

Thus

$$A_n = \frac{2}{n\pi} \frac{1}{\sinh(\frac{n\pi b}{a})} \int_0^a f(x) \cos(\frac{n\pi x}{a}) dx, \quad n > 0.$$

When $n = 0$, we have $A_0 = 0$. Hence

$$u(x, y) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{a}) \cosh(\frac{n\pi y}{a}).$$

We have completed the solutions of the BVP as indicated in Equation (1.2). Next, we use a similar process to solve BVP (1.3). Let

$$u(x, y) = X(x)Y(y).$$

We get $Y'' + \lambda Y = 0$, and $X'' - \lambda X = 0$. Hence, we conclude that $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfuction $Y_0(y) = 1$, and $X_0(x) = 1$. Moreover, $\lambda_m = (\frac{m\pi}{b})^2$ is the eigenvalue with corresponding eigenfuction $Y_n(m) = \cos(\frac{m\pi y}{b})$, Therefore,

$$X_m(x) = A_m \cosh(\frac{m\pi x}{b}).$$

Hence

$$u(x, y) = \sum_{m=0}^{\infty} B_m \cos(\frac{m\pi y}{b}) \cosh(\frac{m\pi x}{b}).$$

Since

$$u_x(a, y) = k(y),$$

$$k(y) = \sum_{m=0}^{\infty} B_m \cos(\frac{m\pi y}{b}) \sinh(\frac{m\pi a}{b})(\frac{m\pi}{b}),$$

we have

$$B_m = \frac{2}{m\pi} \frac{1}{\sinh(\frac{m\pi a}{b})} \int_0^b k(y) \cos(\frac{m\pi y}{b}) dy, \quad n > 0.$$

When $n = 0$, we have $B_0 = 0$. Thus

$$u(x, y) = \sum_{n=1}^{\infty} B_m \cos(\frac{m\pi x}{b}) \cosh(\frac{m\pi y}{b}).$$

We have thus completed the process of solving BVP (1.3). We now move on to solve BVP (1.4). The boundary value condition implies $X^{'}(0) = X^{'}(a) = Y^{'}(b) = 0$. Then we get the following $X'' + \lambda X = 0$ with $X^{'}(0) = 0, X^{'}(a) = 0$, and $Y'' - \lambda Y = 0$ with $Y^{'}(b) = 0$. After solving those two ordinary differential equations, we get $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfuction $X_0(x) = 1$, and for $\lambda_n = \left(\frac{n\pi}{a}\right)^2$, we have

$$X_n(x) = \cos(\frac{n\pi x}{a}).$$

The equation $Y'' - \lambda Y = 0$ with $Y^{'}(b) = 0$ gives $Y_0 = 1$, when $\lambda_0 = 0$. For $\lambda_n = (\frac{n\pi}{a})^2$, we have

$$Y_n(y) = \cosh \frac{(n\pi(b-y))}{a}.$$

Hence

$$u(x, y) = \sum_{n=0}^{\infty} C_n \cos(\frac{n\pi x}{a}) \cosh \frac{(n\pi(b-y))}{a}. \tag{1.9}$$

Since $u_y(x, 0) = h(x)$, we have

$$h(x) = -\sum_{n=0}^{\infty} C_n \cos(\frac{n\pi x}{a}) \sinh(\frac{n\pi b}{a})(\frac{n\pi}{a}).$$

Also, we have

$$C_n = -\frac{2}{n\pi} \frac{1}{\sinh(\frac{n\pi b}{a})} \int_0^a h(x) \cos(\frac{n\pi x}{a}) dx.$$

When $n = 0$, we have $C_0 = 0$.

$$u(x, y) = \sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{a}) \cosh \frac{(n\pi(b-y))}{a}.$$

This completes solving BVP (1.4). Finally, we embark on solving the fourth one referred to as (1.5). The boundary value condition gives $X^{'}(a) = Y^{'}(0) = Y^{'}(b) = 0$, then we get the following $Y^{''} + \lambda Y = 0$ with $Y^{'}(0) = 0, Y^{'}(b) = 0$, and $X^{''} - \lambda X = 0$ with $X^{'}(a) = 0$. We see that $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfunction $Y_0(y) = 1$, and for $\lambda_m = (\frac{m\pi}{b})^2$, we have $Y_m(y) = \cos(\frac{m\pi y}{b})$. For the equation $X'' - \lambda X = 0$ with $X'(a) = 0$, we get $X_0 = 1$ when $\lambda_0 = 0$, and for $\lambda_m = (\frac{m\pi}{b})^2$, we have

$$X_m(x) = \cosh\frac{(m\pi(a-x))}{b}.$$

Hence

$$u(x,y) = \sum_{m=0}^{\infty} D_m \cos(\frac{m\pi y}{b})\cosh\frac{(m\pi(a-x))}{b}. \tag{1.10}$$

Since $u_x(0,y) = g(y)$, we have

$$g(y) = -\sum_{m=0}^{\infty} D_m \cos(\frac{m\pi y}{b})\sinh(\frac{m\pi a}{b})(\frac{m\pi}{b}).$$

$$D_m = -\frac{2}{m\pi}\frac{1}{\sinh(\frac{m\pi a}{b})}\int_0^b g(y)\cos(\frac{m\pi y}{b})dy.$$

When $m = 0$, we impose the condition $D_0 = 0$ to get

$$u(x,y) = \sum_{m=1}^{\infty} D_m \cos(\frac{m\pi y}{b})\cosh\frac{(m\pi(a-x))}{b}.$$

This completes solving the fourth BVP (1.5). Adding all the four solutions together, we have:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi x}{a})\cosh(\frac{n\pi y}{a})$$

$$+\sum_{m=1}^{\infty} B_m \cos(\frac{m\pi y}{b})\cosh(\frac{m\pi x}{b})$$

$$+\sum_{n=1}^{\infty} C_n \cos(\frac{n\pi x}{a})\cosh\frac{(n\pi(b-y))}{a}$$

$$+\sum_{m=1}^{\infty} D_m \cos(\frac{m\pi y}{b}) \cosh \frac{(m\pi(a-x))}{b},$$

where the coefficients are,

$$A_n = \frac{2}{n\pi} \frac{1}{\sinh(\frac{n\pi b}{a})} \int_0^a f(x) \cos(\frac{n\pi x}{a}) dx;$$

$$B_m = \frac{2}{m\pi} \frac{1}{\sinh(\frac{m\pi a}{b})} \int_0^b k(y) \cos(\frac{m\pi y}{b}) dy;$$

$$C_n = -\frac{2}{n\pi} \frac{1}{\sinh(\frac{n\pi b}{a})} \int_0^a h(x) \cos(\frac{n\pi x}{a}) dx;$$

$$D_m = -\frac{2}{m\pi} \frac{1}{\sinh(\frac{m\pi a}{b})} \int_0^b g(y) \cos(\frac{m\pi y}{b}) dy.$$

2.2. Neumann Problem on Cubes

This section is devoted to solving Neumann problem on a three dimensional cube $[0, a] \times [0, b] \times [0, c]$. The solution process is in principle similar to a BVP on a rectangle. However, there are technical details that call for attentions. As such, we write down the whole process in its entirety. We first write down the BVP we are sovling:

$$\triangle u = 0,$$

$$\begin{aligned} &u_y(x, 0, z) = g(x, z), \quad u_y(x, b, z) = t(x, z), \quad 0 < y < b, \\ &u_z(x, y, 0) = k(x, y), \quad u_z(x, y, c) = j(x, y), \quad 0 < z < c, \\ &u_x(0, y, z) = s(y, z), \quad u_x(a, y, z) = f(y, z), \quad 0 < x < a. \end{aligned} \tag{2.11}$$

To solve this problem, we need to break it into six separate problems. After we solve each one, we add them together. Let's state the six problems:

$$\triangle u = 0,$$

$$\begin{aligned} u_y(x,0,z) &= 0, & u_y(x,b,z) &= 0, \\ u_z(x,y,0) &= 0, & u_z(x,y,c) &= 0, \\ u_x(0,y,z) &= 0, & u_x(a,y,z) &= f(y,z). \end{aligned} \tag{2.12}$$

$$\triangle u = 0,$$

$$\begin{aligned} u_y(x,0,z) &= 0, & u_y(x,b,z) &= 0, \\ u_z(x,y,0) &= 0, & u_z(x,y,c) &= j(x,y), \\ u_x(0,y,z) &= 0, & u_x(a,y,z) &= 0. \end{aligned} \tag{2.13}$$

$$\triangle u = 0,$$

$$\begin{aligned} u_y(x,0,z) &= 0, & u_y(x,b,z) &= t(x,z), \\ u_z(x,y,0) &= 0, & u_z(x,y,c) &= 0, \\ u_x(0,y,z) &= 0, & u_x(a,y,z) &= 0. \end{aligned} \tag{2.14}$$

$$\triangle u = 0,$$

$$
\begin{aligned}
&u_y(x,0,z)=0, && u_y(x,b,z)=0,\\
&u_z(x,y,0)=0, && u_z(x,y,c)=0,\\
&u_x(0,y,z)=s(y,z), && u_x(a,y,z)=0.
\end{aligned} \tag{2.15}
$$

$$\triangle u=0,$$

$$
\begin{aligned}
&u_y(x,0,z)=0, && u_y(x,b,z)=0,\\
&u_z(x,y,0)=k(x,y), && u_z(x,y,c)=0,\\
&u_x(0,y,z)=0, && u_x(a,y,z)=0.
\end{aligned} \tag{2.16}
$$

$$\triangle u=0,$$

$$
\begin{aligned}
&u_y(x,0,z)=g(x,z), && u_y(x,b,z)=0,\\
&u_z(x,y,0)=0, && u_z(x,y,c)=0,\\
&u_x(0,y,z)=0, && u_x(a,y,z)=0.
\end{aligned} \tag{2.17}
$$

We first solve (2.12). Again we use separation of variable to write

$$u(x,y,z)=X(x)Y(y)Z(z).$$

The boundary condition implies that

$$X'(0)=Y'(0)=Y'(b)=Z'(0)=Z'(c)=0.$$

We get that

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}=0.$$

Note that x, y, z are independent variables. Hence then exists a constant k, such that

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\frac{Z''(z)}{Z(z)} = k.$$

It follows that

$$Z'' + kZ = 0.$$

Let

$$\frac{X''(x)}{X(x)} - k = -\frac{Y''(y)}{Y(y)} = \lambda.$$

We conclude that $Y'' + \lambda Y = 0$ and $X'' + (\lambda + k)X = 0$. Now we have three ordinary differential equations. To find the right eigenvalues, we solve the following two equations: $Z'' + kZ = 0$ and $Y'' + \lambda Y = 0$. For $Z'' + kZ = 0$, we get , $k_0 = 0$ is the eigenvalue with corresponding eigenfunction $Z_0(z) = 1$ and $k_m = (\frac{m\pi}{c})^2$ with corresponding eigenfunction

$$Z_m(z) = \cos(\frac{m\pi z}{c}).$$

For $Y'' + \lambda Y = 0$, we see that $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfunction $Y_0(y) = 1$. For $\lambda_n = (\frac{n\pi}{b})^2$ we have eigenfunction

$$Y_n(y) = \cos(\frac{n\pi y}{b}).$$

After solving the following $X'' - (\lambda + k)X = 0$ with $X^{'}(0) = 0$, we get

$$X_{mn}(x) = A_{mn}\sinh(\sqrt{(\frac{m\pi}{c})^2 + (\frac{n\pi}{b})^2})x + B_{mn}\cosh(\sqrt{(\frac{m\pi}{c})^2 + (\frac{n\pi}{b})^2})x.$$

Let

$$R_{m,n} = \sqrt{(\frac{m\pi}{c})^2 + (\frac{n\pi}{b})^2}.$$

Then

$$X'_{mn}(x) = R_{m,n}A_{mn}\cosh(R_{m,n})x + R_{m,n}B_{mn}\sinh(R_{m,n}x).$$

Since $X'(0) = 0$, we have $A_{mn} = 0$. Hence

$$u(x, y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{mn}\cosh(R_{m,n}x)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}).$$

Since $u_x(a, y, z) = f(y, z)$, we have

$$f(y, z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} R_{m,n}B_{mn}\sinh(R_{m,n}a)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}).$$

The coefficient is

$$B_{mn} = \frac{4}{bcR_{m,n}\sinh(R_{m,n}a)}\int_0^b\int_0^c f(y, z)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c})dydz,$$

for $n > 0$, $m > 0$. When $n = m = 0$, we have $B_{00} = 0$. Thus

$$u(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\cosh(R_{m,n}x)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}),$$

Similarly, we find the solution of (2.13) in the following:

$$u(x, y, z) = \sum_{n=1}^{\infty}\sum_{h=1}^{\infty} C_{nh}\cosh(R_{n,h})z)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a}).$$

where

$$C_{nh} = \frac{4}{abR_{n,h}\sinh(R_{n,h}c)}\int_0^a\int_0^b j(y, z)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a})dxdy,$$

for $n > 0$, $h > 0$. Going through a similar process, we find the solution of (2.14) in the following:

$$u(x,y,z)=\sum_{m=1}^{\infty}\sum_{h=1}^{\infty}A_{mh}\cosh(R_{m,h}y)\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c}),$$

where the coefficients are

$$A_{hm}=\frac{4}{acR_{m,h}\sinh(R_{m,h}b)}\int_0^a\int_0^c t(y,z)\cos(\frac{h\pi y}{a})\cos(\frac{m\pi z}{c})dxdz,$$

for $h>0$, $m>0$.

Next, we solve (2.15). Let

$$u(x,y,z)=X(x)Y(y)Z(z).$$

The boundary condition implies that

$$X'(a)=Y'(0)=Y'(b)=Z'(0)=Z'(c)=0.$$

We get that

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}+\frac{Z''(z)}{Z(z)}=0.$$

Assume

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=-\frac{Z''(z)}{Z(z)}=k.$$

Then

$$Z''+kZ=0.$$

Let

$$\frac{X''(x)}{X(x)}-k=-\frac{Y''(y)}{Y(y)}=\lambda.$$

We conclude that $Y''+\lambda Y=0$, $X''-(\lambda+k)X=0$. Now we have three ordinary differential equation after we solve $Y''+\lambda Y=0$, and $Z''+kZ=0$. The (BVP) implies that, for $Y'' + \lambda Y = 0$, we get $\lambda_0 = 0$ is the eigenvalue with corresponding eigenfunction $Y_0(y) = 1$, and $\lambda_n = (\frac{n\pi}{b})^2$ is the eigenvalue with corresponding eigenfunction

$$Y_n(y) = \cos(\frac{n\pi y}{b}).$$

For $Z'' + kZ = 0$, we get $K_0 = 0$ is the eigenvalue with corresponding eigenfunction $Z_0(x) = 1$, and $K_m = (\frac{m\pi}{c})^2$ is the eigenvalue with corresponding eigenfunction

$$Z_m(z) = \cos(\frac{m\pi z}{c}).$$

Now, let us solve

$$X'' + (\lambda + k)X = 0$$

with $X^{'}(a) = 0$, we get

$$X_{mn} = D_{mn}\sinh(\sqrt{(\frac{m\pi}{c})^2 + (\frac{n\pi}{b})^2}\ x) + H_{mn}\cosh(\sqrt{(\frac{m\pi}{c})^2 + (\frac{n\pi}{b})^2}\ x),$$

and

$$X^{'}_{mn}(x) = R_{m,n}D_{mn}\cosh(R_{m,n}x) + R_{m,n}H_{mn}\sinh(R_{m,n}x).$$

Since $X^{'}(a) = 0$, we have

$$X^{'}_{mn}(a) = R_{m,n}D_{mn}\cosh(R_{m,n}a) + R_{m,n}H_{mn}\sinh(R_{m,n}x).$$

Hence

$$X_{mn}(x) = \cosh(R_{m,n}(a - x)).$$

We conclude that

$$u(x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn}\cosh(R_{m,n}(a - x))\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}).$$

Since $u_x(0,y,z) = s(y,z)$, we have

$$s(y,z) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(R_{m,n})D_{mn}\sinh(R_{m,n}(a))\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}),$$

$$D_{mn} = -\frac{4}{bcR_{m,n}\sinh(R_{m,n}a)}\int_0^b\int_0^c s(y,z)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c})dydz,$$

for $n > 0$, $m > 0$. When $n = m = 0$, we have $D_{00} = 0$. Hence

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\cosh(R_{m,n}(a-x))\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}).$$

The processes for solving the fifth BVP (2.16) and the sixth one (2.17) are similar to the fourth one. The result for the fifth one is:

$$u(x,y,z) = \sum_{h=1}^{\infty}\sum_{n=1}^{\infty} E_{hn}\cosh(R_{h,n}(c-z))\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a}),$$

with coefficients

$$E_{hn} = -\frac{4}{abR_{h,n}\sinh(R_{h,n}c)}\int_0^a\int_0^b k(x,y)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a})dxdz,$$

for $n > 0$, $h > 0$. The result for the sixth one (2.17) is:

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{h=1}^{\infty} P_{mh}\cosh(R_{m,h}(b-y))\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c}),$$

with coefficients

$$P_{mh} = -\frac{4}{acR_{m,h}\sinh(R_{m,h}b)}\int_0^a\int_0^c g(x,z)\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c})dxdz,$$

for $h > 0$, $m > 0$. Hence the final solution for BVP (2.11), it is the sum of all the solutions as following

$$
\begin{aligned}
u(x,y,z) = &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{mn}\cosh(R_{m,n}x)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}) \\
&+\sum_{n=1}^{\infty}\sum_{h=1}^{\infty} C_{nh}\cosh(R_{n,h}z)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a}) \\
&+\sum_{m=1}^{\infty}\sum_{h=1}^{\infty} A_{mh}\cosh(R_{m,h}y)\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c}) \\
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\cosh(R_{m,n}(a-x))\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c}) \\
&+\sum_{h=1}^{\infty}\sum_{n=1}^{\infty} E_{hn}\cosh(R_{h,n}(c-z))\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a}) \\
&+\sum_{m=1}^{\infty}\sum_{h=1}^{\infty} P_{mh}\cosh(R_{m,h}(b-y))\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c})
\end{aligned}
$$

where the coefficients are

$$
\begin{aligned}
B_{mn} =&\frac{4}{bcR_{m,n}\sinh(R_{m,n}a)}\int_0^b\int_0^c f(y,z)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c})dydz; \\
C_{nh} =&\frac{4}{abR_{n,h}\sinh(R_{n,h}c)}\int_0^a\int_0^b j(y,z)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a})dxdy; \\
A_{hm} =&\frac{4}{acR_{h,m}\sinh(R_{h,m}b)}\int_0^a\int_0^c t(y,z)\cos(\frac{h\pi y}{a})\cos(\frac{m\pi z}{c})dxdz;
\end{aligned}
$$

$$D_{mn} = -\frac{4}{bcR_{m,n}\sinh(R_{m,n}a)}\int_0^b\int_0^c s(y,z)\cos(\frac{n\pi y}{b})\cos(\frac{m\pi z}{c})dydz;$$

$$E_{hn} = -\frac{4}{abR_{h,n}\sinh(R_{h,n}c)}\int_0^a\int_0^b k(x,y)\cos(\frac{n\pi y}{b})\cos(\frac{h\pi x}{a})dxdz;$$

$$P_{mh} = -\frac{4}{acR_{m,h}\sinh(R_{m,h}b)}\int_0^a\int_0^c g(x,z)\cos(\frac{h\pi x}{a})\cos(\frac{m\pi z}{c})dxdz.$$

$n > 0, \; m > 0, \; h > 0$.

2.3. Neumann Problem on n-d Cubes

In this section, we study Neumann problem on $\Omega_n = [0, L]^n$, $n \geq 2$, where $L > 0$ is fixed. We express the boundary of Ω_n as follows.

$$\Omega_{n,i}^{(0)} = \Omega_n|_{x_i=0}; \quad \Omega_{n,i}^{(L)} = \Omega_n|_{x_i=L}; \quad 1 \leq i \leq n.$$

We will use the notations

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad 1 \leq i \leq n.$$

The boundary value problem we study can be stated as follows.

$$\begin{aligned}
&\triangle u = 0,\\
&u_{x_i}(x_1, ..., x_{i-1}, 0, x_{i+1} ..., x_n) = f_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),\\
&(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \Omega_{n,i}^{(0)},\\
&u_{x_i}(x_1, ..., x_{i-1}, L, x_{i+1}, ..., x_n) = g_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n),\\
&(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \Omega_{n,i}^{(L)}.
\end{aligned}$$

The eigenvalues are $\lambda_{m_k} = (\frac{m_k \pi x_k}{L})^2$ for $k = 1, 2, \ldots n$, with their corresponding eigenfunctions $\cos(\frac{m_k \pi x_k}{L})$. Let

$$R_m = \sqrt{\sum_{i=1}^{n} (\frac{m_i \pi}{L})^2},$$

and let

$$\prod_{k \neq i} \sum_{m_k=1}^{\infty} = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_{i-1}=1}^{\infty} \sum_{m_{i+1}=1}^{\infty} \cdots \sum_{m_n=1}^{\infty}, \quad k \neq i.$$

This problem has general solution of the form:

$$\begin{aligned} & u(x_1, ...x_{i-1}, x_i, x_{i+1}..., x_n) \\ = & \sum_{i=1}^{n} u_i(x_1, ..., x_{i-1}, x_i, x_{i+1}..., x_n), \end{aligned}$$

where for each i, we have

$$\begin{aligned} & u_i(x_1, ...x_{i-1}, x_i, x_{i+1}..., x_n) \\ = & \prod_{k \neq i} \sum_{m_k=1}^{\infty} \prod_{k \neq i} \cos(\frac{m_k \pi x_k}{L})[B_i^* \ \cosh(R_m x_i) + C_i^* \cosh(R_m(L - x_i))]. \end{aligned}$$

The coefficients are as follows.

$$B_i^* = B_{m_1, \ldots, m_{i-1} m_{i+1}, \ldots, m_n}.$$

$$C_i^* = C_{m_1, \ldots, m_{i-1} m_{i+1}, \ldots, m_n}.$$

$$u_{x_i}(x_1, ..., x_{i-1}, L, x_{i+1}, ..., x_n) = f_i(x_1, ..., x_{i-1} x_{i+1}, ..., x_n).$$

$$u_{x_i}(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) = g_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Closed forms of the coefficients can be written as

$$B_i^* = \frac{2^{n-1}}{L^{n-1} R_m \sinh(R_m L)} \int_0^L ... \int_0^L \prod_{k \neq i} \cos(\frac{m_k \pi x_k}{L})\ f_i\ dx_1 ... dx_{i-1} dx_{i+1} ... dx_n.$$

$$C_i^* = -\frac{2^{n-1}}{L^{n-1} R_m \sinh(R_m L)} \int_0^L ... \int_0^L \prod_{k \neq i} \cos(\frac{m_k \pi x_k}{L})\ g_i\ dx_1 ... dx_{i-1} dx_{i+1} ... dx_n.$$

2.4. Neumann Problem for Circle

The circular nature of the boundary naturally leads us to use polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$. The relations

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

are also very useful in the transition from rectangular coordinates to polor coordinates. By using the chain rule, we find that Laplacian operator in rectangular coordinates

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

becomes

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Now, consider the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$
$$u_r(a,\theta) = f(\theta), \quad 0 < r < a, \quad 0 \ \leq \ \theta < 2\pi.$$

These are normal derivatives on the circle $r = a$. Let

$$u(r,\theta) = R(r)\Theta(\theta).$$

Then

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0,$$

or

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0.$$

Suppose

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.$$

Then, we get two ordinary differential equations:

$$r^2R'' + rR' - \lambda R = 0,$$

$$\Theta^{''} + \lambda\Theta = 0.$$

There is no homogeneous boundary condition, so the solution must be bounded and periodic in θ with period 2π. Let's start by solving

$$\Theta^{''} + \lambda\Theta = 0.$$

First case, if $\lambda < 0$, let $\lambda = -u^2$ then the equation becomes

$$\Theta^{''} - u^2\Theta = 0.$$

This equation has solution of the form:

$$\Theta(\theta) = Ae^{u\theta} + Be^{-u\theta}.$$

Since $\Theta(\theta)$ is periodic only if $A = B = 0$, λ cannot be negative.

Second case, if $\lambda = 0$, then $\Theta''(\theta) = 0$. Hence

$$\Theta(\theta) = A + B\theta.$$

In this case the solution can be periodic only if $B = 0$, so that $\Theta(\theta)$ is constant.

Third case, $\lambda > 0$ let $\lambda = u^2$, then the equation becomes

$$\Theta'' + u^2\Theta = 0.$$

After we solve this equation, we get

$$\Theta = a\cos(u\theta) + b\sin(u\theta).$$

Note that Θ must be periodic, so we have

$$\Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta),$$

for some $n = 0, 1, 2, \ldots$. Next, we have

$$r^2R'' + rR' - n^2R = 0.$$

This is an Euler equation, let $r = e^z$. We get $\frac{d^2R}{dz^2} - n^2R = 0$. Hence

$$R(z) = \begin{cases} c_1 + c_2 z, & \text{if } n = 0, \\ c_1 e^{nz} + c_2 e^{-nz}, & \text{if } n = 1, 2, \ldots. \end{cases}$$

Therefore,

$$R(r) = \begin{cases} c_1 + c_2, & \text{if } n = 0, \\ c_1 r^n + c_2 r^{-n}, & \text{if } n = 1, 2, ... \end{cases}$$

To get a bounded solution, we set set $c_1 = 1$, and $c_2 = 0$. Thus, we have

$$R(r) = \begin{cases} 1 & \text{if } n = 0, \\ r^n & \text{if } n = 1, 2, \ldots. \end{cases}$$

It follows that

$$u_0(r,\theta) = a_0, \ \textit{and} \ u_n(r,\theta) = r^n[a_n\cos(n\theta) + b_n\sin(n\theta)], n = 1, 2, \ldots.$$

Hence

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n a_n \cos(n\theta) + b_n \sin(n\theta).$$

$$u_r(r,\theta) = \sum_{n=1}^{\infty} \frac{n}{r^{n-1}}[a_n\cos(n\theta) + b_n\sin(n\theta)],$$

since $u_r(a,\theta) = f(\theta)$ implies that

$$f(\theta) = \sum_{n=1}^{\infty} \frac{n}{a^{n-1}}[a_n\cos(n\theta) + b_n\sin(n\theta)].$$

Where the coefficients are

$$a_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} f(\theta)\cos(n\theta)d\theta, \quad n > 0;$$

$$b_n = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} f(\theta)\sin(n\theta)d\theta, \quad n > 0.$$

2.5. Neumann Problem for Cylinder

We refer readers who are interested in the Dirichlet Problem to [5] and [8]. Here we will solve the Neumann problem. Consider the boundary value problem:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r \le a;\ 0 \le \theta \le 2\pi;\ 0 \le z \le L.$$

On the top, we have

$$u_z(r, \theta, L) = f(r, \theta).$$

On the bottom, we have

$$u_z(r, \theta, 0) = g(r, \theta).$$

On the side, we have

$$u_r(a, \theta, z) = h(\theta, z).$$

We need to divide this into three separate problems. After we solve each one, we add them together. Let us state the three problems:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r \le a;\ 0 \le \theta \le 2\pi;\ 0 \le z \le L.$$

$$\begin{aligned} &u_z(r, \theta, L) = f(r, \theta), \\ &u_z(r, \theta, 0) = 0, \\ &u_r(a, \theta, z) = 0 \qquad\qquad 0 \le r \le a,\ 0 \le \theta \le 2\pi. \end{aligned} \tag{5.18}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \le r \le a;\ 0 \le \theta \le 2\pi;\ 0 \le z \le L.$$

$$u_z(r,\theta,L)=0,$$
$$u_z(r,\theta,0)=g(r,\theta),$$
$$u_r(a,\theta,z)=0, \qquad 0\le r\le a, 0\le\theta\le 2\pi. \tag{5.19}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial z^2}=0, \quad 0\le r\le a;\ 0\le\theta\le 2\pi;\ 0\le z\le L.$$

$$u_z(r,\theta,L)=0,$$
$$u_z(r,\theta,0)=0,$$
$$u_r(a,\theta,z)=h(\theta,z), \qquad 0\le\theta\le 2\pi,\ 0\le z\le L. \tag{5.20}$$

By using separation of variables, let

$$u(r,\theta,z)=R(r)\Theta(\theta)Z(z).$$

Substitute the above equation into (2.5.1). Differentiating and dividing on both sides of it by $R\Theta Z$, we get

$$\frac{((rR^{'}))^{'}}{rR}+\frac{\Theta^{''}}{r^2\Theta}+\frac{Z^{''}}{Z}=0.$$

Let

$$\frac{((rR^{'}))^{'}}{rR}+\frac{\Theta^{''}}{r^2\Theta}=-\frac{Z^{''}}{Z}=-\lambda.$$

We get the equation,

$$Z^{''}-\lambda Z=0.$$

We still have

$$\frac{((rR^{'}))^{'}}{rR}+\frac{\Theta^{''}}{r^2\Theta}=-\lambda.$$

We multiply this by r^2 to get

$$\frac{r^2R^{''}+rR^{'}+\lambda r^2}{R}=-\frac{\Theta^{''}}{\Theta}=k.$$

Therefore,

$$\Theta^{''}+k\Theta=0,$$

and

$$r^2R^{''}+rR^{'}+(\lambda r^2-k)R=0.$$

For the first equation, we solve the boundary value condition in (5.18).

$$\Theta^{''}+k\Theta=0.$$

This equation has periodic boundary condition. For the first case, $k<0$, let $k=-u^2$. Then the equation becomes

$$\Theta^{''}-u^2\Theta=0.$$

This equation has solution of the form:

$$\Theta(\theta)=Ae^{u\theta}+Be^{-u\theta}.$$

Since $\Theta(\theta)$ is periodic only if $A=B=0$, k cannot be negative.

For the second case, $k=0$, we get $\Theta^{''}(\theta)=0$, and hence

$$\Theta(\theta)=A+B\theta.$$

In this case, the solution can be periodic only if $B=0$ so that $\Theta(\theta)$ is constant.

For the third case, $k > 0$, let $k = u^2$. Then the equation becomes

$$\Theta^{''} + u^2\Theta = 0.$$

After we solve this equation, we get

$$\Theta = a \ \cos(u\theta) + b \ \sin(u\theta).$$

Hence $k = u^2 = n^2$ is the eigenvalue, and

$$\Theta_n(\theta) = a_n \ \cos(n\theta) + b_n \ \sin(n\theta),$$

for some $n = 0, 1, 2, \ldots,$ is the corresponding eigenfuction. Now let us solve

$$r^2 R^{''} + rR^{'} + (\lambda r^2 - \nu^2)R = 0.$$

This is a Bessel equation, and the boundary conditions imply that $Z'(0) = 0$, and $R'(a) = 0$. We get

$$R(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r).$$

Since $Y_n(\sqrt{\lambda} r)$ is not defined when $r = 0$, so we let $c_2 = 0$. Hence

$$R(r) = c_1 J_n(\sqrt{\lambda} r) = 0,$$

where J_n is a Bessel function of the first kind. Since $R^{'}(a) = 0$, we let $c_1 = 1$ when $r = a$, to get $R(r) = J_n^{'}(\sqrt{\lambda} a) = 0$. In this case, we have $\sqrt{\lambda} a = z_{nm}^{'}$, where $z_{nm}^{'}$ is the m^{th} solution of $J_n^{'}(z_{nm}^{'}) = 0$. Therefore, $\lambda = (\frac{z_{nm}{'}}{a})^2$ are the eigenvalues, and $J_n(z_{nm}^{'} r)$ are the corresponding eigenfunctions. Since $Z^{''} - \lambda Z = 0$, we get

$Z'' - (\frac{z'_{nm}}{a})^2 Z = 0$. Since $Z'(0) = 0$, we have

$$Z(z) = \cosh(\frac{z'_{nm}z}{a}).$$

Hence

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\frac{z'_{nm}r}{a})\cosh(\frac{z'_{nm}z}{a})[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)].$$

Since $u_z(r,\theta,L) = f(r,\theta)$, we get

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{z'_{nm}}{a} J_n(\frac{z'_{nm}r}{a})\sinh(\frac{z'_{nm}L}{a})[a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)].$$

Now, for $n = 0$, we have

$$\sum_{m=1}^{\infty} A_{0m}\frac{z'_{0m}}{a} J_0(\frac{z'_{0m}r}{a})\sinh(\frac{z'_{nm}L}{a}) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta.$$

For $n \geq 1$, we have

$$\sum_{m=1}^{\infty} A_{nm}\frac{z'_{nm}}{a} J_n(\frac{z'_{nm}r}{a})\sinh(\frac{z'_{nm}L}{a}) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(m\theta)d\theta;$$

$$\sum_{m=1}^{\infty} B_{nm}\frac{z'_{nm}}{a} J_n(\frac{z'_{nm}r}{a})\sinh(\frac{z'_{nm}L}{a}) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(m\theta)d\theta.$$

Here the coefficients are: For $n = 0$, $m \geq 1$,

$$A_{0m} = \frac{\frac{1}{2\pi}\int_0^{2\pi}\int_0^a rf(r,\theta)J_0(\frac{z'_{0m}r}{a})drd\theta}{\frac{z'_{0m}}{a}\sinh(\frac{z'_{nm}L}{a})\int_0^a rJ_0(\frac{z'_{0m}r}{a})^2dr},$$

and for $n \geq 1$ $m \geq 1$,

$$A_{nm} = \frac{\frac{1}{\pi}\int_0^{2\pi}\int_0^a rf(r,\theta)J_n(\frac{z'_{nm}r}{a})\cos(n\theta)drd\theta}{\frac{z'_{nm}}{a}\sinh(\frac{z'_{nm}L}{a})\int_0^a rJ_n(\frac{z'_{nm}r}{a})^2dr};$$

$$B_{nm} = \frac{\frac{1}{\pi}\int_0^{2\pi}\int_0^a rf(r,\theta)J_n(\frac{z'_{nm}r}{a})\sin(n\theta)drd\theta}{\frac{z'_{nm}}{a}\sinh(\frac{z'_{nm}L}{a})\int_0^a rJ_n(\frac{z'_{nm}r}{a})^2dr}.$$

Next, we solve (5.19). The boundary conditions imply that $Z'(L) = 0$, and $R'(a) = 0$. We can see that $R'(a) = 0$, $\lambda = (\frac{z'_{nm}}{a})^2$ are the eigenvalues, and $J_n(z'_{nm}r)$ are the corresponding eigenfunctions. Therefore,

$$Z'' - (\frac{z'_{nm}}{a})^2 Z = 0.$$

Since $Z'(L) = 0$, we get $\cosh(\frac{z'_{nm}(L-z)}{a})$. We conclude that

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\frac{z'_{nm}r}{a})\cosh(\frac{z'_{nm}(L-z)}{a})[C_{nm}\cos(n\theta) + D_{nm}\sin(n\theta)].$$

Since $u_z(r,\theta,0) = g(r,\theta)$, we have

$$g(r,\theta) = -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{z'_{nm}}{a}J_n(\frac{z'_{nm}r}{a})\sinh(\frac{z'_{nm}(L)}{a})[C_{nm}\cos(n\theta) + D_{nm}\sin(n\theta)].$$

Hence, for $n = 0$, we have

$$\sum_{m=1}^{\infty} C_{0m}\frac{z'_{0m}}{a}J_0(\frac{z'_{0m}r}{a})\sinh(\frac{z'_{nm}(L)}{a}) = -\frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)d\theta.$$

For $n \geq 1$, we have

$$\sum_{m=1}^{\infty} C_{nm}\frac{z'_{nm}}{a}J_n(\frac{z'_{nm}r}{a})\sinh(\frac{z'_{nm}(L)}{a}) = -\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\cos(m\theta)d\theta;$$

$$\sum_{m=1}^{\infty} D_{nm}\frac{z_{nm}^{'}}{a}J_n(\frac{z_{nm}^{'}r}{a})\sinh(\frac{z_{nm}^{'}(L)}{a}) = -\frac{1}{\pi}\int_0^{2\pi} g(r,\theta)\sin(m\theta)d\theta.$$

Here the coefficients are: For $n = 0$, and $m \geq 1$,

$$C_{0m} = -\frac{\frac{1}{2\pi}\int_0^{2\pi}\int_0^a rg(r,\theta)J_0(\frac{z_{0m}^{'}r}{a})drd\theta}{\frac{z_{0m}^{'}}{a}\sinh(\frac{z_{0m}^{'}L}{a})\int_0^a rJ_0(\frac{z_{0m}^{'}r}{a})^2dr},$$

and for $n \geq 1$, $m \geq 1$,

$$C_{nm} = -\frac{\frac{1}{\pi}\int_0^{2\pi}\int_0^a rg(r,\theta)J_n(\frac{z_{nm}^{'}r}{a})\cos(n\theta)drd\theta}{\frac{z_{nm}^{'}}{a}\sinh(\frac{z_{nm}^{'}L}{a})\int_0^a rJ_n(\frac{z_{nm}^{'}r}{a})^2dr};$$

$$D_{nm} = -\frac{\frac{1}{\pi}\int_0^{2\pi}\int_0^a rg(r,\theta)J_n(\frac{z_{nm}^{'}r}{a})\sin(n\theta)drd\theta}{\frac{z_{nm}^{'}}{a}\sinh(\frac{z_{nm}^{'}L}{a})\int_0^a rJ_n(\frac{z_{nm}^{'}r}{a})^2dr}.$$

Finally, let us solve (5.20). The boundary conditions imply that $Z^{'}(0) = 0$, and $Z^{'}(L) = 0$. Now

$$Z^{''} - \lambda Z = 0.$$

After we solve this equation, we get $\lambda = 0$ is the eigenvalue, and $Z_0 = 1$ is the corresponding eigenfunction. This gives the solution of Neumann Problem for the Circle. In addition, $\lambda = -(\frac{m\pi z}{L})^2$ are eigenvalues, and $Z_m = \cos(\frac{m\pi z}{L})$ are the corresponding eigenfunctions for $m = 1, 2, \ldots$. Now by using

$$r^2R^{''} + rR^{'} + (\lambda r^2 - n^2)R = 0,$$

we derive

$$r^2R^{''} + rR^{'} + \left(-(\frac{m\pi z}{L})^2r^2 - n^2\right)R = 0.$$

Again, this is a Bessel function. Hence $R(r) = J_n(\frac{m\pi}{L}ri)$. Since $I_n(r) = \frac{1}{i^n}J_n(ir)$, where I_n is called modified Bessel function of the first kind, we see that $R(r) =$

$I_n(\frac{m\pi}{L}r)$. Therefore, we have

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n(\frac{m\pi}{L}r)\cos(\frac{m\pi z}{L})[E_{nm}\cos(n\theta) + F_{nm}\sin(n\theta)].$$

Since

$$u_r(a,\theta,z) = h(\theta,z),$$

we get

$$h(\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[\frac{nL}{m\pi}I_n(\frac{m\pi}{L}a) + I_{n+1}(\frac{m\pi}{L}a)]\cos(\frac{m\pi z}{L})[E_{nm}\cos(n\theta) + F_{nm}\sin(n\theta)].$$

The coefficients are: For $n = 0$, and $m \geq 1$,

$$E_{0m} = \frac{1}{[I_1(\frac{m\pi}{L}a)]}\int_0^{2\pi}\int_0^L h(\theta,z)\cos(\frac{m\pi z}{L})\cos(n\theta)dzd\theta;$$

and for $n \geq 1$, and $m \geq 1$,

$$E_{nm} = \frac{2}{[\frac{nL}{m\pi}I_n(\frac{m\pi}{L}a) + I_{n+1}(\frac{m\pi}{L}a)]}\int_0^{2\pi}\int_0^L h(\theta,z)\cos(\frac{m\pi z}{L})\cos(n\theta)dzd\theta;$$

$$F_{nm} = \frac{2}{[\frac{nL}{m\pi}I_n(\frac{m\pi}{L}a) + I_{n+1}(\frac{m\pi}{L}a)]}\int_0^{2\pi}\int_0^L h(\theta,z)\cos(\frac{m\pi z}{L})\sin(n\theta)dzd\theta.$$

For $m = 0$, we use the result of Neumann Problem for the circle to achieve the same goal. Hence the solution for this problem is the sum of all three solutions:

$$u(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\frac{z'_{nm}r}{a})\cosh(\frac{z'_{nm}z}{a})[A_{nm}\cos(n\theta) + B_{nm}\sin(n\theta)]$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\frac{z'_{nm}r}{a})\cosh(\frac{z'_{nm}(L-z)}{a})[C_{nm}\cos(n\theta) + D_{nm}\sin(n\theta)]$$

$$+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I_n(\frac{m\pi}{L}r)\cos(\frac{m\pi z}{L})[E_{nm}\cos(n\theta)+F_{nm}\sin(n\theta)]$$

$$+\sum_{n=1}^{\infty}\frac{n}{r^{n-1}}[E_{n0}\cos(n\theta)+F_{n0}\sin(n\theta)].$$

The coefficients are: For $n \geq 0$, and $m \geq 1$

$$A_{0m} = \frac{\int_0^{2\pi}\int_0^a rf(r,\theta)J_0(\frac{z'_{0m}r}{a})drd\theta}{\pi a z'_{0m}\sinh(\frac{z'_{0m}L}{a})J_1(z'_{0m})^2};$$

$$A_{nm} = \frac{2\int_0^{2\pi}\int_0^a rf(r,\theta)J_n(\frac{z'_{nm}r}{a})\cos(n\theta)drd\theta}{a\pi z'_{nm}\sinh(\frac{z'_{nm}L}{a})rJ_{n+1}(z'_{nm})^2};$$

$$B_{nm} = \frac{2\int_0^{2\pi}\int_0^a rf(r,\theta)J_n(\frac{z'_{nm}r}{a})\sin(n\theta)drd\theta}{a\pi z'_{nm}\sinh(\frac{z'_{nm}L}{a})J_{n+1}(z'_{nm})^2};$$

$$C_{0m} = -\frac{\int_0^{2\pi}\int_0^a rg(r,\theta)J_0(\frac{z'_{0m}r}{a})drd\theta}{a\pi z'_{0m}\sinh(\frac{z'_{0m}L}{a})J_1(z'_{0m})^2};$$

$$C_{nm} = -\frac{2\int_0^{2\pi}\int_0^a rg(r,\theta)J_{n+1}(\frac{z'_{nm}r}{a})\cos(n\theta)drd\theta}{a\pi z'_{nm}\sinh(\frac{z'_{nm}L}{a})J_{n+1}(z'_{nm})^2};$$

$$D_{nm} = -\frac{2\int_0^{2\pi}\int_0^a rg(r,\theta)J_n(\frac{z'_{nm}r}{a})\sin(n\theta)drd\theta}{a\pi z'_{nm}\sinh(\frac{z'_{nm}L}{a})J_{n+1}(z'_{nm})^2};$$

$$E_{0m} = \frac{1}{[I_1(\frac{m\pi}{L}a)]}\int_0^{2\pi}\int_0^L h(\theta,z)\cos(\frac{m\pi z}{L})\cos(n\theta)dzd\theta;$$

$$E_{nm} = \frac{2}{[\frac{nL}{m\pi} I_n(\frac{m\pi}{L} a) + I_{n+1}(\frac{m\pi}{L} a)]} \int_0^{2\pi} \int_0^L h(\theta, z) \cos(\frac{m\pi z}{L}) \cos(n\theta) dz d\theta;$$

$$E_{0m} = \frac{1}{[I_1(\frac{m\pi}{L} a)]} \int_0^{2\pi} \int_0^L h(\theta, z) \cos(\frac{m\pi z}{L}) \cos(n\theta) dz d\theta;$$

$$F_{nm} = \frac{2}{[\frac{nL}{m\pi} I_n(\frac{m\pi}{L} a) + I_{n+1}(\frac{m\pi}{L} a)]} \int_0^{2\pi} \int_0^L h(\theta, z) \cos(\frac{m\pi z}{L}) \sin(n\theta) dz d\theta.$$

For $n \geq 0$, and $m = 0$ we have

$$E_{n0} = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} \int_0^L h(\theta, z) \cos(n\theta) dz d\theta;$$

$$F_{n0} = \frac{1}{n\pi a^{n-1}} \int_0^{2\pi} \int_0^L h(\theta, z) \sin(n\theta) dz d\theta.$$

3. DIRICHLET PROBLEM ON UNBOUNDED DOMAINS: APPLICATIONS OF FOURIER TRANSFORM

The Fourier transform decomposes a function of time into frequencies. The inverse Fourier transform uses the contributions of all the different frequencies to reconstruct the original function, which means that we are able to use the information on the frequencies to find the time when a corresponding frequency occurs. The opposite is also true. Generally speaking, Fourier transforms are defined on an unbounded domain. As such, cautions are needed as to choose the right kinds of functions to integrate. Concepts and theories of Fourier transform and inverse Fourier transform we use in this chapter can all be found in [5, 6] .

3.1. Fourier Transform Formula

We follow the notations from Haberman [5] to define Fourier transform pair.

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{iwx}dx,$$

$$f(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx}dw.$$

The 1st integral is called the Fourier transform and the 2nd the inverse Fourier transform. Calculate the inverse Fourier transform of $F(w) = e^{-|w|}$. Using the definition of the inverse Fourier transform, we have

$$f(x) = \int_{-\infty}^{\infty} e^{-|w|}e^{-iwx}dw$$

$$= \int_{-\infty}^{0} e^{w}e^{-iwx}dw + \int_{0}^{\infty} e^{-w}e^{-iwx}dw$$

$$= \int_{-\infty}^{0} e^{w(1-ix)} dw + \int_{0}^{\infty} e^{-w(1+ix)} dw$$

$$= \frac{1}{(1-ix)} + \frac{1}{(1+ix)} = \frac{2}{x^2+1}.$$

3.2. Fourier Sine and Cosine Transform

We define the Fourier Cosine and Sine transform, respectively, by

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos wx dx;$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin wx dx.$$

For sufficiently smooth functions, we can use the inverse transforms to recover the original function $f(x)$ according to the following formula:

$$f(x) = \int_{0}^{\infty} [A(w) \cos wx + B(w) \sin wx] dx.$$

3.3. Unbounded Problem for Rectangle

We consider the Boundry value problem (BVP)

$$\triangle u = 0, \quad -\infty < x < \infty; \quad 0 < y < \infty,$$

$$u(x, 0) = f(x), \quad |\ u(x, y)\ | \ < M.$$

By using separation of variables, let $u(x, y) = X(x)Y(y)$.

Then, we have

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Let

$$\frac{Y''(y)}{Y(y)} = -\frac{X''(x)}{X(x)} = \lambda.$$

We get two ordinary differential equations:

$$X'' + \lambda X = 0, \tag{3.1}$$

and

$$Y'' - \lambda Y = 0. \tag{3.2}$$

Setting $\lambda = w^2$, then we have

$$X'' + w^2 X = 0.$$

The solution is:

$$X(x) = a_1 \cos(wx) + b_1 \sin(wx).$$

Now, we need to solve

$$Y'' - w^2 Y = 0.$$

We have

$$Y(y) = a_2 e^{wy} + b_2 e^{-wy}.$$

Since $u(x, y) = X(x)Y(y)$, we have

$$u(x, y) = (a_1 \cos(wx) + b_1 \sin(wx))(a_2 e^{wy} + b_2 e^{-wy}).$$

Since $w > 0$, and $y \to \infty$, we must have $a_2 = 0$. It follows that

$$u(x, y) = \int_0^\infty e^{-wy}[\cos(wx) + \sin(wx)]dw.$$

Since $u(x,0) = f(x)$, we have

$$f(x) = \int_0^\infty [A(w)\cos(wx) + B(w)\sin(wx)]dw.$$

Thus, from Fourier Sine and Cosine transforms, we find that

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)dx;$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)dx.$$

3.4. Convolution Theorem

Suppose that $f(x)$ and $g(x)$ satisfy the conditions required by the Fourier Transform and the inverse Fourier Transform. We write

$$f(x) = \int_{-\infty}^{\infty} F(w)e^{-iwx}dw,$$

$$F(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{iwx}dx.$$

$$g(x) = \int_{-\infty}^{\infty} G(w)e^{-iwx}dw,$$

$$G(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{iwx}dx.$$

Assume $H(w) = F(w)G(w)$. We calculate the inverse Fourier transform of $H(w)$. We have

$$h(x) = \int_{-\infty}^{\infty} H(w)e^{-iwx}dw$$

$$=\int_{-\infty}^{\infty} H(w)e^{-iwx}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)[\int_{-\infty}^{\infty} g(z)e^{iwz}dz]e^{-iwx}dw$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(z)f(x-z)dz.$$

Hence

$$h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} g(z)f(x-z)dz.$$

This is called the convolution of $f(x)$ and $g(x)$.

3.5. Application of Convolution Theorem

On the unbounded rectangle, we write

$$u(x,y)=\int_{0}^{\infty} e^{-wy}[A(w)\cos(wx)+B(w)\sin(wx)]\ dw,$$

where

$$A(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)\ dx;$$

$$B(w)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(wx)\ dx.$$

After we plug the coefficients into the general formula, we get

$$u(x,y)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} e^{-wy}[f(z)\cos(wz)\cos(wx)+f(z)\sin(wz)\sin(wx)]\ dwdz$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} e^{-wy}f(z)\cos w(z-x)\ dwdz.$$

Let

$$g(x,y) = \int_0^\infty e^{-wy} \cos w(x)\ dw$$

$$= \int_0^\infty e^{-wy} \frac{e^{iwx} + e^{-iwx}}{2}\ dw$$

$$= \frac{1}{2}\left(\frac{1}{y - ix} + \frac{1}{y + xy}\right)$$

$$= \frac{y}{y^2 + x^2}.$$

Now we apply the convolution theorem to get

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z) g(x-z,y) dz.$$

Let

$$f(z) = \begin{cases} 2 & \text{if} \quad z > 0, \\ 0 & \text{if} \quad z < 0 \end{cases}$$

Then we have

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2y}{y^2 + (z-x)^2} dz$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}\left(\frac{x}{y}\right)\right].$$

Let

$$\phi = \tan^{-1}\left(\frac{x}{y}\right),$$

and since

$$\tan^{-1}\left(\frac{x}{y}\right) = \left(\frac{\pi}{2}\right) - \tan^{-1}\left(\frac{y}{x}\right),$$

hence $u(x,y) = 1 - \frac{\phi}{\pi}$.

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