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## Residues, Bernoulli Numbers and Finding Sums

Mohammed Saif Alotaibi

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# **RESIDUES, BERNOULLI NUMBERS AND FINDING SUMS**

A Masters Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

Mohammed Saif Alotaibi

May 2017

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# RESIDUES, BERNOULLI NUMBERS AND FINDING SUMS

Mathematics

Missouri State University, May 2017

Master of Science

Mohammed Saif Alotaibi

## ABSTRACT

A large number of infinite sums, such as  $\sum_{k=0}^{\infty} \frac{1}{k^2}$ , cannot be found by the methods of real analysis. However, many of them can be evaluated using the theory of residues. In this thesis we characterize several methods of summations using residues, including methods integrating residues and the Bernoulli numbers. In fact, with this technique we derive some summation formulas for particular Finite Sums and Infinite Series that are difficult or impossible to solve by the methods of real analysis.

**KEYWORDS:** analytic function, homotopy, singularity, pole, zero, residue, Bernoulli numbers, finite sums, infinite series.

This abstract is approved as to form and content

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Dr. Shelby J. Kilmer  
Chairperson, Advisory Committee  
Missouri State University

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By

Mohammed Saif Alotaibi

A Masters Thesis  
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Approved:

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Dr. Shelby J. Kilmer

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Julie Masterson, Dr. Dean, Graduate College

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Finally, I must express my very profound gratitude to my parents, who supported me through all my years of study, with all its difficult circumstances, and in spite of the large distance between us. In addition to my parents, I would like to acknowledge all the members of my family, specifically my wife, for providing unfailing support and continuous encouragement throughout my years of study. This achievement would not have been possible without them.

I dedicate this thesis to my daughter Yara.

## TABLE OF CONTENTS

1: Introduction.....	1
2: Preliminaries .....	2
2.1 Differentiation.....	2
2.2 Integrals and Contours .....	6
2.3 Homotopy .....	9
3: Residues.....	18
3.1 Laurent Series .....	18
3.2 Singular Points.....	22
3.3 Definition of Residue.....	23
3.4 Residue at Infinity.....	24
3.5 The Cauchy Residue Theorem.....	25
3.6 Zeros and Poles.....	26
3.7 Residue at a Pole.....	27
3.7.1 Residue at a Pole of Order $m$ .....	27
3.7.2 Residues at Simple Poles .....	29
4: Bernoulli Numbers.....	34
4.1 The Bernoulli Numbers.....	34
4.2 Results.....	37
5: Summing Series by Residues.....	41
5.1 Foundations.....	41
5.2 Finite Sums .....	47
5.3 Infinite Series .....	51
5.3.1 Non-integer Singularities.....	51
5.3.2 Integer Singularities.....	57
5.3.3 Singularities at Zero.....	64
6: Mittag-Leffler Expansion Theorem.....	68
Bibliography .....	73

## CHAPTER 1

### INTRODUCTION

After a brief review of the more important basic concepts of complex analysis, we present Residues and Cauchy's Residue Theorem. Many consider this theorem to be the most important theorem of complex analysis and it is the cornerstone of this thesis. Cauchy's Residue Theorem is not only important in complex analysis but has an important role in real analysis, one that may possibly outweigh its importance in complex analysis. In order to make full use of the Residue Theorem, we derive various methods of calculating residues. Some of our techniques rely on Bernoulli numbers, so we define and explore their properties, before using them to obtain some important infinite sums. We conclude this thesis with two chapters evaluating both finite and infinite sums using these methods.



## CHAPTER 2

### PRELIMINARIES

#### 2.1 Differentiation

In complex variables the derivative is defined the same way as in the real number system. It is, therefore, not surprising that the usual differentiation rules, such as the sum and difference rules, hold when taking derivatives of complex functions.

DEFINITION 2.1. *Given  $G \subset \mathbb{C}$ , let  $f : G \rightarrow \mathbb{C}$  be a complex valued function and let  $z_0 \in G$ . The derivative of  $f$  at  $z_0$  is*

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

*when this limit exists. If so, we say  $f$  is differentiable at  $z_0$ .*

DEFINITION 2.2. *Let  $f : G \rightarrow \mathbb{C}$  be a complex valued function. If  $f$  is differentiable at  $a$  for every  $a \in G$  and these derivatives are continuous, the function  $f$  is said to be analytic on  $G$ . If  $f$  is analytic on the whole complex plane  $\mathbb{C}$ ,  $f$  is said to be entire.*

As an example, the complex polynomials are entire functions.

DEFINITION 2.3. *A disc or a ball centered at  $z_0$  with radius  $r$  is*

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

DEFINITION 2.4. *A set  $G \subset \mathbb{C}$  is open, if for every  $z \in G$  there exist  $r > 0$  such that  $B(z, r) \subset G$ .*

Using the definition above, it is easy to prove a subset of  $\mathbb{C}$  is open if and only if it includes no points of its boundary. It follows that the region inside a closed contour is open.

The following theorem is called Taylor's theorem in honor of the English mathematician Brook Taylor, who discovered its first form. This result is fundamental in the proofs of Cauchy's theorems and many other important theorems in complex variables, as well as in many other area of mathematics.

**THEOREM 2.5.** *If  $f$  is analytic on a disc  $B(z_0, r)$ , then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for all  $z \in B$ , where each

$$a_n = \frac{f^{(n)}(z)}{n!} \quad \text{is unique.}$$

The proof of the part of Taylor's theorem giving the existence of the series is much like, but simpler, than the proof of Laurent's Theorem, which will be included in a later chapter.

**THEOREM 2.6.** *Suppose  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$  has a radius of convergence  $R$ .*

*Then  $f$  can be differentiated term by term inside  $B(z_0, R)$ . That is*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

*Moreover  $f'$  has a radius of convergence  $R$ , as well.*

*Proof.* Without loss of generality we assume  $z_0 = 0$ . Since

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)a_{n+1}}{na_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = R,$$

we see, by the ratio test, that the second series has the same radius of convergence as the first.

Now let  $z \in B(0, R)$  and let  $\varepsilon > 0$ . There exists  $r > 0$  such that

$z \in B(0, r) \subset B(0, R)$ . Let

$$S_n(\xi) = \frac{\sum_{k=0}^n a_k z^k - \sum_{k=0}^n a_k \xi^k}{z - \xi} = \frac{1}{z - \xi} \sum_{k=0}^n a_k (z^k - \xi^k),$$

and,

$$R_n(\xi) = \frac{1}{z - \xi} \sum_{k=n+1}^{\infty} a_k (z^k - \xi^k).$$

$S_n(\xi)$  denotes the  $n^{\text{th}}$  partial sum of  $\frac{f(z)-f(\xi)}{z-\xi}$  and  $R_n(\xi)$  the corresponding remainder. Since  $r < R$ , the series  $\sum_{k=0}^{\infty} k|a_k|r^{k-1}$  converges and so there exist  $N \in \mathbb{N}$  such that

$$\sum_{k=N}^{\infty} k|a_k|r^{k-1} < \frac{\varepsilon}{3}.$$

It follows that for every  $\omega \in B(z, r)$ ,

$$\begin{aligned} |R_n(\omega)| &= \left| \sum_{k=N}^{\infty} \frac{a_k (z^k - \omega^k)}{z - \omega} \right| \\ &\leq \sum_{k=N}^{\infty} |a_k| |z^{k-1} + z^{k-2}\omega + \dots + \omega^{k-1}| \\ &\leq \sum_{k=N}^{\infty} k|a_k|r^{k-1} \\ &< \frac{\varepsilon}{3}. \end{aligned}$$

Let  $S'_N$  denote the  $N^{\text{th}}$  partial sum of  $\sum_{k=1}^{\infty} k a_k z^{k-1}$  and  $R'$  the corresponding remainder. Thus,

$$|R'| \leq \sum_{k=N}^{\infty} k|a_k|r^{k-1} < \frac{\varepsilon}{3}.$$

Now, since the partial sums of  $f$  are polynomials,

$$\lim_{\omega \rightarrow z} S_N = S'_N.$$

Therefore, we have  $\delta > 0$  with  $\delta < r$  such that when  $|\omega - z| < \delta$ ,  $|S_N - S'_N| < \frac{\varepsilon}{3}$ . It

now follows that when  $|\omega - z| < \delta$ ,

$$\begin{aligned} \left| \frac{f(z) - f(\omega)}{z - \omega} - \sum_{n=1}^{\infty} n a_n z^{n-1} \right| &= |S_N(\omega) + R_N(\omega) - S'_N - R'_N| \\ &\leq |S_N - S'_N| + |R_n(\omega)| + |R'_n| \\ &< \varepsilon, \end{aligned}$$

which finishes the proof. □

The following corollary is an example of how important Taylor's theorem is in complex variables.

**COROLLARY 2.7.** *If  $f$  is analytic on an open set  $G$ , then  $f$  is infinitely differentiable on  $G$ .*

*Proof.* Let  $z_0 \in G$ . Since  $G$  is open, it contains a ball centered at  $z_0$ . By Taylor's theorem,  $f$  has a valid power series on that ball. By the previous theorem  $f'$  has a power series form which is differentiable on the ball as well. Continuing inductively  $f$  must be infinitely differentiable. □

## 2.2 Integrals and Contours

Integration of functions along contours in the Complex plane will play an important role in our methods. Some of the concepts and theorems given in this section will become powerful tools for proving important theorems in later sections.

DEFINITION 2.8. Let  $C$  be a curve in  $\mathbb{C}$ . We say  $\gamma : [a, b] \rightarrow C$  parameterizes  $C$ , if  $\gamma$  is a continuous surjection. Furthermore,  $C$  is smooth, if it has a differentiable parameterization with a non-zero continuous derivative. The orientation of  $C$  is given by its parameterization;  $\gamma(a)$  is “before”  $\gamma(b)$ .

When  $\gamma : [a, b] \rightarrow C$  parameterizes  $C$ , it is easy to see  $\gamma(a + b - t)$  is also a parameterization of  $C$  but with the opposite orientation. We generally refer to  $-C$  when switching to the parameterization giving the opposite orientation.

DEFINITION 2.9. A curve  $C$  is a contour, if it is the union of finitely many smooth curves  $C_1, C_2, \dots, C_n$ , and the end point of  $C_k$  coincides with the starting point of  $C_{k+1}$ , for  $k = 1, 2, \dots, n - 1$ . We write  $C = C_1 + C_2 + \dots + C_n$ .

DEFINITION 2.10. A contour  $C$  is closed, if its starting point and endpoint are the same.

DEFINITION 2.11. A closed contour is positively oriented, when its parameterization traverses it in the counterclockwise direction.

DEFINITION 2.12. A contour  $C$  is simple and sometimes called a Jordan arc, if it never cross itself, except possible at the endpoints.

DEFINITION 2.13. When  $g : [a, b] \rightarrow \mathbb{C}$ ,

$$\int_a^b g(t)dt = \int_a^b \operatorname{Re}[g(t)]dt + i \int_a^b \operatorname{Im}[g(t)]dt,$$

where the integrals on the right are defined as in elementary calculus.

DEFINITION 2.14. When  $\gamma : [a, b] \rightarrow C$  parameterizes a smooth curve  $C$  and  $f$  is defined on  $C$ , we define the integral of  $f$  on  $C$ , by

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt,$$

and when  $C = C_1 + C_2 + \cdots + C_n$  is a contour, the contour integral of  $f$  on  $C$  is

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

Since the values of these integrals are independent of the particular parameterization used, DEFINITION 2.13 above is valid. To get an idea of how a proof would run, let

$g$  be real and consider  $\gamma : [a, b] \rightarrow \operatorname{Domain}(g)$  and  $\sigma : [c, d] \rightarrow \operatorname{Domain}(g)$ , with

$\gamma(a) = \sigma(c)$  and  $\gamma(b) = \sigma(d)$ . By the substitution principle we have

$$\int_a^b g(\gamma(t))\gamma'(t)dt = \int_{\gamma(a)}^{\gamma(b)} g(u)du = \int_{\sigma(c)}^{\sigma(d)} g(u)du = \int_a^b g(\sigma(t))\sigma'(t)dt.$$

A complete proof would require combining real and imaginary parts and so on.

More of this type of reasoning can show that the integration rules from elementary calculus, such as the sum, difference and constant multiple rules all hold.

THEOREM 2.15. If  $\gamma : [a, b] \rightarrow C$  is smooth and length of  $C$ ,  $L(C)$ , is finite, then

$$\int_a^b |\gamma'(t)| dt = L(C).$$

*Proof.* Since the length of  $C \in \mathbb{C}$  is the same as the length of  $\langle \operatorname{Re}\gamma, \operatorname{Im}\gamma \rangle \in \mathbb{R}^2$ , this

follows immediately from the arc length formula in elementary calculus. Thus

$$L(C) = \int_a^b \sqrt{(\operatorname{Re}\gamma'(t))^2 + (\operatorname{Im}\gamma'(t))^2} dt = \int_a^b |\gamma'(t)| dt.$$

The proof is complete. □

The following corollary will be indispensable as we proceed. It's proof is immediate.

COROLLARY 2.16. [4] *If the integral of  $f$  on  $C$  exists, the length of  $C$  is finite and  $f$  is bounded on  $C$ , then*

$$\left| \int_{\gamma} f(z) dz \right| \leq L(C) M_f,$$

where  $L(C)$  is the arclength of  $C$  and  $M_f$  is the maximum value of  $|f|$  on  $C$ .

The field of complex variables has an analog of the fundamental theorem of calculus from real analysis.

DEFINITION 2.17. *The function  $F$  is a primitive of the function  $f$  on the set  $G$ , if for all  $z \in G$ ,*

$$F'(z) = f(z).$$

THEOREM 2.18. [4] *Let  $C$  be a contour in an open set  $G$ , with endpoints  $\alpha$  and  $\beta$ . If  $F$  is a primitive of  $f$  on  $G$ , then*

$$\int_C f(z) dz = F(\beta) - F(\alpha).$$

*Proof.* Let  $\gamma : [a, b] \rightarrow C$  parameterize a smooth curve  $C$  from  $\alpha$  to  $\beta$ . Then

$$\begin{aligned} \int_C f(z) dz &= \int_C F'(z) dz = \int_a^b F'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (F \circ \gamma)'(t) dt = F \circ \gamma(b) - F \circ \gamma(a) \\ &= F(\beta) - F(\alpha). \end{aligned}$$

Now consider a contour  $C = C_1 + C_2 + \cdots + C_n$ , with connections at  $z_1, z_2, \dots, z_{n-1}$ , respectively. Then from the smooth case, we have

$$\int_C f(z) dz = F(z_1) - F(\alpha) + F(z_2) - F(z_1) \cdots + F(\beta) - F(z_{n-1}) = F(\beta) - F(\alpha).$$

The proof is complete. □

Since  $F(\beta) = F(\alpha)$ , on a closed contour, we then have the following immediate corollary.

**COROLLARY 2.19.** [4] *Let  $C$  be a closed contour in an open set  $G$ . If  $F$  is a primitive of  $f$  on  $G$ , then*

$$\int_C f(z)dz = 0.$$

The following corollary is often called the first version of Cauchy's theorem.

**COROLLARY 2.20.** *If  $C$  is a closed contour in  $B(z_0, r)$  and  $f$  is analytic on  $B(z_0, r)$ , then*

$$\int_C f(z)dz = 0.$$

*Proof.* Since  $f$  is analytic on  $B(z_0, r)$ , it has a Taylor series valid on  $B(z_0, r)$ .

Taking the antiderivative term by term yields a primitive for  $f$ . Proof is immediate by the previous corollary. □

### 2.3 Homotopy

**DEFINITION 2.21.** *Two curves,  $C$  and  $C'$ , from  $A$  to  $B$  are homotopic in  $G \subset \mathbb{C}$ , if there exists continuous  $\Psi : [0, 1]^2 \rightarrow G$ , such that*

$$\Psi(s, 0) = A \text{ for every } s \in [0, 1],$$

$$\Psi(s, 1) = B \text{ for every } s \in [0, 1],$$

$\Psi(0, t)$  parameterizes  $C$  and



$\Psi(1, t)$  parameterizes  $C'$ .

We will write  $C \sim C'$ , when  $C$  and  $C'$  are homotopic and  $\psi$  is sufficiently differentiable to produce smooth curves.

Note: for each fixed  $s \in [0, 1]$ ,  $\Psi(s, t) : [0, 1] \rightarrow G$  parameterizes some curve in  $G$  from  $A$  to  $B$ . The intuition is that  $\Psi$  “continuously morphs”  $C$  to  $C'$ .

**THEOREM 2.22.** [1] *If  $f$  is analytic on  $G$  and  $C \sim C'$  in  $G$ , then*

$$\int_C f(z)dz = \int_{C'} f(z)dz.$$

*Proof.* Let  $C$  and  $C'$  be homotopic curves from  $A$  to  $B$  in an open set  $G$ , with  $\Psi$  as in definition (2.21). Since  $\Psi([0, 1]^2)$  is compact and  $\mathbb{C} - G$ , is closed the distance between them is  $r$  for some  $r > 0$ . This means  $f$  is analytic on  $B(z, r)$  for every  $z \in G$ . Moreover, since  $\Psi$  is continuous and  $[0, 1]^2$  is compact,  $\Psi$  is uniformly continuous on  $[0, 1]^2$ . It follows that there exists  $\delta > 0$  such that when

$$\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} < \delta, \text{ then, } |\Psi(s_2, t_2) - \Psi(s_1, t_1)| < r.$$

Choose  $n \in \mathbb{N}$ , so that  $\frac{\sqrt{2}}{n} < \delta$ . Then partition  $[0, 1]^2$  into  $n^2$  congruent squares.

Note that if  $(s_1, t_1)$  and  $(s_2, t_2)$  are in the same square,  $\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} < \delta$ .

If  $k$  is fixed in  $\{0, 1, 2, \dots, n - 1\}$ , then  $\Psi(\frac{k}{n}, t)$  and  $\Psi(\frac{k+1}{n}, t)$  parameterize curves,  $C_k$  and  $C_{k+1}$  from  $A$  to  $B$  in  $G$ .

For each  $j \in \{0, 1, 2, \dots, n - 1\}$ , define  $S_j \in [0, 1]^2$  to be the boundary of the  $\frac{1}{n} \times \frac{1}{n}$  square with bottom left corner  $(\frac{k}{n}, \frac{j}{n})$ . Let  $\xi_j$  denote the closed contour,  $\Psi(S_j)$ ,

traversed counterclockwise. Since  $\text{diam } S_j = \frac{\sqrt{2}}{n} < \delta$ , each  $\xi_j \subset B(\Psi(\frac{k}{n}, \frac{j}{n}), r)$ . It follows by theorem (2.20), that  $\int_{C_j} f(z) dz = 0$ , for each  $j \in \{0, 1, 2, \dots, n-1\}$ .

Each consecutive pair,  $\xi_j$  and  $\xi_{j+1}$ , share sides traversed in opposite directions and integrals over those sides add to zero. Thus for each  $k \in \{0, 1, 2, \dots, n-1\}$ ,

$$\begin{aligned} \int_{C_k} f(z) dz - \int_{C_{k+1}} f(z) dz &= \int_{C_k - C_{k+1}} f(z) dz \\ &= \int_{\xi_1 + \xi_2 + \dots + \xi_{n-1}} f(z) dz \\ &= \int_{\xi_1} f(z) dz + \int_{\xi_2} f(z) dz + \dots + \int_{\xi_{n-1}} f(z) dz \\ &= 0. \end{aligned}$$

It follows that

$$\int_C f(z) dz = \int_{\xi_1} f(z) dz = \int_{\xi_2} f(z) dz \dots = \int_{C'} f(z) dz.$$

The proof is complete. □

**DEFINITION 2.23.** A closed curve  $C$  in  $G$  is homotopic to zero, if  $C$  is homotopic to a constant curve. In other words take  $C'$  in definition (2.21) to be one point  $z_0$  and its parameterization to be of constant value  $z_0$ .

**DEFINITION 2.24.** A region  $G$  is simply connected, if  $G$  is open and every closed curve in  $G$  is homotopic to zero.

**THEOREM 2.25.** If two contours have the same beginning and end points and the same orientation in a simply connected region  $G$ , they are homotopic in  $G$ .

*Proof.* Let  $C_1$  and  $C_2$  be two contours from  $a$  to  $b$  in  $G$ . Let  $C = C_1 - C_2$ , which is a closed contour in  $G$ . Let  $\gamma : [0, 1] \rightarrow G$ , be a parameterization of  $C$  going from  $a$  to  $b$  and back to  $a$  again. Without loss of generality we assume  $\gamma(1/2) = b$ . It follows that

$\gamma : [0, 1/2] \rightarrow G$  parameterizes  $C_1$ ,

$\gamma : [1/2, 1] \rightarrow G$  parameterizes  $-C_2$ , and thus

$\gamma(\frac{2-t}{2}) : [0, 1] \rightarrow G$  parameterizes  $C_2$ .

Since  $G$  is simply connected,  $C \sim 0$ , that is there exists  $z_0 \in G$  and a homotopy

$\Psi : [0, 1]^2 \rightarrow G$ , such that  $\Psi(0, t) = \gamma(t)$  for all  $t$  and  $\Psi(1, t) = z_0$  for all  $t$ .

Define

$\Psi_1 : [0, 1/2] \times [0, 1] \rightarrow G$ , by  $\Psi_1(s, t) = \Psi(2s, t/2)$  and

$\Psi_2 : [1/2, 1] \times [0, 1] \rightarrow G$ , by  $\Psi_2(s, t) = \Psi(2 - 2s, \frac{2-t}{2})$ .

As compositions of continuous functions, both are continuous on their domains.

Now define  $\Phi : [0, 1]^2 \rightarrow G$ , by

$$\Phi(s, t) = \begin{cases} \Psi_1(s, t), & \text{if } s \leq 1/2 \\ \Psi_2(s, t), & \text{if } s \geq 1/2 \end{cases}.$$

Since  $\Psi_1$  and  $\Psi_2$  are continuous, to see  $\Phi$  is continuous, it only remains to see

$\Psi_1 = \Psi_2$  on the intersection of their domains. To that end note that for all  $t$

$$\Psi_1(1/2, t) = \Psi(1, t/2) = z_0 = \Psi(1, (2-t)/2) = \Psi_2(1/2, t).$$

Now it remains to show  $\Phi$  transforms  $C_1$  to  $C_2$ . For all  $t \in [0, 1]$

$$\Phi(0, t) = \Psi_1(0, t) = \Psi(0, t/2) = \gamma(t/2),$$

which parameterizes  $C_1$ . Moreover for all  $t \in [0, 1]$

$$\Phi(1, t) = \Psi_2(1, t) = \Psi(0, (2-t)/2) = \gamma((2-t)/2),$$

which parameterizes  $C_2$ . □

The following theorem is one of the most famous and important theorems of all complex analysis.

**THEOREM 2.26** (Cauchy-Goursat Theorem). [4] *Let  $C$  be a simple closed contour in a simply connected set  $G$ . If a function  $f(z)$  is analytic at all points interior to and on  $C$ , then*

$$\int_C f(z)dz = 0.$$

*Proof.* Take two distinct points  $a$  and  $b$  on  $C$ . This forms two contour curves  $C_1$  and  $C_2$  from  $a$  to  $b$  in  $G$ , with  $C = C_1 - C_2$ . Since  $G$  is simply connected, by theorem (2.25),  $C_1$  and  $C_2$  are homotopic. Thus by theorem (2.22),

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Therefore,

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0.$$

The proof is complete. □

One of the most famous theorems of complex analysis will now be established.

**THEOREM 2.27** (Cauchy Integral Formula). [4] *Let  $C$  be a positively oriented simple closed contour, and let  $f$  be analytic function everywhere inside and on  $C$ . If  $a$  is any point interior to  $C$ , then*

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - a} d\xi.$$

*Proof.* Let  $G$  represent the interior of  $C$  and let  $a \in G$  be given. Since  $G \cup C$  is compact and  $f$  is continuous,  $M_f$ , the maximum value of  $|f(\xi) - f(a)|$  on  $G \cup C$  exists.  $G$  is open, so there exists  $R > 0$  such that  $B(a, R) \subset G$  and  $\delta > 0$  such that  $|\xi - z| > \delta$ , whenever  $\xi \in C$  and  $z \in B(a, R)$ . Let

$$r = \frac{1}{2} \min \left\{ R, \frac{\varepsilon \delta}{2\pi M_f} \right\}.$$

Let  $\gamma$  be the positively oriented simple closed contour around the boundary of  $B(a, r)$  and note that

$$\int_{\gamma} \frac{d\xi}{\xi - a} = \int_0^{2\pi} \frac{ire^{it}}{a + re^{it} - a} dt = 2\pi i.$$

It follows that

$$\begin{aligned} \left| 2\pi i f(a) - \int_{\gamma} \frac{f(\xi)}{\xi - a} d\xi \right| &= \left| 2\pi i f(a) - \int_{\gamma} \frac{f(\xi)}{\xi - a} d\xi \right| \\ &= \left| f(a) \int_{\gamma} \frac{d\xi}{\xi - a} - \int_{\gamma} \frac{f(\xi)}{\xi - a} d\xi \right| \\ &= \left| \int_{\gamma} \frac{f(a) - f(\xi)}{\xi - a} d\xi \right| \\ &\leq \int_{\gamma} \left| \frac{f(a) - f(\xi)}{\xi - a} \right| d\xi \\ &\leq \frac{M_f}{\delta} 2\pi r \\ &< \varepsilon. \end{aligned}$$

Therefore, by THEOREM 2.22

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - a} d\xi = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - a} d\xi.$$

The proof is complete. □

The following theorem is a generalization of the Cauchy Integral Formula.

**THEOREM 2.28** (Cauchy's Integral Formula for derivatives). [4] *Let  $C$  be a positively oriented simple closed contour, and let  $f$  be an analytic function everywhere inside and on  $C$ . If  $a$  is any point interior to  $C$ , then for all  $n = 0, 1, 2, \dots$ ,*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(\xi) d\xi}{(\xi - a)^{n+1}}. \quad (2.1)$$

*Proof.* We proceed by induction. Cauchy's integral formula, previously proven, verifies (2.1) for  $n = 0$ . We assume

$$f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(\xi) d\xi}{(\xi - a)^n},$$

for some  $n$ .

Let  $G$  represent the interior of  $C$ . Let  $a \in G$  and  $n \in \mathbb{N}$  be given. Since  $G \cup C$  is compact and  $f$  is continuous,  $M_f$ , the maximum value of  $|f(\xi)|$  on  $G \cup C$  exists.  $G$  is open, so there exists  $R > 0$  such that  $B(a, R) \subset G$  and  $\delta > 0$  such that  $|\xi - z| > \delta$ , whenever  $\xi \in C$  and  $z \in B(a, R)$ . Let

$$r = \frac{1}{2} \min \left\{ R, \frac{\varepsilon \delta^{n+1}}{4n\pi M_f} \right\}$$

and let  $\gamma$  be the positively oriented simple closed contour around the boundary of  $B(a, r)$ . Define  $F$  on  $G$  by

$$F(z) = \int_{\gamma} \frac{f(\xi)d\xi}{(\xi - z)^n}.$$

For  $z \in B(a, r)$ ,

$$\begin{aligned} & \left| \frac{F(z) - F(a)}{z - a} - n \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi \right| \\ &= \left| \int_{\gamma} \frac{f(\xi)}{z - a} \left( \frac{1}{(\xi - z)^n} - \frac{1}{(\xi - za)^n} \right) - \frac{nf(\xi)}{(\xi - a)^{n+1}} d\xi \right| \\ &= \left| \int_{\gamma} \frac{f(\xi)}{z - a} \left( \frac{1}{\xi - z} - \frac{1}{\xi - a} \right) \left( \sum_{k=0}^{n-1} \frac{1}{(\xi - z)^{n-1-k}(\xi - a)^k} \right) - \frac{nf(\xi)}{(\xi - a)^{n+1}} d\xi \right| \\ &= \left| \int_{\gamma} f(\xi) \left[ \left( \frac{1}{(\xi - z)(\xi - a)} \right) \left( \sum_{k=0}^{n-1} \frac{1}{(\xi - z)^{n-1-k}(\xi - a)^k} \right) - \frac{nf(\xi)}{(\xi - a)^{n+1}} \right] d\xi \right| \\ &\leq \int_{\gamma} |f(\xi)| \frac{1}{|\xi - z||\xi - a|} \sum_{k=0}^{n-1} \frac{1}{|\xi - z|^{n-1-k}|\xi - a|^k} + \frac{n|f(\xi)|}{|\xi - a|^{n+1}} d\xi \\ &< \int_{\gamma} M_k \frac{1}{\delta^2} \frac{n}{\delta^{n-1}} + \frac{nM_k}{\delta^{n+1}} d\xi \\ &\leq \frac{2nM_k}{\delta^{n+1}} 2\pi r \\ &< \varepsilon. \end{aligned}$$

Thus,

$$F'(a) = \lim_{z \rightarrow a} \frac{F(z) - F(a)}{z - a} = n \int_{\gamma} \frac{f(\xi)}{(\xi - a)^{n+1}} d\xi.$$

It follows that

$$f^{(n)}(a) = \frac{(n-1)!F'(a)}{2\pi i} = \frac{n!}{2\pi i} \int_C \frac{f(\xi)d\xi}{(\xi-a)^{n+1}}$$

and the proof is complete by induction. □



## CHAPTER 3

### RESIDUES

In the previous chapter the Cauchy-Goursat Theorem says that if the function  $f$  is analytic at all points interior to and on a simple closed contour  $C$ , the integral of  $f$  on  $C$  is zero. But, what if  $f$  fails to be analytic at a finite number of isolated points interior to  $C$ ? In order to answer this question, we define the concept of residue and present Cauchy's Residue Theorem. This theorem will contribute to the evaluation of integrals of some non-analytic functions and depends on finding specific numbers called residues.

In order to find the residue of a function  $f(z)$  that is not analytic at some  $z_0$ , we expand it into a series of positive and negative powers of  $(z - z_0)$ . The theorem allowing us to do this is Laurent's Theorem.

#### 3.1 Laurent Series

DEFINITION 3.1. An annulus is a region in the complex plane defined by

$$\{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\} \quad \text{or} \quad \{z \in \mathbb{C} : R_1 < |z - z_0|\}.$$

When  $R_1 = 0$ , the region is often called a punctured disc. When a property holds for all  $z$  in a punctured disc with its center at  $z_0$ , we say that property holds near  $z_0$ .

THEOREM 3.2. (LAURENT'S THEOREM) [2] If  $f$  is analytic on an annulus  $D$  and  $C$  is any positively oriented simple closed curve in the interior of  $D$  about  $z_0$ , then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,$$

for all  $z \in D$ , where each

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}}.$$

*Proof.* Let  $R_1$  and  $R_2$  be the inner and outer radii of  $D$ . Let  $z \in C$  and consider the simple closed curves  $C_2$  traversing  $\{z : |z - z_0| = r_2\}$  counterclockwise and  $C_1$  traversing  $\{z : |z - z_0| = r_1\}$  clockwise in  $D$ , where  $R_1 < r_1 < |z - z_0| < r_2 < R_2$ . Let  $C_3$  be any radial line segment not containing  $z$  and going from  $C_1$  to  $C_2$ . Thus by Cauchy's Integral Formula,

$$\begin{aligned} 2\pi i f(z) &= \int_{C_2 - C_3 - C_1 + C_3} \frac{f(\xi)d\xi}{\xi - z} \\ &= \int_{C_2} \frac{f(\xi)d\xi}{\xi - z} - \int_{C_1} \frac{f(\xi)d\xi}{\xi - z} \\ &= \int_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0) - (z - z_0)} - \int_{C_1} \frac{f(\xi)d\xi}{(\xi - z_0) - (z - z_0)} \\ &= \int_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0)(1 - \frac{z - z_0}{\xi - z_0})} - \int_{C_1} \frac{f(\xi)d\xi}{(z - z_0)(1 - \frac{\xi - z_0}{z - z_0})} \\ &= \int_{C_2} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi - \int_{C_1} \frac{f(\xi)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0}\right)^n d\xi \\ &= \int_{C_2} f(\xi) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} d\xi - \int_{C_1} f(\xi) \sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z - z_0)^n} d\xi. \end{aligned}$$

Since for all  $\xi \in C_2$ ,  $|z - z_0| < |\xi - z_0|$ , and for all  $\xi \in C_1$ ,  $|\xi - z_0| < |z - z_0|$ , the geometric series above are absolutely convergent. We can therefore interchange the

order of summation and integration. Thus

$$2\pi i f(z) = \sum_{n=0}^{\infty} \left( (z - z_0)^n \int_{C_2} \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}} \right) - \sum_{n=1}^{\infty} \left( \frac{1}{(z - z_0)^n} \int_{C_1} \frac{f(\xi) d\xi}{(\xi - z_0)^{-n+1}} \right).$$

Since  $-C_1$  and  $C_2$  are both homotopic to  $C$ , we can replace each of them by  $C$  and we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where each

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}}.$$

The proof is complete. □

**DEFINITION 3.3.** *If for all  $z$  in an annulus  $D$ ,*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

*we call this series, the Laurent series of  $f$  on  $D$ .*

In many instances we will obtain a Laurent series for a function and need to know that it is the same series given in Laurent's Theorem. We will see definition (3.3) designates just the one series.

**LEMMA 3.4.** *If there exists  $r > 0$ , such that  $\sum_{n=-\infty}^{\infty} \xi_n (z - a)^n = 0$ , for every  $z \in B(a, r)$ , then  $\xi_n = 0$  for all  $n \in \mathbb{Z}$ .*

*Proof.* Let  $C$  be any simple positively oriented closed contour around  $a$  and inside  $B(a, R)$ . First note that by each of Cauchy's Integral Formulas (2.28) applied to any constant function,  $f(z) = \xi$ , we have that

$$\int_C \xi(z-a)^n dz = \begin{cases} 2\pi i, & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Moreover by the Cauchy-Goursat Theorem 2.26,

$$\text{if } n \geq 0, \text{ then } \int_C \xi(z-a)^n dz = 0.$$

Thus for each  $n \in \mathbb{N}$ ,

$$\begin{aligned} 0 &= \int_C (z-a)^{k+1} \sum_{n=-\infty}^{\infty} \xi_n(z-a)^n dz \\ &= \int_C \sum_{n=-\infty}^{\infty} \xi_n(z-a)^{n+(k+1)} dz \\ &= \int_C \sum_{n=-(k+1)}^{\infty} \xi_n(z-a)^{n+(k+1)} dz + \int_C \sum_{n=(k+2)}^{\infty} \frac{\xi_{-n}}{(z-a)^{n-(k+1)}} dz \\ &= \int_C \sum_{n=0}^{\infty} \xi_{n-(k+1)}(z-a)^n dz + \int_C \sum_{n=1}^{\infty} \frac{\xi_{-n+(k+1)}}{(z-a)^n} dz \\ &= \sum_{n=0}^{\infty} \int_C \xi_{n-(k+1)}(z-a)^n dz + \int_C \frac{\xi_k}{z-a} dz + \sum_{n=2}^{\infty} \int_C \frac{\xi_{-n+(k+1)}}{(z-a)^n} dz \\ &= 0 + 2\pi i \xi_k + 0 \\ &= 2\pi i \xi_k \\ &= \xi_k. \end{aligned}$$

The proof is complete. □

THEOREM 3.5. (*The Uniqueness Theorem*)[2] *If there exists  $r > 0$ , such that*

$$\sum_{n=-\infty}^{\infty} \alpha_n(z-a)^n = \sum_{n=-\infty}^{\infty} \beta_n(z-a)^n,$$

*for every  $z \in B(a, r)$ , then  $\alpha_n = \beta_n$  for all  $n \in \mathbb{Z}$ .*

*Proof.* For all  $z \in B(a, r)$ , we have

$$\sum_{n=-\infty}^{\infty} (\alpha_n - \beta_n)(z-a)^n = \sum_{n=-\infty}^{\infty} \alpha_n(z-a)^n - \sum_{n=-\infty}^{\infty} \beta_n(z-a)^n = 0.$$

Thus by LEMMA 3.4, we have  $\alpha_n = \beta_n$  for all  $n \in \mathbb{Z}$ . □

### 3.2 Singular Points

DEFINITION 3.6. *A function  $f$  has an isolated singularity at  $z_0$ , if there exists*

*$R > 0$  such that  $f$  is analytic on the punctured disc  $\{z : 0 < |z - z_0| < R\}$  but not at  $z_0$ .*

DEFINITION 3.7. *An isolated singularity,  $z_0$ , of  $f$  is removable, if there exists a function  $g$  and  $R > 0$  such that  $g$  is analytic on  $B(z_0, R)$  and  $f(z) = g(z)$  on the punctured disc  $\{z : 0 < |z - z_0| < R\}$ .*

DEFINITION 3.8. *Let  $z_0$  be an isolated singular point of  $f(z)$ . Then  $z_0$  is a pole of order  $m$  of  $f$ , if there exists a natural number  $m$  and  $r > 0$  such that*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

for a function  $\phi$ , that is analytic on  $\{z : |z - z_0| < r\}$ , with  $\phi(z_0) \neq 0$ .  $z_0$  is a simple pole when  $m = 1$ .

DEFINITION 3.9. An isolated singularity,  $z_0$ , of  $f$  is essential, if it is neither removable nor a pole.

### 3.3 Definition of Residue

Let  $z_0$  be an isolated singularity of a function  $f$ , which is analytic on  $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$ . Then  $f$  has a Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n,$$

where each

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)d\xi}{(\xi - z_0)^{(n+1)},}$$

for any positively oriented simple closed curve  $C$  in the interior of  $D$ .

DEFINITION 3.10. When  $f$  has a Laurent series representation as in (3.3), and  $z_0$  is an isolated singular point of  $f$ , the residue of  $f$  at  $z_0$  is

$$\text{Res}(f, z_0) = A_{-1} = \frac{1}{2\pi i} \int_C f(z)dz.$$

### 3.4 Residue at Infinity

DEFINITION 3.11. If  $f$  is analytic on  $\{z : |z| > R\}$ , for some  $R > 0$ , then we say  $f$  has an isolated singularity at  $\infty$ .

DEFINITION 3.12. Let  $f$  be analytic on  $\{z : |z| > R\}$  and let  $C$  be the positively oriented circle  $\{z : |z| = R\}$ . When all the singularities of  $f$ , except  $\infty$ , are inside  $C$ , we define

$$\operatorname{Res}(f, \infty) = \frac{1}{2\pi i} \int_{-C} f(z) dz.$$

THEOREM 3.13. If  $f$  is analytic on  $\{z : |z| > R\}$ , for some  $R > 0$ , with all the singularities of  $f$ , except  $\infty$ , inside  $\{z : |z| < R\}$  then

$$\operatorname{Res}(f, \infty) = -\operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right].$$

*Proof.* let  $C$  be the positively oriented circle  $\{z : |z| = R\}$ . By Laurent's Theorem  $f$  has a valid Laurent series representation outside  $C$  we denote by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

For all  $z$  such that  $0 \neq |1/z| < 1/R$ , we have that  $|z| > R$ , hence  $f(1/z)/z^2$  is analytic at  $z$ . Moreover,

$$\frac{f(1/z)}{z^2} = \sum_{n=-\infty}^{\infty} a_n z^{-n-2} = \sum_{n=-\infty}^{\infty} a_{-n} z^{n-2} = \sum_{n=-\infty}^{\infty} a_{-n-2} z^n,$$

which must be the Laurent series of  $f(1/z)/z^2$ , valid on  $\{z : 0 < |z| < R\}$ , by the

uniqueness theorem. From this, we see

$$\operatorname{Res}\left[\frac{f(1/z)}{z^2}, 0\right] = a_{-(-1)-2} = a_{-1} = \frac{1}{2\pi i} \int_C f(z) dz = -\operatorname{Res}(f, \infty),$$

completing the proof. □

### 3.5 The Cauchy Residue Theorem

The Residue Theorem was discovered by Augustin-Louis Cauchy in 1814 and immediately became a powerful tool in complex analysis for computing line integrals. The Residue Theorem soon became very importance in real analysis as a tool for evaluating some difficult real integrals, and then, as we show, in finding infinite sums, as well as other applications.

**THEOREM 3.14.** [2] *Suppose that  $f$  is an analytic function on and inside a simple closed positively oriented curve  $C$ , except at finitely many isolated singularities  $z_1, \dots, z_n$  inside  $C$ . Then*

$$\int_C f(z) dz = 2\pi i \sum_{i=1}^n \operatorname{Res}(f, z_i).$$

*Proof.* Let  $C$  be a simple closed positively oriented curve, and suppose  $f$  is an analytic function inside and on  $C$ . Consider circles,  $C_1, C_2, \dots, C_n$ , centered at  $z_1, \dots, z_n$ , where each circle,  $C_i$ , has radius  $r_i$ , sufficiently small, so that  $C_1, \dots, C_n$  are disjoint and in the interior of  $C$ . We construct a simple closed positively oriented curve  $C'$  that surrounds all the points  $z_i$  along each circle  $C_i$  and joins these small circles with segments.



Since the curve  $C'$  follows each segment two times with opposite orientation it is enough to sum the integrals of  $f$  around the small circles. By the definition of residue we have

$$\int_C f(z)dz = \int_{C'} f(z)dz = \sum_{i=1}^n \int_{C_i} f(z)dz = 2\pi i \sum_{i=1}^n \text{Res}(f, z_i).$$

The proof is complete. □

### 3.6 Zeros and Poles

Since the zeros of the denominator of a quotient function cause the function not to be analytic, there is an obvious relationship between zeros and poles. In this section we explore this relationship.

DEFINITION 3.15. *When  $f$  is analytic at  $z_0$ ,  $f$  has a zero of order  $n$  at  $z_0$ , if*

$$f(z) = (z - z_0)^n q(z),$$

*for some function  $g$  such that  $q(z_0) \neq 0$  and  $q$  is analytic on  $B(z_0, \varepsilon)$  for some  $\varepsilon > 0$ .*

THEOREM 3.16. [2] *If  $z_0$  is a pole of  $f$ , then  $\lim_{z \rightarrow z_0} f(z) = \infty$ .*

*Proof.* Let  $n$  be the order of  $z_0$ . Then there exists a function  $\phi(z)$ , such that

$\phi(z_0) \neq 0$ ,  $\phi$  is analytic near  $z_0$  and

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}.$$

Therefore,  $\lim_{z \rightarrow z_0} f(z) = \infty$ . □

**THEOREM 3.17.** [2] *Assume that  $g(z)$  and  $h(z)$  are analytic functions at  $z = z_0$ ,  $h(z)$  has a zero of order  $n$  at  $z = z_0$  and  $g(z_0) \neq 0$ . Then*

$$f(z) = \frac{g(z)}{h(z)}$$

*has a pole of order  $n$  at  $z = z_0$ .*

*Proof.* Since  $h(z)$  has a zero of order  $n$  at  $z_0$ ,  $h(z) = (z - z_0)^n q(z)$ , where  $q(z_0) \neq 0$ , and  $q$  is analytic near  $z_0$ . Thus

$$f(z) = \frac{g(z)}{(z - z_0)^n q(z)} = \frac{g(z)/q(z)}{(z - z_0)^n}.$$

We have that  $g(z)/q(z)$  is analytic near  $z_0$  and not zero at  $z_0$ . We conclude that  $f(z)$  has a pole of order  $n$ . □

### 3.7 Residue at a Pole

In the previous section we saw that the residue of a function  $f(z)$  with an isolated singularity at a point  $z_0$  could be found within the Laurent expansion of  $f$  as the coefficient of the  $(z - z_0)^{-1}$  term. That can often be difficult. This section contains theorems for finding residues with alternative techniques that are often more convenient to use.

#### 3.7.1 Residue at a Pole of Order $m$

**THEOREM 3.18.** [2] *Let  $f$  be analytic on the punctured disc  $\{z : 0 < |z - z_0| < r\}$  for some  $r > 0$ . Then if  $f$  has a pole of order  $m$  at  $z_0$ , then*

$$\operatorname{Res}(f, z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, \quad (3.1)$$

where  $\phi$  is as given in definition (3.8).

*Proof.* Since  $f$  has a pole of order  $m$ , there exists a natural number  $m$  and  $r > 0$  such that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m},$$

for a function  $\phi$ , that is analytic on  $B(z_0, r)$ , with  $\phi(z_0) \neq 0$ . It follows that

$$\begin{aligned} f(z) &= \frac{1}{(z - z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z - z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!} (z - z_0)^{n-m} \\ &= \sum_{n=-m}^{\infty} \frac{\phi^{(n+m)}(z_0)}{(n+m)!} (z - z_0)^n. \end{aligned}$$

Since Laurent series are unique, this is the Laurent series of  $f$ . Therefore,

$$\operatorname{Res}(f, z_0) = A_{-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

The proof is complete. □

**COROLLARY 3.19.** [2] *If  $f(z)$  has a pole of order  $m$  at  $z_0$ , then*

$$\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \left[ \frac{(z - z_0)^m f(z)}{(m-1)!} \right].$$

*Proof.* Since  $\phi$  is analytic on  $B(z_0, r)$ ,  $\phi^{(m-1)}$  is continuous. Moreover,

$\phi(z) = (z - z_0)^m f(z)$  on  $0 < |z - z_0| < r$ , hence

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \\ &= \lim_{z \rightarrow z_0} \frac{\phi^{(m-1)}(z)}{(m-1)!} \\ &= \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} \left[ \frac{(z - z_0)^m f(z)}{(m-1)!} \right]. \end{aligned}$$

The proof is complete. □

### 3.7.2 Residues at Simple Poles

COROLLARY 3.20. [2] *If  $z_0$  is a simple pole of  $f(z)$ , then*

$$\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

*Proof.* Since a simple pole is of order  $m = 1$ , this is immediate from COROLLARY

3.19. □

THEOREM 3.21. [2] *Let  $p(z)$  and  $q(z)$  both be analytic at  $z_0$  and suppose  $q(z_0) = 0$ ,*

*$p(z_0) \neq 0$ , and  $q'(z_0) \neq 0$ . If  $f(z) = \frac{p(z)}{q(z)}$ , then  $z_0$  is a simple pole of  $f(z)$  and*

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

*Proof.* First, we need to show that  $z_0$  is a zero of order 1. Suppose that  $q$  has a zero

of order  $n \geq 2$  at  $z_0$ , then  $q(z) = (z - z_0)^n \phi(z)$  for an analytic function  $\phi$ . So,

$$\begin{aligned} q'(z) &= n(z - z_0)^{n-1} \phi(z) + (z - z_0)^n \phi'(z) \\ &= (z - z_0)[n(z - z_0)^{n-2} \phi(z) + (z - z_0)^{n-1} \phi'(z)]. \end{aligned}$$

Since  $n \geq 2$ , then  $n - 2 \geq 0$ . So,  $q'$  has a zero, and  $q'(z_0) \neq 0$ . Order of  $q$ 's zero is 1.

Now, by THEOREM 3.17,  $z_0$  is a simple pole. Thus by COROLLARY 3.21 and

because  $q(z_0) = 0$ , we have

$$\begin{aligned} \text{Res}(f, z_0) &= \lim_{z \rightarrow z_0} (z - z_0) \frac{p(z)}{q(z)} \\ &= \lim_{z \rightarrow z_0} \frac{(z - z_0)p(z)}{q(z) - q(z_0)} \\ &= \lim_{z \rightarrow z_0} p(z) \lim_{z \rightarrow z_0} \frac{z - z_0}{q(z) - q(z_0)} \\ &= \frac{p(z_0)}{q'(z_0)}. \end{aligned}$$

The proof is complete. □

**THEOREM 3.22.** [2] *If  $g(z)$  is analytic at  $z_0$  and  $f(z)$  has a simple pole at  $z_0$ , then*

$$\text{Res}(fg, z_0) = g(z_0)\text{Res}(f, z_0).$$

*Proof.* Since  $g$  is analytic at  $z_0$ , it's easy to see  $fg$  also has a simple pole at  $z_0$ .

Therefore, by COROLLARY 3.20, we have

$$\begin{aligned}
\operatorname{Res}(fg, z_0) &= \lim_{z \rightarrow z_0} [(z - z_0)f(z)g(z)] \\
&= \lim_{z \rightarrow z_0} [(z - z_0)f(z)] \lim_{z \rightarrow z_0} g(z) \\
&= g(z_0)\operatorname{Res}(f, z_0).
\end{aligned}$$

The proof is complete. □

LEMMA 3.23. *Suppose that  $f$  is analytic and not identically zero in a region  $G$ .*

*i. If  $z_0$  is a zero of  $f$  of order  $k \geq 1$ , then  $f'/f$  has a simple pole at  $z_0$  and*

$$\operatorname{Res}(f'/f, z_0) = k.$$

*ii. If  $z_0$  is a pole of  $f$  of order  $k \geq 1$ , then  $f'/f$  has a simple pole at  $z_0$  and*

$$\operatorname{Res}(f'/f, z_0) = -k.$$

*Proof.* (i) Since  $f$  has a zero of order  $k$ , there exist a function  $\phi$  and  $R > 0$  such that  $f(z) = \phi(z)(z - z_0)^k$ ,  $\phi(z_0) \neq 0$  and  $\phi$  is analytic in  $B(z_0, R)$ . For all  $z$  in  $B(z_0, R)$ , we have

$$\frac{f'(z)}{f(z)} = \frac{k\phi(z)(z - z_0)^{k-1} + \phi(z)'(z - z_0)^k}{\phi(z)(z - z_0)^k} = \frac{k}{z - z_0} + \frac{\phi'(z)}{\phi(z)}.$$

However  $\phi(z_0) \neq 0$  and  $\phi'/\phi$  is analytic at  $z_0$ , hence  $\phi'/\phi$  has a convergent Taylor series. Thus,

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Therefore, we conclude that  $f'/f$  has a simple pole at  $z_0$ , and

$$\operatorname{Res}(f'/f, z_0) = k.$$

(ii) Since  $f$  has a pole of order  $k$ , we have  $f(z) = \phi(z)/(z - z_0)^k$  and  $R > 0$  such that  $\phi(z_0) \neq 0$  and  $\phi$  is analytic in  $B(z_0, R)$ . For any  $z$  in  $B(z_0, R)$ ,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{\phi'(z)(z - z_0)^k - k\phi(z)(z - z_0)^{k-1}}{(z - z_0)^{2k}} \cdot \frac{(z - z_0)^k}{\phi(z)} \\ &= \frac{\phi'(z)(z - z_0)^k - k\phi(z)(z - z_0)^{k-1}}{\phi(z)(z - z_0)^k} \\ &= \frac{\phi'(z)}{\phi(z)} - \frac{k}{z - z_0}. \end{aligned}$$

Since  $\phi(z_0) \neq 0$  and  $\phi'/\phi$  is analytic at  $z_0$ , it has a convergent Taylor series. So,

$$\frac{f'(z)}{f(z)} = \frac{-k}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Hence,  $f'/f$  has a simple pole at  $z_0$ , and

$$\operatorname{Res}(f'/f, z_0) = -k,$$

and the proof is complete. □

**THEOREM 3.24.** *If  $p$  is a polynomial of degree at least 2, and  $z_1, z_2, \dots, z_n$  are the zeros of  $p$ , then*

$$\sum_{j=1}^n \operatorname{Res}\left(\frac{1}{p(z)}, z_j\right) = 0.$$

*Proof.* Suppose that  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ , where  $a_n \neq 0$  and  $n \geq 2$ . By

the fundamental theorem of algebra,  $p$  has at most  $n$  different zeros. Let  $C$  be a circle centered at 0 with radius  $R$ , sufficiently large that every singularity of  $1/p$  is inside  $C$ . We now consider

$$\frac{1}{z^2} \frac{1}{p\left(\frac{1}{z}\right)} = \frac{1}{z^2} \frac{1}{\frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + a_0} = \frac{1}{z^2} \frac{z^n}{a_n + a_{n-1}z + \cdots + a_0 z^n}$$

Since  $n \geq 2$ , the singularity at  $z = 0$  is removable. Therefore, by the Cauchy-Goursat theorem 2.26,

$$\int_C \frac{1}{z^2} \frac{1}{p\left(\frac{1}{z}\right)} dz = 0.$$

Thus by Cauchy's Residue Theorem 3.14, the definition of residue at infinity 3.12, and THEOREM 3.13, we have

$$\begin{aligned} \sum_{j=1}^n \operatorname{Res} \left( \frac{1}{p(z)}, z_j \right) &= \frac{1}{2\pi i} \int_C \frac{1}{p(z)} dz \\ &= -\operatorname{Res} \left( \frac{1}{p(z)}, \infty \right) \\ &= \operatorname{Res} \left( \frac{1}{z^2 p\left(\frac{1}{z}\right)}, 0 \right) \\ &= 0. \end{aligned}$$

This finishes the proof. □



## CHAPTER 4

### BERNOULLI NUMBERS

Bernoulli numbers have long been used in algebra and number theory. In this section we define and explore properties of Bernoulli numbers in the framework of complex analysis. In the next chapter, we use them to obtain some important infinite sums.

#### 4.1 The Bernoulli Numbers

DEFINITION 4.1. *The Bernoulli numbers  $\{B_n\}_1^\infty$  are defined recursively by,*

$$B_0 = 1 \quad \text{and}$$

$$B_n = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \quad \text{for } n \geq 1.$$

LEMMA 4.2. [2] *Let  $F(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$  and let  $f(z) = 1/F(z)$ . Then  $f(z)$  is analytic on  $B(0, 2\pi)$  and*

$$f(z) = \begin{cases} \frac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}. \quad (4.1)$$

*Proof.* Since  $F(0) = 1$ , for  $z \neq 0$ ,

$$\frac{e^z - 1}{z} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} = F(z).$$

Thus (4.1) follows, and from that, we see  $f$  is analytic when  $e^z \neq 1$ , that is, when  $z \neq 2\pi ik$  for some  $k \in \mathbb{Z} \setminus \{0\}$ . Therefore,  $f(z)$  is analytic for all  $z$  such that  $|z| < 2\pi$ . □

**THEOREM 4.3.** [2] *Let  $F(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$  and let  $f(z) = 1/F(z)$ . Then for all*

*$z \in B(0, 2\pi)$*

$$f(z) = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

where  $\{B_n\}_1^{\infty}$  are the Bernoulli numbers.

*Proof.* By LEMMA 4.2,  $f$  is analytic on  $B(0, 2\pi)$ . Therefore,  $f$  has a convergent Maclaurin Series on  $B(0, 2\pi)$ , say

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

Since  $f(z)$  and  $F(z) = (e^z - 1)/z$  are reciprocals, we have

$$1 = F(z)f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

By the Cauchy product theorem [2], we then have for each  $z \in B(0, 2\pi)$ ,

$$1 = \sum_{k=1}^{\infty} c_k z^k,$$

where for each  $n$ ,

$$c_n = \sum_{k=0}^n \frac{a_k}{k!} \frac{1}{(n-k)!} = \sum_{k=0}^n \frac{a_k}{k!(n-k+1)!}.$$

The Maclaurin series for 1 has all zero coefficients, except  $c_0 = 1$ , hence by the uniqueness of Taylor series, for all  $n \neq 0$ ,  $c_n = 0$ . Thus for  $n \geq 1$ ,

$$0 = c_n = \sum_{k=0}^n \frac{a_k}{k!(n-k)!} = \frac{a_n}{n!} + \sum_{k=0}^{n-1} \frac{a_k}{k!(n-k+1)!}.$$

It follows that,

$$a_n = -n! \sum_{k=0}^{n-1} \frac{a_k}{k!(n-k+1)!} = \frac{-1}{n+1} \sum_{k=0}^{n-1} \frac{(n+1)!}{k!(n-k+1)!} a_k = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} a_k.$$

Since  $a_0 = B_0$  and  $\{a_n\}_0^\infty$  and  $\{B_n\}_0^\infty$  have the same recursion formula, we conclude they are the same sequence.  $\square$

DEFINITION 4.4. *In light of LEMMA 4.2 and THEOREM 4.3, we call  $z/(e^z - 1)$  the generating function for the Bernoulli numbers and write*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

*assuming the value 1 at the removable singularity at zero.*

COROLLARY 4.5. [2] *The odd Bernoulli numbers are zero except  $B_1$ .*

*Proof.* It is easy to see the function  $f(z) = \frac{z}{e^z - 1} + \frac{z}{2} - 1$  is even. Since for  $z \neq 0$ ,

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n,$$

we see

$$\sum_{n=2}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} + \frac{z}{2} - 1$$

is even as well. Therefore,

$$\sum_{n=2}^{\infty} \frac{B_n}{n!} z^n = \sum_{n=2}^{\infty} (-1)^n \frac{B_n}{n!} z^n.$$

It follows that when  $k > 2$  is odd  $B_k = -B_k$ , and therefore zero. □

## 4.2 Results

In this section we use the results of the previous section to find several of our main results. These sums, found using Bernoulli numbers, will also become powerful tools in the evaluation of other series in the next chapter.

**RESULT 4.6.** *When  $0 < |z| < \pi$ ,*

$$z \coth z = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

*Proof.* First note that when  $0 < |z| < \pi$ ,

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} \frac{e^z + 1}{e^z - 1} = \frac{z}{2} \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2} \coth \frac{z}{2}.$$

When  $0 < |z| < \pi$ , by DEFINITION 4.4, we have

$$z \coth z = \frac{2z}{e^{2z} - 1} + z = \sum_{n=0}^{\infty} \frac{B_n}{n!} (2z)^n + \frac{2z}{2} = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} (2z)^n.$$

Now, since  $B_{2n+1} = 0$ , for all  $n \geq 1$ , by COROLLARY 4.5, this simplifies to

$$z \coth z = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

This finishes the proof. □

RESULT 4.7. *When  $|z| < \pi$ ,*

$$z \cot z = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

*Proof.* In RESULT 4.7 replace  $z$  by  $iz$ , and since  $iz \coth(iz) = z \cot z$ , we have,

$$z \cot z = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}(i)^{2n}}{(2n)!} z^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

This finishes the proof. □

RESULT 4.8. When  $|z| < \frac{\pi}{2}$ ,

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1}.$$

*Proof.* Since  $\tan z = \frac{1}{\cot z} = \cot z - \frac{\cot^2 z - 1}{\cot z} = \cot z - 2 \cot(2z)$ , we have by RESULT 4.7, that

$$\begin{aligned} \tan z &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1} - 2 \sum_{n=0}^{\infty} (-1)^n \frac{2^{4n-1} B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}(1-2^{2n})B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1}. \end{aligned}$$

This finishes the proof. □

RESULT 4.9. When  $|z| < \pi$ ,

$$\csc z = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2^{2n}-2)B_{2n}}{(2n)!} z^{2n-1}.$$

*Proof.* Since  $\csc z = 1/\sin z$ , we have for all  $z$  such that  $|z| < \pi$ ,

$$\csc 2z = \frac{1}{2 \sin z \cos z} = \frac{\csc^2 z}{2 \cot z} = \cot z - \frac{\cot^2 z - 1}{2 \cot z} = \cot z - \cot 2z.$$

So,

$$\begin{aligned}\csc z &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{2^{2n-1} (2n)!} z^{2n-1} - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(2 - 2^{2n}) B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2^{2n} - 2) B_{2n}}{(2n)!} z^{2n-1}.\end{aligned}$$

□

## CHAPTER 5

### SUMMING SERIES BY RESIDUES

This chapter contains more of our main results, making use of the tools presented in the previous chapters. We will use the theory of residues to develop a powerful technique to find sums of the form  $\sum_k f(k)$ , where  $f(z) = \frac{q(z)}{p(z)}$  is a rational function with degree  $p(z)$  - degree  $q(z) \geq 2$ .

#### 5.1 Foundations

In this section we develop the technique that will produce more of our main results. The concept is of capturing an ever widening set of singularities inside contours, obtaining corresponding finite sums, and then deducing the desired infinite sum.

DEFINITION 5.1. *For convenience we will refer to a contour  $C_n$  as a basic contour, provided*

1.  $C_n$  is positively oriented.
2.  $C_n$  is simple.
3.  $C_n$  is piecewise smooth.
4.  $C_n$  is centered at the origin.
5.  $C_n$  is on a square of side  $2n + 1$  or on a circle of radius  $n + 1/2$  for any  $n \in \mathbb{N}$ .



LEMMA 5.2. Let  $r_n = n + 1/2$  for any  $n \in \mathbb{N}$  and let  $0 < \varepsilon < 1$ . If  $(x, y)$  is on the intersection of

$$(x - r_n)^2 + y^2 = \varepsilon^2 \quad \text{and} \quad x^2 + y^2 = r_n^2$$

for any  $n \in \mathbb{N}$ , then  $|y| \geq \varepsilon/2$ .

*Proof.* Solving this system of equations yields

$$y = \pm \varepsilon \sqrt{1 - \frac{\varepsilon^2}{(2r_n)^2}}.$$

Since  $r_n \geq 3/2$  for all  $n$  and  $\varepsilon < 1$ ,

$$\sqrt{1 - \frac{\varepsilon^2}{(2r_n)^2}} \geq \sqrt{1 - \frac{1}{9}} = \sqrt{8/9} > \frac{1}{2}.$$

Therefore,  $|y| \geq \varepsilon/2$ . □

LEMMA 5.3. [2] *There exists  $B > 0$  such that whenever  $z$  is on any basic contour  $C_n$ ,*

$$|\cot(\pi z)| < B \quad \text{and} \quad |\csc(\pi z)| < B.$$

*Proof.* For any  $z = x + iy$ , since  $\cot$  and  $\csc$  are odd functions, we will, without loss of generality, assume  $y \geq 0$ . It follows that

$$|e^{2\pi iz}| = |e^{2\pi ix - 2\pi y}| = |e^{2\pi ix} e^{-2\pi y}| = e^{-2\pi y} \leq 1. \quad (5.1)$$

Let  $z \in B(\frac{1}{2}, \frac{1}{4})$ . We have  $1/4 < x < 3/4$ , hence  $\cos(2\pi x) < 0$ . It follows that

$$\begin{aligned}
|e^{2\pi iz} - 1| &\geq |Re(e^{2\pi ix - 2\pi y} - 1)| \\
&= |Re(e^{-\pi y} (\cos(2\pi x) + i \sin(2\pi x)) - 1)| \\
&= |e^{-\pi y} \cos(2\pi x) - 1| \\
&= 1 - e^{-\pi y} \cos(2\pi x) \\
&\geq 1.
\end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we have for all  $z \in B(\frac{1}{2}, \frac{1}{4})$ ,

$$|\cot(\pi z)| = \left| \frac{e^{\pi iz} + e^{-\pi iz}}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1} \right| \leq |e^{2\pi iz}| + 1 \leq 2 \tag{5.3}$$

and

$$|\csc(\pi z)| = \left| \frac{1}{e^{\pi iz} - e^{-\pi iz}} \right| = \left| \frac{e^{\pi iz}}{e^{2\pi iz} - 1} \right| \leq 1. \tag{5.4}$$

Now, consider  $z \in B(n + \frac{1}{2}, \frac{1}{4})$  for any integer  $n \neq 0$ . Since  $z - n \in B(\frac{1}{2}, \frac{1}{4})$  and both  $|\cot|$  and  $|\csc|$  are  $\pi$ -periodic, we see

$$|\cot(\pi z)| = |\cot(\pi(z - n))| \leq 2 \tag{5.5}$$

and

$$|\csc(\pi z)| = |\csc(\pi(z - n))| \leq 1 \tag{5.6}$$

for all  $z \in S = \bigcup_{n \neq 0} B(n + \frac{1}{2}, \frac{1}{4})$ .

For  $z$  on a square with vertices at  $\pm(n + \frac{1}{2}) \pm i(n + \frac{1}{2})$  but  $z \notin S$ , we see  $|y| \geq 1/4$ .

If  $z$  is on a circle of radius  $r_n = n + 1/2$  but  $z \notin S$ , then  $|y| \geq |b|$ , where  $(a, b)$  is a point of intersection of

$$(x - r_n)^2 + y^2 = 1/16 \quad \text{and} \quad x^2 + y^2 = r_n^2.$$

By LEMMA 5.2,  $|y| \geq |a| \geq 1/8$ . Noting that  $e^{-2\pi y} < 1$ , it follows that

$$|\cot(\pi z)| = \left| \frac{e^{2\pi i x} e^{-2\pi y} + 1}{e^{2\pi i x} e^{-2\pi y} - 1} \right| \leq \frac{e^{-2\pi y} + 1}{|e^{-2\pi y} - 1|} \leq \frac{e^{-2\pi y} + 1}{1 - e^{-2\pi y}} \leq \frac{e^{-2\pi/8} + 1}{1 - e^{-\pi/4}}. \quad (5.7)$$

and

$$|\csc(\pi z)| = \left| \frac{e^{\pi i z}}{e^{2\pi i z} - 1} \right| = \frac{e^{\pi i x} e^{-\pi y}}{e^{2\pi i x} e^{-2\pi y} - 1} \leq \frac{e^{-\pi y} + 1}{|e^{-2\pi y} - 1|} \leq \frac{e^{-\pi/8} + 1}{1 - e^{-\pi/4}}. \quad (5.8)$$

Taking  $B$  to be the minimum of the bounds given in lines (5.5), (5.6), (5.7) and (5.8) yields the desired bound for all  $z$  on all basic contours.  $\square$

LEMMA 5.4. [2] *Let  $n$  be a positive integer, and let  $C_n$  be a basic contour. If*

*$f(z) = \frac{p(z)}{q(z)}$  is a rational function with degree  $q(z)$  - degree  $p(z) \geq 2$ , then,*

$$\lim_{n \rightarrow \infty} \int_{C_n} \frac{p(z)}{q(z)} \cot(\pi z) dz = 0 \quad (5.9)$$

and

$$\lim_{n \rightarrow \infty} \int_{C_n} \frac{p(z)}{q(z)} \csc(\pi z) dz = 0. \quad (5.10)$$

*Proof.* By LEMMA 5.3  $|\cot(\pi z)|$  and  $|\csc(\pi z)|$  are bounded by some  $B > 0$  on  $C_n$ .

The function  $12zf(z)$  is a rational function whose numerator is of degree at least one less than the degree of its denominator. Thus given any  $\varepsilon > 0$ , there exists a

number  $N$  such that when  $|z| \geq N$ ,

$$|2\pi z f(z)| < |12z f(z)| < \frac{\varepsilon}{B}.$$

We also assume that when  $n \geq N$ , each pole of  $f$  is inside  $C_n$ . Let  $n \geq N$ . In the case  $C_n$  is a square, given any  $z$  on  $C_n$ , we have  $1 \leq n < |z|$ . It follows that

$12|z| = 8|z| + 4|z| > 8n + 4$ . Thus since  $n \geq N$ ,  $|z| > N$  and hence

$$\left| \int_{C_n} f(z) \cot(\pi z) dz \right| \leq \int_{C_n} \frac{12|z|}{8n+4} |f(z)| B dz \leq \frac{L(C_n)}{8n+4} \frac{\varepsilon}{B} B = \varepsilon,$$

proving (5.9) in the case  $C_n$  is a square. For the case when  $C_n$  is a circle,

$|z| = n + 1/2$ , hence

$$\left| \int_{C_n} f(z) \cot(\pi z) dz \right| \leq \int_{C_n} \frac{2\pi|z|}{2\pi(n+1/2)} |f(z)| B dz \leq \frac{L(C_n)}{2\pi(n+1/2)} \frac{\varepsilon}{B} B = \varepsilon,$$

finishing (5.9).

The proof for both cases of (5.10) are the same as for (5.9), and are omitted.  $\square$

**THEOREM 5.5.** [2] *Suppose that  $f$  is analytic at an integer  $k$ , then*

- i.*  $\operatorname{Res}(f(z) \cot(\pi z), k) = \frac{1}{\pi} f(k).$
- ii.*  $\operatorname{Res}(f(z) \csc(\pi z), k) = \frac{(-1)^n}{\pi} f(k).$
- iii.*  $\operatorname{Res}(f(z) \tan(\pi z), \frac{2k+1}{2}) = \frac{1}{\pi} f\left(\frac{2k+1}{2}\right).$
- iv.*  $\operatorname{Res}(f(z) \sec(\pi z), \frac{2k+1}{2}) = \frac{(-1)^n}{\pi} f\left(\frac{2k+1}{2}\right).$

*Proof.* (i) Since  $\sin(\pi z) = 0$  if and only if  $z$  is an integer  $k$  and  $\cos(\pi k) \neq 0$ . We see by THEOREM 3.21,  $\cot(\pi z) = \cos(\pi z)/\sin(\pi z)$  has a simple pole at each integer  $k$  and

$$\operatorname{Res}(\cot(\pi z), k) = \frac{\cos(\pi k)}{\frac{d}{dz} \sin(\pi k)} = \frac{\cos(\pi k)}{\pi \cos(\pi k)} = \frac{1}{\pi}.$$

Therefore, by THEOREMS 3.21, 3.22,

$$\operatorname{Res}(f(z) \cot(\pi z), k) = f(k) \operatorname{Res}(\cot(\pi z), k) = \frac{f(k)}{\pi}.$$

(ii) Recall that  $\csc(\pi z) = 1/\sin(\pi z)$ , and as in part (i) above,  $\csc(\pi z) = 1/\sin(\pi z)$  has a simple pole at each integer  $k$ . So, by THEOREM 3.21,

$$\operatorname{Res}(\csc(\pi z), k) = \operatorname{Res}\left(\frac{1}{\sin(\pi z)}, k\right) = \frac{1}{\pi \cos(\pi k)} = \frac{(-1)^n}{\pi}.$$

Now, by THEOREMS 3.21, 3.22,

$$\operatorname{Res}(f(z) \csc(\pi z), k) = f(k) \operatorname{Res}(\csc(\pi z), k) = \frac{(-1)^n}{\pi} f(k).$$

(iii) Recall that  $\tan(\pi z) = \sin(\pi z)/\cos(\pi z)$ . Note that the zeros of  $\cos(\pi z)$  are  $\frac{2k+1}{2}$  where  $k$  is an integer. By THEOREM 3.21, those zeros are simple poles of  $\tan$  and

$$\operatorname{Res}\left(\tan(\pi z), \frac{2k+1}{2}\right) = \frac{\sin\left(\frac{2\pi k+\pi}{2}\right)}{\pi \sin\left(\frac{2\pi k+\pi}{2}\right)} = \frac{1}{\pi}.$$

Now, by THEOREM 3.22,

$$\begin{aligned} \operatorname{Res}\left(f(z) \tan(\pi z), \frac{2k+1}{2}\right) &= f\left(\frac{2k+1}{2}\right) \operatorname{Res}\left(\tan(\pi z), \frac{2k+1}{2}\right) \\ &= \frac{1}{\pi} f\left(\frac{2k+1}{2}\right). \end{aligned}$$

(iv) Recall that  $\sec(\pi z) = 1/\cos(\pi z)$ , and from the previous part

$\sec(\pi z) = 1/\cos(\pi z)$  has a simple pole at each  $\frac{2k+1}{2}$ . So, by THEOREM 3.21,

$$\operatorname{Res}\left(\sec(\pi z), \frac{2k+1}{2}\right) = \operatorname{Res}\left(\frac{1}{\cos(\pi z)}, \frac{2k+1}{2}\right) = \frac{1}{\sin\left(\frac{2\pi k+\pi}{2}\right)} = \frac{(-1)^n}{\pi}.$$

Now, by THEOREM 3.22,

$$\begin{aligned}\operatorname{Res}\left(f(z)\sec(\pi z), \frac{2k+1}{2}\right) &= f\left(\frac{2k+1}{2}\right)\operatorname{Res}\left(\sec(\pi z), \frac{2k+1}{2}\right) \\ &= \frac{(-1)^n}{\pi}f\left(\frac{2k+1}{2}\right).\end{aligned}$$

The proof is complete. □

THEOREM 5.6. *For every integer  $k$ ,  $\pi \coth(\pi z)$  has a simple pole at  $z = ik$  and*

$$\operatorname{Res}(\pi \coth(\pi z), ik) = 1.$$

*Proof.* If we let  $\sinh(\pi z) = 0$ , then  $e^{\pi z} - e^{-\pi z} = 0$ , hence  $e^{2\pi z} = 1$ . Therefore, the zeros of  $\sinh(\pi z)$  are  $z = ik$  for every integer  $k$ . Since  $\cosh \pi ik \neq 0$ , by theorem 3.21 the poles of  $\pi \coth(\pi z) = \pi \cosh(\pi z)/\sinh(\pi z)$  are simple and

$$\operatorname{Res}(\pi \coth(\pi z), ik) = \frac{\pi \cosh(\pi z)}{\frac{d}{dz} \sinh(\pi z)} = \frac{\pi \cosh(\pi k)}{\pi \cosh(\pi k)} = 1.$$

The proof is complete. □

## 5.2 Finite Sums

LEMMA 5.7. [3] *Let  $C_n$  be a basic contour. If  $f$  is analytic on  $C_n$ , except at finitely many singularities  $z_1, \dots, z_m$ , all inside  $C_n$  none of which are integers, then*

$$\sum_{k=-n}^n f(k) = \frac{1}{2i} \int_{C_n} f(z) \cot(\pi z) dz - \pi \sum_{j=1}^m \operatorname{Res}(f(z) \cot(\pi z), z_j)$$

and

$$\sum_{k=-n}^n (-1)^k f(k) = \frac{1}{2i} \int_{C_n} f(z) \csc(\pi z) dz - \pi \sum_{j=1}^m \operatorname{Res}(f(z) \csc(\pi z), z_j).$$

*Proof.*

By THEOREM 5.5,

$$\operatorname{Res}(f(z) \cot(\pi z), k) = \frac{f(k) \cos(\pi k)}{\pi \cos(\pi k)} = \frac{1}{\pi} f(k).$$

Moreover, since each  $z_j$  is inside  $C_n$ , we have, by the Cauchy Residue theorem, 3.14,

that

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_n} f(z) \cot(\pi z) dz &= \sum_{k=-n}^n \operatorname{Res}(f(z) \cot(\pi z), k) + \sum_{j=1}^m \operatorname{Res}(f(z) \cot(\pi z), z_j) \\ &= \frac{1}{\pi} \sum_{k=-n}^n f(k) + \sum_{j=1}^m \operatorname{Res}(f(z) \cot(\pi z), z_j). \end{aligned}$$

Therefore, we conclude that

$$\sum_{k=-n}^n f(k) = \frac{1}{2i} \int_{C_n} f(z) \cot(\pi z) dz - \pi \sum_{j=1}^m \operatorname{Res}(f(z) \cot(\pi z), z_j).$$

The proof of the second assertion is almost the same as the proof of the first. The

only difference is that by THEOREM 5.5,

$$\text{Res}(f(z) \csc(\pi z), k) = \frac{f(z)}{\pi \cos(\pi z)} = \frac{(-1)^k}{\pi} f(k).$$

The rest of the proof is exactly the same. □

DEFINITION 5.8. Let  $\delta > 0$  and suppose  $\alpha < \beta$ . We define,

$$E_{\alpha, \beta} = \lim_{\delta \rightarrow \infty} \left( \int_{\alpha}^{\alpha+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz + \int_{\alpha}^{\alpha-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz \right. \\ \left. - \int_{\beta}^{\beta+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz - \int_{\beta}^{\beta-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz \right).$$

THEOREM 5.9. Suppose that  $f$  is analytic in the region  $G = \{z : \alpha \leq \text{Re } z \leq \beta\}$ .

Also, for  $z = x + iy$  suppose

$$\lim_{|z| \rightarrow \infty} e^{-2\pi|z|} f(x + iy) = 0, \quad (5.11)$$

uniformly in  $G$ . If  $m - 1 < \alpha < m$ ,  $n < \beta < n + 1$ , ( $m, n \in \mathbb{Z}$ ), then

$$\sum_{k=m}^n f(k) = \int_{\alpha}^{\beta} f(x) dx + E_{\alpha, \beta}. \quad (5.12)$$

*Proof.* Let  $\delta > 0$ , and define  $C = C_1 + C_2$ , where

$$C_1 = [\alpha, \beta] + [\beta, \beta + i\delta] + [\beta + i\delta, \alpha + i\delta] + [\alpha + i\delta, \alpha], \text{ and}$$

$$C_2 = [\alpha, \beta] + [\beta, \beta - i\delta] + [\beta - i\delta, \alpha - i\delta] + [\alpha - i\delta, \alpha].$$



Let  $C_1 = C \cap \{z : \text{Im } z > 0\}$  and  $C_2 = C \cap \{z : \text{Im } z < 0\}$ . Now, since  $f$  has no singularities in  $G$ , By THEOREM 5.7, we have,

$$\sum_{k=m}^n f(k) = \frac{1}{2i} \int_C f(z) \cot \pi z dz.$$

Hence, 
$$\sum_{k=m}^n f(k) = \frac{1}{2i} \int_{C_1} f(z) \cot \pi z dz + \frac{1}{2i} \int_{C_2} f(z) \cot \pi z dz. \quad (5.13)$$

It is easy to verify these identities,

$$\frac{1}{2i} \cot \pi z = \frac{1}{2} + \frac{1}{e^{2\pi iz} - 1}$$

and

$$\frac{1}{2i} \cot \pi z = \frac{-1}{2} - \frac{1}{e^{-2\pi iz} - 1}.$$

Applying these identities to equation (5.13), we have,

$$\begin{aligned} \sum_{k=m}^n f(k) &= \int_{C_1} f(z) \left( \frac{-1}{2} - \frac{1}{e^{-2\pi iz} - 1} \right) dz + \int_{C_2} f(z) \left( \frac{1}{2} + \frac{1}{e^{2\pi iz} - 1} \right) dz \\ &= \int_{\alpha}^{\beta} f(x) dx + \int_{\alpha}^{\alpha+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz + \int_{\alpha}^{\alpha-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz \\ &\quad - \int_{\beta}^{\beta+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz - \int_{\beta}^{\beta-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz + \int_{\alpha}^{\beta} \frac{f(x+i\delta)}{e^{-2\pi i(x+i\delta)} - 1} dx \\ &\quad + \int_{\alpha}^{\beta} \frac{f(x-i\delta)}{e^{2\pi i(x-i\delta)} - 1} dx. \end{aligned}$$

Let  $\delta \rightarrow \infty$ . In light of hypothesis (5.11), we have

$$\begin{aligned} \sum_{k=m}^n f(k) &= \int_{\alpha}^{\beta} f(x)dx + \lim_{\delta \rightarrow \infty} \left( \int_{\alpha}^{\alpha+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz + \int_{\alpha}^{\alpha-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz \right. \\ &\quad \left. - \int_{\beta}^{\beta+i\delta} \frac{f(z)}{e^{-2\pi iz} - 1} dz - \int_{\beta}^{\beta-i\delta} \frac{f(z)}{e^{2\pi iz} - 1} dz \right) \\ &= \int_{\alpha}^{\beta} f(x)dx + E_{\alpha,\beta}. \end{aligned}$$

The proof is complete. □

### 5.3 Infinite Series

#### 5.3.1 Non-integer Singularities

**THEOREM 5.10.** [2] *Suppose that  $f(z) = \frac{q(z)}{p(z)}$  is a rational function with degree  $p(z)$  - degree  $q(z) \geq 2$ . Also, suppose that  $f$  has poles at  $z_1, \dots, z_m$ , none of which are integers. Then*

$$(i) \quad \sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{j=1}^m \text{Res}(f(z) \cot(\pi z), z_j)$$

and

$$(ii) \quad \sum_{k=-\infty}^{\infty} (-1)^k f(k) = -\pi \sum_{j=1}^m \text{Res}(f(z) \csc(\pi z), z_j).$$

*Proof.* (i) Let  $C_n$  be a basic contour and assume  $n$  is sufficiently large so that each  $z_j$  is inside  $C_n$ . Thus by LEMMA 5.7,

$$\int_{C_n} f(z) \cot(\pi z) dz = 2\pi i \sum_{k=-n}^n \frac{1}{\pi} f(k) + 2\pi i \sum_{j=1}^m \text{Res}(f(z) \cot(\pi z), z_j).$$

Now, let  $n \rightarrow \infty$ . By LEMMA 5.4,  $\lim_{n \rightarrow \infty} \int_{C_n} f(z) \cot(\pi z) dz = 0$ , hence

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{j=1}^m \text{Res}(f(z) \cot(\pi z), z_j).$$

(ii) The proof of the second assertion can be proved in the same way as the first, with  $\csc$  in place of  $\cot$  and  $(-1)^k f(k)$  in place of  $f(k)$ .

□

EXAMPLE 5.11. *If  $ia$  is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \coth(a\pi).$$

*Proof.* Let

$$f(z) = \frac{1}{z^2 + a^2} = \frac{1}{(z - z_1)(z - z_2)},$$

where  $z_1 = ai$  and  $z_2 = -ai$ . By DEFINITION 3.8,  $f(z) \cot(\pi z)$  has a simple pole at  $z_1$  and at  $z_2$ . Now, by THEOREM 3.22

$$\text{Res} \left( \frac{\cot(\pi z)}{(z - ia)(z + ia)}, ai \right) = \left[ \frac{\cot(\pi ia)}{(ia + ia)} \right] \text{Res} \left( \frac{1}{(z - ia)}, ai \right) = \frac{\cot(\pi ia)}{2ia}$$

$$= \frac{1}{2ia} \frac{\cos(\pi ia)}{\sin(\pi ia)} = \frac{1}{2ia} \frac{\cosh(\pi a)}{i \sinh(\pi a)} = -\frac{1}{2a} \coth(\pi a).$$

We can calculate the residue at  $z_2 = -ia$  in the same way, obtaining

$$\operatorname{Res} \left( \frac{\cot(\pi z)}{z^2 + a^2}, -ai \right) = -\frac{1}{2a} \coth(\pi a).$$

Therefore, by THEOREM 5.10, we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = -\pi \sum_{j=1}^2 \operatorname{Res} \left( \frac{\cot(\pi z)}{z^2 + a^2}, z_j \right) = \frac{\pi}{a} \coth(\pi a).$$

The proof is complete. □

EXAMPLE 5.12. *If  $ia$  is not an integer, then*

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}.$$

*Proof.* From the previous example we have  $\sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2)} = \frac{\pi}{a} \coth(\pi a)$ . Since

$f(k) = 1/(k^2 + a^2)$  is an even function, we have

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = \frac{1}{2} \left( \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} - \frac{1}{a^2} \right) = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2}.$$

The proof is complete. □

EXAMPLE 5.13. *If  $ia$  is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2)^2} = \frac{\pi}{2a^3} \coth(a\pi) + \frac{\pi^2}{2a^2} \operatorname{csch}^2(a\pi).$$

*Proof.* Let

$$f(z) = \frac{1}{(z^2 + a^2)^2} = \frac{1}{(z - ia)^2(z + ia)^2},$$

which has poles of order 2 at  $z_1 = ia$  and  $z_2 = -ia$ . By DEFINITION 3.8 ,

$f(z) \cot(\pi z)$  has a pole of order 2 at  $ia$  and at  $-ia$ . Hence, by THEOREM 3.19, we have

$$\begin{aligned} \operatorname{Res} \left( \frac{\cot(\pi z)}{(k^2 + a^2)^2}, ia \right) &= \lim_{z \rightarrow ia} \frac{d}{dz} \left( \frac{(z - ia)^2 \cot(\pi z)}{(z - ia)^2(z + ia)^2} \right) \\ &= \lim_{z \rightarrow ia} \frac{d}{dz} \left( \frac{\cot(\pi z)}{(z + ia)^2} \right) \\ &= \lim_{z \rightarrow ia} \frac{-\pi(z + ia) \operatorname{csc}^2(\pi z) - 2 \cot(\pi z)}{(z + ia)^3} \\ &= \frac{-2\pi ia \operatorname{csc}^2(\pi ia) - 2 \cot(\pi ia)}{(2ia)^3} \\ &= -\frac{\pi i^2 \operatorname{csc}^2(\pi ia)}{4a^2} - \frac{i \cot(\pi ia)}{4a^3} \\ &= -\frac{\pi \operatorname{csh}^2(\pi a)}{4a^2} - \frac{\coth(\pi a)}{4a^3}. \end{aligned}$$

An almost identical calculation yields

$$\operatorname{Res} \left( \frac{\cot(\pi z)}{(k^2 + a^2)^2}, -ia \right) = -\frac{\pi \operatorname{csh}^2(\pi a)}{4a^2} - \frac{\operatorname{coth}(\pi a)}{4a^3}.$$

Hence by THEOREM 5.10, we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + a^2)^2} = -\pi \sum_{j=1}^2 \operatorname{Res} \left( \frac{\cot(\pi z)}{(k^2 + a^2)^2}, z_j \right) = \frac{\pi^2 \operatorname{csh}^2(\pi a)}{2a^2} + \frac{\pi \operatorname{coth}(\pi a)}{2a^3}.$$

The proof is complete. □

EXAMPLE 5.14. *If  $a > 0$  is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k + a)^2} = \pi^2 \operatorname{csc}(\pi a) \cot(\pi a).$$

*Proof.* Let  $f(z) = (z + a)^{-2}$ . Since  $-a$  is not an integer, by DEFINITION 3.8,

$f(z) \operatorname{csc}(\pi z)$  has a pole of order 2 at  $z = -a$ . Hence, by THEOREM 3.19

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{(z + a)^2} \operatorname{csc}(\pi z), -a \right) &= \lim_{z \rightarrow -a} \frac{d}{dz} \left[ \frac{(z + a)^2}{(z + a)^2} \operatorname{csc}(\pi z) \right] \\ &= \lim_{z \rightarrow -a} [-\pi \cot(\pi z) \operatorname{csc}(\pi z)] \\ &= -\pi \cot(-\pi a) \operatorname{csc}(-\pi a) \\ &= -\pi \cot(\pi a) \operatorname{csc}(\pi a). \end{aligned}$$

It then follows by THEOREM 5.10, that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^2} = -\pi (-\pi \cot(\pi a) \csc(\pi a)) = \pi^2 \csc(\pi a) \cot(\pi a).$$

The proof is complete. □

THEOREM 5.15. [3] *Suppose that  $a$ ,  $b$ , and  $t$  are real numbers, and  $|b| < |a|$ , then*

$$\sum_{k=-\infty}^{\infty} (-1)^k \frac{at}{\pi^2 k^2 + a^2 t^2} e^{\frac{i\pi b k}{a}} = \frac{\cosh(bt)}{\sinh(at)}.$$

*Proof.* Let

$$f(z) = \frac{at}{\pi^2 z^2 + a^2 t^2} e^{\frac{i\pi b z}{a}} = \frac{at e^{\frac{i\pi b z}{a}}}{\pi^2 (z - z_1)(z - z_2)},$$

where  $z_1 = ait/\pi$ , and  $z_2 = -ait/\pi$ . Since these poles are simple, by THEOREM 3.22,

$$\begin{aligned} \operatorname{Res} \left( \frac{at e^{\frac{i\pi b z}{a}}}{\pi^2 z^2 + a^2 t^2} \csc(\pi z), z_1 \right) &= \left( \frac{at e^{\frac{i\pi b z_1}{a}}}{\pi^2 (z_1 - z_2)} \csc(\pi z_1) \right) \operatorname{Res} \left( \frac{1}{z - z_1}, z_1 \right) \\ &= \frac{1}{2\pi i} e^{-bt} \csc(iat). \end{aligned}$$

An almost identical calculation yields

$$\operatorname{Res} \left( \frac{at}{(z - z_1)(z - z_2)} e^{\frac{i\pi b z}{a}} \csc(\pi z), z_2 \right) = \frac{1}{2\pi i} e^{bt} \csc(iat),$$

as well. Hence by THEOREM 5.10, we have

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^k \frac{at}{\pi^2 k^2 + a^2 t^2} e^{\frac{i\pi b k}{a}} &= -\pi \sum_{j=1}^2 \operatorname{Res} \left( \frac{a t e^{i\pi b z/a}}{\pi^2 z^2 + a^2 t^2} \csc(\pi z), z_j \right) \\
&= -\pi \left[ \frac{1}{2\pi i} e^{-bt} \csc(iat) + \frac{1}{2\pi i} e^{bt} \csc(iat) \right] \\
&= \frac{e^{bt} + e^{-bt}}{-2i \sin(iat)} \\
&= \frac{\cosh(bt)}{\sinh(at)}.
\end{aligned}$$

This finishes the proof. □

### 5.3.2 Integer Singularities

**THEOREM 5.16.** *Suppose that  $f(z) = \frac{p(z)}{q(z)}$  is a rational function, with poles  $\{z_1, z_2, \dots, z_n\}$ , some of which may be integers, and let  $S = \mathbb{Z} \setminus \{z_1, z_2, \dots, z_n\}$ .*

*Then,*

$$\sum_{k \in S} f(k) = -\pi \sum_{j=1}^n \operatorname{Res}(f(z) \cot(\pi z), z_j).$$

*Proof.* If  $k \in S$ , then by DEFINITION 3.8,  $f(z) \cot(\pi z)$  has simple pole at  $k$  and  $\operatorname{Res}(f(z) \cot(\pi z), k) = \frac{1}{\pi} f(k)$ . Now, consider  $n$  such that all singularities of  $f$  are on the inside of  $C_n$ . Then, by LEMMA 5.7, we have



$$\begin{aligned}
\int_{C_n} f(z) \cot(\pi z) dz &= 2\pi i \sum \{ \text{all the residues in } C_n \} \\
&= 2\pi i \sum_{k \in S, |k| \leq N} \frac{1}{\pi} f(k) + 2\pi i \sum_{z_j \in R} \text{Res}(f(z) \cot(\pi z), z_j),
\end{aligned}$$

Let  $n \rightarrow \infty$ . Then by LEMMA 5.4

$$2\pi i \sum_{k \in S} \frac{1}{\pi} f(k) + 2\pi i \sum_{z_j \in R} \text{Res}(f(z) \cot(\pi z), z_j) = 0.$$

We conclude,

$$\sum_{k \in S} f(k) = -\pi \sum_{z_j \in R} \text{Res}(f(z) \cot(\pi z), z_j).$$

The proof is complete. □

EXAMPLE 5.17. *Euler's famous sum:*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

*Proof.* Let  $f(z) = 1/z^2$ , then by DEFINITION 3.8 the function  $f(z) \cot(\pi z)$  has a pole at  $z = 0$  of order 3. By THEOREM 3.19 and L'Hôpital's rule we have

$$\begin{aligned}
\text{Res} \left( \frac{1}{z^2} \cot(\pi z), 0 \right) &= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d^2}{dz^2} \left[ \frac{z^3 \cot(\pi z)}{z^2} \right] \\
&= \frac{1}{2} \lim_{z \rightarrow 0} \frac{d}{dz} [-\pi z \csc^2(\pi z) + \cot(\pi z)] \\
&= \frac{1}{2} \lim_{z \rightarrow 0} [2\pi^2 z \cot(\pi z) \csc^2(\pi z) - 2\pi \csc^2(\pi z)] \\
&= \lim_{z \rightarrow 0} \left[ \frac{\pi^2 z \cos(\pi z)}{\sin^3(\pi z)} - \frac{\pi}{\sin^2(\pi z)} \right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{z \rightarrow 0} \left[ \frac{\pi^2 z \cos(\pi z) - \pi \sin(\pi z)}{\sin^3(\pi z)} \right] \\
&= \lim_{z \rightarrow 0} \left[ \frac{-\pi^3 z \sin(\pi z) + \pi^2 \cos(\pi z) - \pi^2 \cos(\pi z)}{3\pi \sin^2(\pi z) \cos(\pi z)} \right] \\
&= \lim_{z \rightarrow 0} \left[ \frac{-\pi^2 z}{3 \sin(\pi z) \cos(\pi z)} \right] \\
&= -\frac{\pi}{3}.
\end{aligned}$$

Now, taking  $S = \mathbb{Z} \setminus \{0\}$  in THEOREM 5.16, we have,

$$\sum_{k \neq 0} \frac{1}{k^2} = -\pi \operatorname{Res} \left( \frac{1}{z^2} \cot(\pi z), 0 \right) = -\pi \left( \frac{-\pi}{3} \right) = \frac{\pi^2}{3}.$$

Since  $1/z^2$  is an even function, we see

$$2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k \neq 0} \frac{1}{k^2} = \frac{\pi^2}{3}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

The proof is complete. □

EXAMPLE 5.18. *If  $a$  is not an integer, then*

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k^2 + a^2)} = \frac{3 + a^2\pi^2 - 3\pi a \coth(\pi a)}{6a^4}.$$

*Proof.* Let

$$f(z) = \frac{1}{z^2(z^2 + a^2)}.$$

By DEFINITION 3.8 the function  $f(z) \cot(\pi z)$  has a pole of order 3 at  $z_1 = 0$  and simple poles at each of  $z_2 = ia$  and  $z_3 = -ia$ . Note that  $z_1$  is an integer, whereas  $z_2$  and  $z_3$  are not. Thus by THEOREM 3.22, we have

$$\operatorname{Res} \left( \frac{\cot(\pi z)}{z^2(z - ia)(z + ia)}, ai \right) = \frac{-i \coth(a\pi)}{(-a^2)(2ia)} \operatorname{Res} \left( \frac{1}{z - ia}, ai \right) = \frac{\coth(a\pi)}{2a^3}.$$

In the same way we found the previous residue,

$$\operatorname{Res} \left( \frac{\cot(\pi z)}{z^2(z - ia)(z + ia)}, -ai \right) = \frac{\coth(a\pi)}{2a^3}.$$

In finding the residue of the function  $f(z) \cot(\pi z)$  at the pole  $z_1 = 0$ , we will make use of the Bernoulli form of the Taylor series for  $z \cot z$ , RESULT 4.7, obtaining

$$\pi z \cot(\pi z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} (\pi z)^{2n}.$$

From this we obtain the Laurent series for  $\cot(\pi z)$  :

$$\begin{aligned} \cot(\pi z) &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k} \pi^{2k-1}}{(2k)!} z^{2k-1} \\ &= \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \dots \end{aligned}$$

Moreover, as a geometric series,

$$\frac{1}{z^2(z^2 + a^2)} = \frac{1}{a^2 z^2} \frac{1}{1 + (z/a)^2} = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-2}}{a^{2k+2}} = \frac{1}{a^2 z^2} - \frac{1}{a^4} + \frac{z^2}{a^6} + \dots$$

It follows that,

$$\begin{aligned}
\frac{\cot(\pi z)}{z^2(z^2 + a^2)} &= \left( \frac{1}{a^2 z^2} - \frac{1}{a^4} + \frac{z^2}{a^6} + \dots \right) \times \left( \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \dots \right) \\
&= \left( \frac{1}{a^2 \pi z^3} - \frac{\pi}{3a^2 z} + \frac{\pi^3 z}{45a^2} + \dots \right) \\
&\quad + \left( \frac{-1}{a^4 \pi z} + \frac{\pi z}{3a^4} - \frac{\pi^3 z^3}{45a^4} - \dots \right) \\
&\quad + \left( \frac{z}{\pi a^6} - \frac{\pi z^3}{3a^6} + \frac{\pi^3 z^5}{45a^6} + \dots \right) \\
&\quad + \dots \\
&= \frac{z^{-3}}{a^2 \pi} - \frac{3 + a^2 \pi^2}{3a^4 \pi} z^{-1} + \frac{-\pi^4 a^4 + 15\pi^2 a^2 + 45}{45\pi a^6} z + \dots
\end{aligned}$$

Therefore, by definition,

$$\operatorname{Res} \left( \frac{\cot(\pi z)}{z^2(z^2 + a^2)}, 0 \right) = -\frac{3 + a^2 \pi^2}{3a^4 \pi}.$$

Taking  $S = \mathbb{Z} \setminus \{0\}$  in THEOREM 5.16, we have,

$$\begin{aligned}
\sum_{k \neq 0} \frac{1}{k^2(k^2 + a^2)} &= -\pi \sum_{j=1}^3 \operatorname{Res} \left( \frac{1}{z^2(z^2 + a^2)} \cot(\pi z), z_j \right) \\
&= -\pi \left( \frac{\coth(a\pi)}{2a^3} + \frac{\coth(a\pi)}{2a^3} - \frac{3 + a^2 \pi^2}{3a^4 \pi} \right) \\
&= \frac{3 + a^2 \pi^2 - 3\pi a \coth(\pi a)}{3a^4}.
\end{aligned}$$

Since  $f(z)$  is an even function,

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k^2 + a^2)} = \frac{1}{2} \sum_{k \neq 0} \frac{1}{k^2(k^2 + a^2)} = \frac{3 + a^2\pi^2 - 3\pi a \coth(\pi a)}{6a^4}.$$

The proof is complete. □

**THEOREM 5.19.** *Suppose that  $f(z) = \frac{p(z)}{q(z)}$  is a rational function, with poles  $\{z_1, z_2, \dots, z_n\}$ , some of which may be integers, and let  $S = \mathbb{Z} \setminus \{z_1, z_2, \dots, z_n\}$ .*

*Then,*

$$\sum_{k \in S} (-1)^k f(k) = -\pi \sum_{j=1}^n \operatorname{Res}(f(z) \csc(\pi z), z_j).$$

*Proof.* Since  $\csc(\pi z)$  and  $\cot(\pi z)$  have the same denominator and the theorem has the same hypotheses otherwise, the proof for this theorem is similar to the the previous one and will be omitted. □

**EXAMPLE 5.20.** *If  $a$  is an integer, and  $a \neq 0$ . Then*

$$\sum_{k \in \mathbb{Z} \setminus \{0, a\}} \frac{(-1)^k}{k^2(k-a)} = \frac{6 + a^2\pi^2 - 12(-1)^{a+1}}{6a^3}.$$

*Proof.* Let

$$f(z) = \frac{1}{z^2(z-a)},$$

then by DEFINITION 3.8 the function  $f(z) \csc(\pi z)$  has a pole of order 3 at  $z_1 = 0$  and a pole of order 2 at  $z_2 = a$ . Note that  $z_1$  and  $z_2$  are integers.

To find the residues of the function  $f(z) \csc(\pi z)$  at the pole  $z_2 = a$ , by Theorem 3.19 we have,

$$\operatorname{Res}\left(\frac{\csc(\pi z)}{z^2(z-a)}, a\right) = \lim_{z \rightarrow a} \frac{d}{dz} \left[ \frac{(z-a)^2 \csc(\pi z)}{z^2(z-a)} \right] = \frac{2(-1)^{a+1}}{a^3\pi}.$$

Now, to find the residue of the function  $f(z) \csc(\pi z)$  at the pole  $z_1 = 0$ , we will use the sum identity for the cosecant 4.9

$$\begin{aligned} \csc(\pi z) &= \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2^{2k} - 2) B_{2k} \pi^{2k-1}}{(2k)!} z^{2k-1} \\ &= \frac{1}{\pi z} + \frac{\pi z}{6} - \frac{7\pi^3 z^3}{360} - \dots \end{aligned}$$

Also, we need the Taylor expansion for  $z^{-2}(z-a)^{-1}$ ,

$$\frac{1}{z^2} \frac{1}{z-a} = \frac{1}{az^2} \frac{-1}{1 - \left(\frac{z}{a}\right)} = - \sum_{k=0}^{\infty} \frac{z^{k-2}}{a^{k+1}}.$$

Hence,

$$- \sum_{k=0}^{\infty} \frac{z^{k-2}}{a^{k+1}} = -\frac{1}{az^2} - \frac{1}{a^2 z} - \frac{1}{a^3} + \dots$$

Now, we will find the product of these two summations,

$$\begin{aligned} \csc(\pi z) \left( \frac{1}{z^2} \frac{1}{z-a} \right) &= \left( \frac{1}{\pi z} + \frac{\pi z}{6} - \frac{7\pi^3 z^3}{360} - \dots \right) \times \left( -\frac{1}{az^2} - \frac{1}{a^2 z} - \frac{1}{a^3} + \dots \right) \\ &= \left( -\frac{1}{a\pi z^3} - \frac{1}{a^2 \pi z^2} - \frac{1}{a^3 \pi z} + \dots \right) \\ &\quad + \left( -\frac{\pi}{6az} - \frac{\pi}{6a^2} - \frac{\pi z}{6a^3} - \dots \right) \\ &\quad + \left( -\frac{7\pi^3 z}{360a} - \frac{7\pi^3 z^2}{360a^2} - \frac{7\pi^3 z^3}{360a^3} - \dots \right) \\ &\quad + \dots \end{aligned}$$

Hence, we see that the coefficient of  $\frac{1}{z}$  is  $\left(-\frac{6+a^2\pi^2}{6a^3\pi}\right)$ , which is the residue of the function  $f(z) \csc(\pi z)$  at the pole  $z_1 = 0$ . Hence,

$$\operatorname{Res} \left( \frac{1}{z^2(z-a)} \csc(\pi z), 0 \right) = -\frac{6 + a^2\pi^2}{6a^3\pi}.$$

Now, let  $S = \mathbb{Z} \setminus \{0, a\}$ . By THEOREM 5.19 we have,

$$\begin{aligned} \sum_{k \in \mathbb{Z} \setminus \{0, a\}} \frac{(-1)^k}{k^2(k-a)} &= -\pi \sum_{j=1,2} \operatorname{Res} \left( \frac{1}{z^2(z-a)} \csc(\pi z), z_j \right) \\ &= -\pi \left( \frac{2(-1)^{a+1}}{a^3\pi} - \frac{6 + a^2\pi^2}{6a^3\pi} \right) \\ &= \frac{6 + a^2\pi^2 - 12(-1)^{a+1}}{6a^3}. \end{aligned}$$

The proof is complete. □

### 5.3.3 Singularities at Zero

THEOREM 5.21. [2] *If  $n$  is a positive integer, and  $\{B_k\}$  are the Bernoulli numbers, then*

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{2^{2n-1} B_{2n} \pi^{2n}}{(2n)!}.$$

*Proof.* Let  $f(z) = 1/z^{2n}$ , then by DEFINITION 3.8 the function  $f(z) \cot(\pi z)$  has a pole of order  $2n + 1$  at the singularity  $z = 0$ . By RESULT 4.7, we have the Laurent series,

$$\frac{\cot(\pi z)}{z^{2n}} = \frac{1}{\pi z^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k} \pi^{2k}}{(2k)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k} \pi^{2k-1}}{(2k)!} z^{2k-2n-1}.$$

When  $2k - 2n - 1 = -1$ ,  $k = n$ , hence

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right) = \frac{(-1)^n 2^{2n} B_{2n} \pi^{2n-1}}{(2n)!}.$$

Considering  $S = \mathbb{Z} \setminus \{0\}$  in THEOREM 5.16, we have

$$\sum_{k \neq 0} \frac{1}{k^{2n}} = -\pi \operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right) = \frac{(-1)^{n-1} 2^{2n} B_{2n} \pi^{2n}}{(2n)!}.$$

Since  $f(k) = 1/k^{2n}$  is an even function,

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{1}{2} \sum_{k \neq 0} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} 2^{2n-1} B_{2n} \pi^{2n}}{(2n)!},$$

completing the proof. □

EXAMPLE 5.22.

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2\pi^2 B_2}{2!} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4} = -\frac{2^3 \pi^4 B_4}{4!} = \frac{\pi^4}{90}.$$

THEOREM 5.23. [2] *If  $n$  is a positive integer, and  $\{B_k\}$  are the Bernoulli numbers, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} = (-1)^n \frac{(2^{2n-1} - 1) B_{2n} \pi^{2n}}{(2n)!}.$$

*Proof.* Let  $f(z) = 1/z^{2n}$ , then by DEFINITION 3.8 the function  $f(z) \csc(\pi z)$  has a pole of order  $2n + 1$  at the singularity  $z = 0$ . By RESULT 4.9, we have the Laurent



series,

$$\csc(\pi z) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2^{2k} - 2)B_{2k}}{(2k)!} \pi^{2k-1} z^{2k-1}.$$

It follows that

$$\frac{\csc(\pi z)}{z^{2n}} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2^{2k} - 2)B_{2k}}{(2k)!} \pi^{2k-1} z^{2k-2n-1}.$$

We obtain the residue from the previous series, for when  $2k - 2n - 1 = -1$ , we see  $n = k$ . Hence,

$$\operatorname{Res} \left( \frac{\csc(\pi z)}{z^{2n}}, 0 \right) = (-1)^{n-1} \frac{(2^{2n} - 2)B_{2n}}{(2n)!} \pi^{2n-1}.$$

Considering  $S = \mathbb{Z} \setminus \{0\}$  in THEOREM 5.16, we have

$$\sum_{k \neq 0} \frac{1}{k^{2n}} = -\pi \operatorname{Res} \left( \frac{\csc(\pi z)}{z^{2n}}, 0 \right) = (-1)^n \frac{(2^{2n} - 2)B_{2n}}{(2n)!} \pi^{2n}.$$

But, since  $f(k) = \frac{(-1)^k}{k^{2n}}$  is an even function,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} = \frac{1}{2} \sum_{k \neq 0} \frac{(-1)^k}{k^{2n}} = (-1)^n \frac{2(2^{2n-1} - 1)B_{2n}\pi^{2n}}{(2n)!}.$$

The proof is complete. □

EXAMPLE 5.24.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}.$$

*Proof.* Taking  $n = 1$  in THEOREM 5.23, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = -\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = \frac{B_2 \pi^2}{2}.$$

Since  $B_2 = 1/6$ , the proof is complete. □

## CHAPTER 6

### MITTAG-LEFFLER EXPANSION THEOREM

The Mittag-Leffner theorem seems unique in its concept, finding an infinite sum form of functions in terms of its singularities and the corresponding residues.

**THEOREM 6.1** (Mittag-Leffler Expansion Theorem). [5] *Let  $f(z)$  be analytic except at distinct simple poles  $\{z_j\}_1^\infty$ , for which  $0 < |z_j| \leq |z_{j+1}|$  for all  $j$ . Denote  $R_j = \text{Res}(f, z_j)$  and let  $\{C_n\}_1^\infty$  be circles of radius  $r_n$ , centered at 0, none of which pass through any  $z_j$  and such that  $r_n \rightarrow \infty$ . Moreover, assume there exists  $B > 0$  such that when  $z \in C_n$  for any  $n$ ,  $|f(z)| < B$ . Then*

$$f(z) = f(0) + \sum_{j=1}^{\infty} R_j \left( \frac{1}{z - z_j} + \frac{1}{z_j} \right).$$

*Proof.* Let  $z_0$  be any complex number except a pole of  $f$ . Define

$$F(z) = \frac{f(z)}{z - z_0},$$

then  $F$  has a simple pole at  $z_0$ , as well as at each  $z_j$ . By THEOREM 3.22, for all  $j \in \mathbb{N}$ ,

$$\text{Res}(F, z_j) = \frac{R_j}{z_j - z_0} \quad \text{and} \quad \text{Res}(F, z_0) = f(z_0).$$

By the Cauchy Residue theorem 3.14, for any  $n$ ,

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z)}{z - z_0} dz = f(z_0) + \sum_{z_j \in C_n} \frac{R_n}{z_n - z_0}. \quad (6.1)$$

Letting  $z_0 = 0$ , in (6.1) yields

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z)}{z} dz = f(0) + \sum_{z_j \in C_n} \frac{R_n}{z_n}. \quad (6.2)$$

Subtracting (6.2) from (6.1), we obtain

$$\begin{aligned} \frac{z_0}{2\pi i} \int_{C_n} \frac{f(z)}{z(z-z_0)} dz &= \frac{1}{2\pi i} \int_{C_n} f(z) \left( \frac{1}{z-z_0} - \frac{1}{z} \right) dz \\ &= f(z_0) - f(0) + \sum_{z_j \in C_n} R_n \left( \frac{1}{z_n - z_0} - \frac{1}{z_n} \right). \end{aligned} \quad (6.3)$$

Since  $|z - z_0| \geq |z| - |z_0| = r_n - |z_0|$ , for all  $z$  on  $C_n$ , we have

$$\left| \int_{C_n} \frac{f(z)}{z(z-z_0)} dz \right| \leq \frac{2\pi r_n B}{r_n(r_n - |z_0|)} = \frac{2\pi B}{r_n - |z_0|},$$

which shows that

$$\lim_{n \rightarrow \infty} \int_{C_n} \frac{f(z)}{z(z-z_0)} dz = 0,$$

as  $n \rightarrow \infty$ , and therefore as  $r_n \rightarrow \infty$ . It then follows from line (6.3) that

$$\begin{aligned} f(z_0) &= f(0) - \lim_{n \rightarrow \infty} \sum_{z_j \in C_n} R_j \left( \frac{1}{z_j - z_0} - \frac{1}{z_j} \right) \\ &= f(0) + \sum_{j=1}^{\infty} R_j \left( \frac{1}{z_0 - z_j} + \frac{1}{z_j} \right). \end{aligned}$$

The proof is complete. □

**EXAMPLE 6.2.** *If  $z \neq k\pi$  for any  $k \in \mathbb{Z}$ , then*

$$\cot z = \frac{1}{z} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - k\pi} + \frac{1}{k\pi} \right).$$

*Proof.* Let  $f(z) = \cot z - 1/z$ . By THEOREM 5.5 we have that  $\cot z$  has simple poles at  $z = k\pi$ , when  $k$  is an integer and that the residues at these poles are 1. It follows that the Laurent series is

$$\cot z = \sum_{k=-1}^{\infty} a_k z^k, \text{ where } a_{-1} = 1.$$

Therefore,

$$\cot z - \frac{1}{z} = \sum_{k=0}^{\infty} a_k z^k,$$

hence  $z = 0$  is a removable singularity. By L'Hospital's rule we have

$$\lim_{z \rightarrow 0} \left( \cot z - \frac{1}{z} \right) = 0,$$

so, without loss of generality,  $f(0) = 0$ . Moreover, by LEMMA 5.3 we have  $\cot z$  is bounded on basic contours  $C_n$ . Hence by Mittag-Leffler Expansion Theorem we have,

$$\cot z = \frac{1}{z} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{1}{z - k\pi} + \frac{1}{k\pi} \right).$$

The proof is complete. □

LEMMA 6.3. *If  $z_j = \frac{\pi}{2}(2j + 1)$  for all  $j \in \mathbb{Z}$ , then*

*i.*  $z_{-j-1} = -z_j$ .

$$ii. \frac{1}{z - z_j} + \frac{1}{z - z_{-j-1}} = \frac{2z}{z^2 - z_j^2}.$$

*Proof.* Given  $z_j = \frac{\pi}{2}(2j + 1)$ ,

$$i. z_{-j-1} = \frac{\pi}{2}[2(-j - 1) + 1] = \frac{\pi}{2}(-2j - 2 + 1) = -\frac{\pi}{2}(2j + 1) = -z_j.$$

Then from (i), we see

$$ii. \frac{1}{z - z_j} + \frac{1}{z - z_{-j-1}} = \frac{1}{z - z_j} + \frac{1}{z + z_j} = \frac{z + z_j + z - z_j}{(z - z_j)(z + z_j)} = \frac{2z}{z^2 - z_j^2}.$$

The proof is complete. □

EXAMPLE 6.4. For all  $z \neq \pi k$  for some  $k \in \mathbb{Z}$ ,

$$\tan z = \sum_{k=0}^{\infty} \frac{2z}{((2k + 1)\frac{\pi}{2})^2 - z^2}.$$

*Proof.* Let  $\{C_n\}_1^{\infty}$  be circles of radius  $\pi n$ , centered at 0. Using the methods of

LEMMA 5.3, it can be shown that there exists  $B > 0$  such that  $|\tan(z)| < B$ , when

$z \in C_n$  for any  $n$ . Denote the singularities of  $\tan$  as

$$\omega_j = (2j + 1)\frac{\pi}{2} \text{ for all } j \in \mathbb{Z},$$

noting  $\tan$  has simple poles with residues of 1 at each  $\omega_j$  and none of them are on

any  $C_n$ . We renumber these singularities in such a way to satisfy the remaining

hypothesis of the Mittag-Leffler Expansion Theorem 6.1. Denote

$$z_k = \begin{cases} \omega_{k/2}, & \text{if } k \text{ is even} \\ \omega_{(1-k)/2}, & \text{if } k \text{ is odd} \end{cases}$$

Therefore,

$$\begin{aligned} \tan z &= \sum_{k=1}^{\infty} \left( \frac{1}{z - z_k} + \frac{1}{z_k} \right) \\ &= \sum_{k=1}^{\infty} \left( \frac{1}{z - z_{2k}} + \frac{1}{z_{2k}} + \frac{1}{z - z_{2k-1}} + \frac{1}{z_{2k-1}} \right) \\ &= \sum_{k=1}^{\infty} \left( \frac{1}{z - \omega_k} + \frac{1}{\omega_k} + \frac{1}{z - \omega_{1-k}} + \frac{1}{\omega_{1-k}} \right) \\ &= \sum_{k=1}^{\infty} \left( \frac{1}{z - \omega_k} + \frac{1}{\omega_k} \right) + \sum_{k=1}^{\infty} \left( \frac{1}{z - \omega_{1-k}} + \frac{1}{\omega_{1-k}} \right) \\ &= \sum_{k=0}^{\infty} \left( \frac{1}{z - \omega_k} + \frac{1}{\omega_k} \right) + \sum_{k=0}^{\infty} \left( \frac{1}{z - \omega_{-1-k}} + \frac{1}{\omega_{-1-k}} \right) \end{aligned}$$

By part (i) of LEMMA 6.3, we have  $\omega_{-1-k} = -\omega_k$ , hence

$$\tan z = \sum_{k=0}^{\infty} \left( \frac{1}{z - \omega_k} + \frac{1}{z - \omega_{-1-k}} \right).$$

Then by the (ii) part of LEMMA 6.3, we have

$$\tan z = \sum_{k=0}^{\infty} \frac{2z}{((2k+1)\frac{\pi}{2})^2 - z^2}.$$

The proof is complete. □

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