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Residues, Bernoulli Numbers and Finding Sums

Mohammed Saif Alotaibi

RESIDUES, BERNOULLI NUMBERS AND FINDING SUMS

A Masters Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

Mohammed Saif Alotaibi

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ABSTRACT

A large number of infinite sums, such as $\sum_{k=0}^{\infty} \frac{1}{k^2}$, cannot be found by the methods of real analysis. However, many of them can be evaluated using the theory of residues. In this thesis we characterize several methods of summations using residues, including methods integrating residues and the Bernoulli numbers. In fact, with this technique we derive some summation formulas for particular Finite Sums and Infinite Series that are difficult or impossible to solve by the methods of real analysis.

KEYWORDS: analytic function, homotopy, singularity, pole, zero, residue, Bernoulli numbers, finite sums, infinite series.

This abstract is approved as to form and content

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Dr. Shelby J. Kilmer
Chairperson, Advisory Committee
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I dedicate this thesis to my daughter Yara.

TABLE OF CONTENTS

1: Introduction 1

2: Preliminaries 2
- 2.1 Differentiation 2
- 2.2 Integrals and Contours 6
- 2.3 Homotopy 9

3: Residues 18
- 3.1 Laurent Series 18
- 3.2 Singular Points 22
- 3.3 Definition of Residue 23
- 3.4 Residue at Infinity 24
- 3.5 The Cauchy Residue Theorem 25
- 3.6 Zeros and Poles 26
- 3.7 Residue at a Pole 27
  - 3.7.1 Residue at a Pole of Order m 27
  - 3.7.2 Residues at Simple Poles 29

4: Bernoulli Numbers 34
- 4.1 The Bernoulli Numbers 34
- 4.2 Results 37

5: Summing Series by Residues 41
- 5.1 Foundations 41
- 5.2 Finite Sums 47
- 5.3 Infinite Series 51
  - 5.3.1 Non-integer Singularities 51
  - 5.3.2 Integer Singularities 57
  - 5.3.3 Singularities at Zero 64

6: Mittag-Leffler Expansion Theorem 68

Bibliography 73

CHAPTER 1

INTRODUCTION

After a brief review of the more important basic concepts of complex analysis, we present Residues and Cauchy's Residue Theorem. Many consider this theorem to be the most important theorem of complex analysis and it is the cornerstone of this thesis. Cauchy's Residue Theorem is not only important in complex analysis but has an important role in real analysis, one that may possibly outweigh its importance in complex analysis. In order to make full use of the Residue Theorem, we derive various methods of calculating residues. Some of our techniques rely on Bernoulli numbers, so we define and explore their properties, before using them to obtain some important infinite sums. We conclude this thesis with two chapters evaluating both finite and infinite sums using these methods.

CHAPTER 2

PRELIMINARIES

2.1 Differentiation

In complex variables the derivative is defined the same way as in the real number system. It is, therefore, not surprising that the usual differentiation rules, such as the sum and difference rules, hold when taking derivatives of complex functions.

Definition 2.1. *Given $G \subset \mathbb{C}$, let $f : G \to \mathbb{C}$ be a complex valued function and let $z_0 \in G$. The derivative of f at z_0 is*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

when this limit exits. If so, we say f is differentiable *at z_0.*

Definition 2.2. *Let $f : G \longrightarrow \mathbb{C}$ be a complex valued function. If f is differentiable at a for every $a \in G$ and these derivatives are continuous, the function f is said to be* analytic *on G. If f is analytic on the whole complex plane $\mathbb{C}$, f is said to be* entire.

As an example, the complex polynomials are entire functions.

Definition 2.3. *A* disc *or a* ball *centered at z_0 with radius r is*

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

Definition 2.4. *A set $G \subset \mathbb{C}$ is* open*, if for every $z \in G$ there exist $r > 0$ such that $B(z, r) \subset G$.*

Using the definition above, it is easy to prove a subset of $\mathbb{C}$ is open if and only if it includes no points of its boundary. It follows that the region inside a closed contour is open.

The following theorem is called Taylor's theorem in honor of the English mathematician Brook Taylor, who discovered its first form. This result is fundamental in the proofs of Cauchy's theorems and many other important theorems in complex variables, as well as in many other area of mathematics.

THEOREM 2.5. *If f is analytic on a disc $B(z_0, r)$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

for all $z \in B$, where each

$$a_n = \frac{f^{(n)}(z)}{n!} \quad \textit{is} \quad \textit{unique}.$$

The proof of the part of Taylor's theorem giving the existence of the series is much like, but simpler, than the proof of Laurent's Theorem, which will be included in a later chapter.

THEOREM 2.6. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has a radius of convergence R. Then f can be differentiated term by term inside $B(z_0, R)$. That is*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

Moreover f' has a radius of convergence R, as well.

Proof. Without loss of generality we assume $z_0 = 0$. Since

$$\lim_{n \to \infty} \left| \frac{(n+1)a_{n+1}}{n a_n} \right| = \lim_{n \to \infty} \frac{n+1}{n} \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = R,$$

we see, by the ratio test, that the second series has the same radius of convergence as the first.

Now let $z \in B(0,R)$ and let $\varepsilon > 0$. There exits $r > 0$ such that

$z \in B(0,r) \subset B(0,R)$. Let

$$S_n(\xi) = \frac{\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n} a_n \xi^k}{z-\xi} = \frac{1}{z-\xi}\sum_{k=0}^{n} a_k(z^k - \xi^k),$$

and,

$$R_n(\xi) = \frac{1}{z-\xi}\sum_{k=n+1}^{\infty} a_k(z^k - \xi^k).$$

$S_n(\xi)$ denotes the n^{th} partial sum of $\frac{f(z)-f(\xi)}{z-\xi}$ and $R_n(\xi)$ the corresponding remainder. Since $r < R$, the series $\sum_{k=0}^{\infty} k|a_k|r^{k-1}$ converges and so there exist $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} k|a_k|r^{k-1} < \frac{\varepsilon}{3}.$$

It follows that for every $\omega \in B(z,r)$,

$$\begin{aligned}
|R_n(\omega)| &= \left|\sum_{k=N}^{\infty} \frac{a_k(z^k - \omega^k)}{z-\omega}\right| \\
&\leqslant \sum_{k=N}^{\infty} |a_k||z^{k-1} + z^{k-2}\omega \ldots \omega^{k-1}| \\
&\leqslant \sum_{k=N}^{\infty} k|a_k|r^{k-1} \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

Let S'_N denote the N^{th} partial sum of $\sum_{k=1}^{\infty} k a_k z^{k-1}$ and R' the corresponding remainder. Thus,

$$|R'| \leqslant \sum_{k=N}^{\infty} k|a_k|r^{k-1} < \frac{\varepsilon}{3}.$$

Now, since the partial sums of f are polynomials,

$$\lim_{\omega \to z} S_N = S'_N.$$

Therefore, we have $\delta > 0$ with $\delta < r$ such that when $|\omega - z| < \delta$, $|S_N - S'_N| < \frac{\varepsilon}{3}$. It now follows that when $|\omega - z| < \delta$,

$$\begin{aligned} \left| \frac{f(z) - f(\omega)}{z - \omega} - \sum_{n=1}^{\infty} n a_n z^{n-1} \right| &= |S_N(\omega) + R_N(\omega) - S'_N - R'_N| \\ &\leqslant |S_N - S'_N| + |R_n(\omega)| + |R'_n| \\ &< \varepsilon, \end{aligned}$$

which finishes the proof. □

The following corollary is an example of how important Taylor's theorem is in complex variables.

Corollary 2.7. *If f is analytic on an open set G, then f is infinitely differentiable on G.*

Proof. Let $z_0 \in G$. Since G is open, it contains a ball centered at z_0. By Taylor's theorem, f has a valid power series on that ball. By the previous theorem f' has a power series form which is differentiable on the ball as well. Continuing inductively f must be infinitely differentiable. □

2.2 Integrals and Contours

Integration of functions along contours in the Complex plane will play an important role in our methods. Some of the concepts and theorems given in this section will become powerful tools for proving important theorems in later sections.

Definition 2.8. *Let C be a curve in $\mathbb{C}$. We say $\gamma : [a, b] \to C$* parameterizes *C, if γ is a continuous surjection. Furthermore, C is* smooth, *if it has a differentiable parameterization with a non-zero continuous derivative. The* orientation *of C is given by its parameterization; $\gamma(a)$ is "before" $\gamma(b)$.*

When $\gamma : [a, b] \to C$ parameterizes C, it is easy to see $\gamma(a + b - t)$ is also a parameterization of C but with the opposite orientation. We generally refer to $-C$ when switching to the parameterization giving the opposite orientation.

Definition 2.9. *A curve C is a* contour, *if it is the union of finitely many smooth curves $C_1, C_2, \ldots C_n$, and the end point of C_k coincides with the starting point of C_{k+1}, for $k = 1, 2, \ldots n - 1$. We write $C = C_1 + C_2 + \cdots + C_n$.*

Definition 2.10. *A contour C is* closed, *if its starting point and endpoint are the same.*

Definition 2.11. *A closed contour is* positively oriented, *when its parameterization traverses it in the counterclockwise direction.*

Definition 2.12. *A contour C is* simple *and sometimes called a* Jorden arc, *if it never cross itself, except possible at the endpoints.*

DEFINITION 2.13. *When* $g : [a, b] \to \mathbb{C}$,

$$\int_a^b g(t)dt = \int_a^b Re[g(t)]dt + i \int_a^b Im[g(t)]dt,$$

where the integrals on the right are defined as in elementary calculus.

DEFINITION 2.14. *When* $\gamma : [a, b] \to C$ *parameterizes a smooth curve* C *and* f *is defined on* C*, we define the* integral of f on C, *by*

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt,$$

and when $C = C_1 + C_2 + \cdots + C_n$ *is a contour, the* contour integral of f on C *is*

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

Since the values of these integrals are independent of the particular parameterization used, DEFINITION 2.13 above is valid. To get an idea of how a proof would run, let g be real and consider $\gamma : [a, b] \to Domain(g)$ and $\sigma[c, d] \to Domain(g)$, with $\gamma(a) = \sigma(c)$ and $\gamma(b) = \sigma(d)$. By the substitution principle we have

$$\int_a^b g(\gamma(t))\gamma'(t)dt = \int_{\gamma(a)}^{\gamma(b)} g(u)du = \int_{\sigma(c)}^{\sigma(d)} g(u)du = \int_a^b g(\sigma(t))\sigma'(t)dt.$$

A complete proof would require combining real and imaginary parts and so on. More of this type of reasoning can show that the integration rules from elementary calculus, such as the sum, difference and constant multiple rules all hold.

THEOREM 2.15. *If* $\gamma : [a, b] \to C$ *is smooth and length of* C*,* $L(C)$*, is finite, then*

$$\int_a^b |\gamma'(t)|\, dt = L(C).$$

Proof. Since the length of $C \in \mathbb{C}$ is the same as the length of $\langle Re\gamma, Im\gamma\rangle \in \mathbb{R}^2$, this follows immediately from the arc length formula in elementary calculus.Thus

$$L(C) = \int_a^b \sqrt{(Re\gamma'(t))^2 + (Im\gamma'(t))^2}dt = \int_a^b |\gamma'(t)|dt.$$

The proof is complete. □

The following corollary will be indispensable as we proceed. It's proof is immediate.

COROLLARY 2.16. [4] *If the integral of f on C exists, the length of C is finite and f is bounded on C, then*

$$\left|\int_\gamma f(z)dz\right| \leqslant L(C)M_f,$$

where $L(C)$ is the arclength of C and M_f is the maximum value of $|f|$ on C.

The field of complex variables has an analog of the fundamental theorem of calculus from real analysis.

DEFINITION 2.17. *The function F is a* <u>primitive</u> *of the function f on the set G, if for all $z \in G$,*

$$F'(z) = f(z).$$

THEOREM 2.18. [4] *Let C be a contour in an open set G, with endpoints α and β. If F is a primitive of f on G, then*

$$\int_C f(z)dz = F(\beta) - F(\alpha).$$

Proof. Let $\gamma : [a, b] \to C$ parameterize a smooth curve C from α to β. Then

$$\begin{aligned}
\int_C f(z)dz &= \int_C F'(z)dz = \int_a^b F'(\gamma(t))\gamma'(t)dt \\
&= \int_a^b (F \circ \gamma)'(t)dt = F \circ \gamma(b) - F \circ \gamma(a) \\
&= F(\beta) - F(\alpha).
\end{aligned}$$

Now consider a contour $C = C_1 + C_2 + \cdots + C_n$, with connections at $z_1, z_2, \ldots, z_{n-1}$, respectively. Then from the smooth case, we have

$$\int_C f(z)dz = F(z_1) - F(\alpha) + F(z_2) - F(z_1) \cdots + F(\beta) - F(z_{n-1}) = F(\beta) - F(\alpha).$$

The proof is complete. $\square$

Since $F(\beta) = F(\alpha)$, on a closed contour, we then have the following immediate corollary.

COROLLARY 2.19. [4] *Let C be a closed contour in an open set G. If F is a primitive of f on G, then*

$$\int_C f(z)dz = 0.$$

The following corollary is often called the first version of Cauchy's theorem.

COROLLARY 2.20. *If C is a closed contour in $B(z_0, r)$ and f is analytic on $B(z_0, r)$, then*

$$\int_C f(z)dz = 0.$$

Proof. Since f is analytic on $B(z_0, r)$, it has a Taylor series valid on $B(z_0, r)$. Taking the antiderivative term by term yields a primitive for f. Proof is immediate by the previous corollary. $\square$

2.3 Homotopy

DEFINITION 2.21. *Two curves, C and C', from A to B are* homotopic *in $G \subset \mathbb{C}$, if there exists continuous $\Psi : [0,1]^2 \to G$, such that*

$\Psi(s, 0) = A$ for every $s \in [0, 1]$,

$\Psi(s, 1) = B$ for every $s \in [0, 1]$,

$\Psi(0, t)$ parameterizes C and

$\Psi(1,t)$ parameterizes C'.

We will write $C \sim C'$, when C and C' are homotopic and ψ is sufficiently differentiable to produce smooth curves.

<u>Note:</u> for each fixed $s \in [0,1]$, $\Psi(s,t) : [0,1] \to G$ parameterizes some curve in G from A to B. The intuition is that Ψ "continuously morphs" C to C'.

THEOREM 2.22. [1] *If f in analytic on G and $C \sim C'$ in G, then*

$$\int_C f(z)dz = \int_{C'} f(z)dz.$$

Proof. Let C and C' be homotopic curves from A to B in an open set G, with Ψ as in definition (2.21). Since $\Psi([0,1]^2)$ is compact and $\mathbb{C} - G$, is closed the distance between them is r for some $r > 0$. This means f is analytic on $B(z,r)$ for every $z \in G$. Moreover, since Ψ is continuous and $[0,1]^2$ is compact, Ψ is uniformly continuous on $[0,1]^2$. It follows that there exists $\delta > 0$ such that when

$$\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} < \delta, then, |\Psi(s_2,t_2) - \Psi(s_1,t_1)| < r.$$

Choose $n \in \mathbb{N}$, so that $\frac{\sqrt{2}}{n} < \delta$. Then partition $[0,1]^2$ into n^2 congruent squares. Note that if (s_1,t_1) and (s_2,t_2) are in the same square, $\sqrt{(s_2 - s_1)^2 + (t_2 - t_1)^2} < \delta$. If k is fixed in $\{0,1,2\ldots,n-1\}$, then $\Psi(\frac{k}{n},t)$ and $\Psi(\frac{k+1}{n},t)$ parameterize curves, C_k and C_{k+1} from A to B in G.

For each $j \in \{0,1,2\ldots,n-1\}$, define $S_j \in [0,1]^2$ to be the boundary of the $\frac{1}{n} \times \frac{1}{n}$ square with bottem left corner $(\frac{k}{n},\frac{j}{n})$. Let ξ_j denote the closed contour, $\Psi(S_j)$, traversed counterclockwise. Since $diam\, S_j = \frac{\sqrt{2}}{n} < \delta$, each $\xi_j \subset B(\Psi(\frac{k}{n}, \frac{j}{n}), r)$. It follows by theorem (2.20), that $\int_{C_j} f(z)\, dz = 0$, for each $j \in \{0, 1, 2 \ldots, n-1\}$. Each consecutive pair, ξ_j and ξ_{j+1}, share sides traversed in opposite directions and integrals over those sides add to zero. Thus for each $k \in \{0, 1, 2 \ldots, n-1\}$,

$$\begin{aligned}
\int_{C_k} f(z)\, dz - \int_{C_{k+1}} f(z)\, dz &= \int_{C_k - C_{k+1}} f(z)\, dz \\
&= \int_{\xi_1 + \xi_2 + \cdots + \xi_{n-1}} f(z)\, dz \\
&= \int_{\xi_1} f(z)\, dz + \int_{\xi_2} f(z)\, dz + \cdots + \int_{\xi_{n-1}} f(z)\, dz \\
&= 0.
\end{aligned}$$

It follows that

$$\int_C f(z)\, dz = \int_{\xi_1} f(z)\, dz = \int_{\xi_2} f(z)\, dz \cdots = \int_{C'} f(z)\, dz.$$

The proof is complete. ☐

Definition 2.23. *A closed curve C in G is* <u>homotopic to zero,</u> *if C is homotopic to a constant curve. In other words take C' in definition (2.21) to be one point z_0 and its parameterization to be of constant value z_0.*

Definition 2.24. *A region G is* <u>simply connected,</u> *if G is open and every closed curve in G is homotopic to zero.*

Theorem 2.25. *If two contours have the same beginning and end points and the same orientation in a simply connected region G, they are homotopic in G.*

Proof. Let C_1 and C_2 be two contours from a to b in G. Let $C = C_1 - C_2$, which is a closed contour in G. Let $\gamma : [0, 1] \to G$, be a parameterization of C going from a to b and back to a again. Without loss of generality we assume $\gamma(1/2) = b$. It follows that

$\gamma : [0, 1/2] \to G$ parameterizes C_1,

$\gamma : [1/2, 1] \to G$ parameterizes $-C_2$, and thus

$\gamma\big(\frac{2-t}{2}\big) : [0, 1] \to G$ parameterizes C_2.

Since G is simply connected, $C \sim 0$, that is there exists $z_0 \in G$ and a homotopy $\Psi : [0, 1]^2 \to G$, such that $\Psi(0, t) = \gamma(t)$ for all t and $\Psi(1, t) = z_0$ for all t.

Define

$\Psi_1 : [0, 1/2] \times [0, 1] \to G$, by $\Psi_1(s, t) = \Psi(2s, t/2)$ and

$\Psi_2 : [1/2, 1] \times [0, 1] \to G$, by $\Psi_2(s, t) = \Psi(2 - 2s, \frac{2-t}{2})$.

As compositions of continuous functions, both are continuous on their domains.

Now define $\Phi : [0, 1]^2 \to G$, by

$$\Phi(s, t) = \begin{cases} \Psi_1(s, t), if s \leqslant 1/2 \\ \Psi_2(s, t), if s \geq 1/2 \end{cases}.$$

Since Ψ_1 and Ψ_2 are continuous, to see Φ is continuous, it only remains to see $\Psi_1 = \Psi_2$ on the intersection of their domains. To that end note that for all t

$$\Psi_1(1/2, t) = \Psi(1, t/2) = z_0 = \Psi(1, (2 - t)/2)) = \Psi_2(1/2, t).$$

Now it remains to show Φ transforms C_1 to C_2. For all $t \in [0, 1]$

$$\Phi(0, t) = \Psi_1(0, t) = \Psi(0, t/2) = \gamma(t/2),$$

which parameterizes C_1. Moreover for all $t \in [0, 1]$

$$\Phi(1, t) = \Psi_2(1, t) = \Psi(0, (2 - t)/2) = \gamma((2 - t)/2),$$

which parameterizes C_2. $\square$

The following theorem is one of the most famous and important theorems of all complex analysis.

Theorem 2.26 (Cauchy-Goursat Theorem). [4] *Let C be a simple closed contour in a simply connected set G. If a function $f(z)$ is analytic at all points interior to and on C, then*

$$\int_C f(z)dz = 0.$$

Proof. Take two distinct points a and b on C. This forms two contour curves C_1 and C_2 from a to b in G, with $C = C_1 - C_2$. Since G is simply connected, by theorem (2.25), C_1 and C_2 are homotopoic. Thus by theorem (2.22),

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Therefore,

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0.$$

The proof is complete. $\square$

One of the most famous theorems of complex analysis will now be established.

Theorem 2.27 (Cauchy Integral Formula). [4] *Let C be a positively oriented simple closed contour, and let f be analytic function everywhere inside and on C. If a is any point interior to C, then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - a}d\xi.$$

Proof. Let G represent the interior of C and let $a \in G$ be given. Since $G \cup C$ is compact and f is continuous, M_f, the maximum value of $|f(\xi) - f(a)|$ on $G \cup C$ exists. G is open, so there exists $R > 0$ such that $B(a, R) \subset G$ and $\delta > 0$ such that $|\xi - z| > \delta$, whenever $\xi \in C$ and $z \in B(a, R)$. Let

$$r = \frac{1}{2} \min \left\{ R, \frac{\varepsilon\delta}{2\pi M_f} \right\}.$$

Let γ be the positively oriented simple closed contour around the boundary of $B(a, r)$ and note that

$$\int_\gamma \frac{d\xi}{\xi - a} = \int_0^{2\pi} \frac{ire^{it}}{a + re^{it} - a} dt = 2\pi i.$$

It follows that

$$\begin{aligned}
\left| 2\pi i f(a) - \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \right| &= \left| 2\pi i f(a) - \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \right| \\
&= \left| f(a) \int_\gamma \frac{d\xi}{\xi - a} - \int_\gamma \frac{f(\xi)}{\xi - a} d\xi \right| \\
&= \left| \int_\gamma \frac{f(a) - f(\xi)}{\xi - a} d\xi \right| \\
&\leqslant \int_\gamma \left| \frac{f(a) - f(\xi)}{\xi - a} \right| d\xi \\
&\leqslant \frac{M_f}{\delta} 2\pi r \\
&< \varepsilon.
\end{aligned}$$

Therefore, by THEOREM 2.22

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - a}d\xi = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - a}d\xi.$$

The proof is complete. □

The following theorem is a generalization of the Cauchy Integral Formula.

THEOREM 2.28 (Cauchy's Integral Formula for derivatives). [4] *Let C be a positively oriented simple closed contour, and let f be an analytic function everywhere inside and on C. If a is any point interior to C, then for all $n = 0, 1, 2 \ldots,$*

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi - a)^{n+1}}. \tag{2.1}$$

Proof. We proceed by induction. Cauchy's integral formula, previously proven, verifies (2.1) for $n = 0$. We assume

$$f^{(n-1)}(a) = \frac{(n-1)!}{2\pi i}\int_\gamma \frac{f(\xi)d\xi}{(\xi - a)^n},$$

for some n.

Let G represent the interior of C. Let $a \in G$ and $n \in \mathbb{N}$ be given. Since $G \cup C$ is compact and f is continuous, M_f, the maximum value of $|f(\xi)|$ on $G \cup C$ exists. G is open, so there exists $R > 0$ such that $B(a, R) \subset G$ and $\delta > 0$ such that $|\xi - z| > \delta$, whenever $\xi \in C$ and $z \in B(a, R)$. Let

$$r = \frac{1}{2}\min\left\{R, \frac{\varepsilon\delta^{n+1}}{4n\pi M_f}\right\}$$

and let γ be the positively oriented simple closed contour around the boundary of $B(a,r)$. Define F on G by

$$F(z) = \int_\gamma \frac{f(\xi)d\xi}{(\xi - z)^n}.$$

For $z \in B(a,r)$,

$$\begin{aligned}
&\left|\frac{F(z)-F(a)}{z-a} - n\int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}d\xi\right| \\
= \; & \left|\int_\gamma \frac{f(\xi)}{z-a}\left(\frac{1}{(\xi-z)^n} - \frac{1}{(\xi-za)^n}\right) - \frac{nf(\xi)}{(\xi-a)^{n+1}}d\xi\right| \\
= \; & \left|\int_\gamma \frac{f(\xi)}{z-a}\left(\frac{1}{\xi-z} - \frac{1}{\xi-a}\right)\left(\sum_{k=0}^{n-1}\frac{1}{(\xi-z)^{n-1-k}(\xi-a)^k}\right) - \frac{nf(\xi)}{(\xi-a)^{n+1}}d\xi\right| \\
= \; & \left|\int_\gamma f(\xi)\left[\left(\frac{1}{(\xi-z)(\xi-a}\right)\left(\sum_{k=0}^{n-1}\frac{1}{(\xi-z)^{n-1-k}(\xi-a)^k}\right) - \frac{nf(\xi)}{(\xi-a)^{n+1}}\right]d\xi\right| \\
\leqslant \; & \int_\gamma |f(\xi)|\frac{1}{|\xi-z||\xi-a|}\sum_{k=0}^{n-1}\frac{1}{|\xi-z|^{n-1-k}|\xi-a|^k} + \frac{n|f(\xi)|}{|\xi-a|^{n+1}}d\xi \\
< \; & \int_\gamma M_k\frac{1}{\delta^2}\frac{n}{\delta^{n-1}} + \frac{nM_k}{\delta^{n+1}}d\xi \\
\leqslant \; & \frac{2nM_k}{\delta^{n+1}}2\pi r \\
< \; & \varepsilon.
\end{aligned}$$

Thus,

$$F'(a) = \lim_{z\to a}\frac{F(z)-F(a)}{z-a} = n\int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}d\xi.$$

It follows that

$$f^{(n)}(a) = \frac{(n-1)!F'(a)}{2\pi i} = \frac{n!}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi-a)^{n+1}}$$

and the proof is complete by induction. $\square$

CHAPTER 3

RESIDUES

In the previous chapter the Cauchy-Goursat Theorem says that if the function f is analytic at all points interior to and on a simple closed contour C, the integral of f on C is zero. But, what if f fails to be analytic at a finite number of isolated points interior to C? In order to answer this question, we define the concept of residue and present Cauchy's Residue Theorem. This theorem will contribute to the evaluation of integrals of some non-analytic functions and depends on finding specific numbers called residues.

In order to find the residue of a function $f(z)$ that is not analytic at some z_o, we expand it into a series of positive and negative powers of $(z - z_0)$. The theorem allowing us to do this is Laurent's Theorem.

3.1 Laurent Series

DEFINITION 3.1. *An* <u>*annulus*</u> *is a region in the complex plane defined by*

$$\{z \in \mathbb{C} : R_1 < |z - z_0)| < R_2\} \quad \textit{or} \quad \{z \in \mathbb{C} : R_1 < |z - z_0|\}.$$

When $R_1 = 0$, the region is often called a <u>*punctured disc.*</u> *When a property holds for all z in a punctured disc with its center at z_0, we say that property holds* <u>*near*</u> *z_0.*

THEOREM 3.2. (LAURENT'S THEOREM) [2] *If f is analytic on an annulus D and C is any positively oriented simple closed curve in the interior of D about z_0, then*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

for all $z \in D$, where each

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}}.$$

Proof. Let R_1 and R_2 be the inner and outer radii of D. Let $z \in C$ and consider the simple closed curves C_2 traversing $\{z : |z - z_0| = r_2\}$ counterclockwise and C_1 traversing $\{z : |z - z_0| = r_1\}$ clockwise in D, where $R_1 < r_1 < |z - z_0| < r_2 < R_2$. Let C_3 be any radial line segment not containing z and going from C_1 to C_2. Thus by Cauchy's Integral Formula,

$$\begin{aligned}
2\pi i f(z) &= \int_{C_2 - C_3 - C_1 + C_3} \frac{f(\xi)d\xi}{\xi - z} \\
&= \int_{C_2} \frac{f(\xi)d\xi}{\xi - z} - \int_{C_1} \frac{f(\xi)d\xi}{\xi - z} \\
&= \int_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0) - (z - z_0)} - \int_{C_1} \frac{f(\xi)d\xi}{(\xi - z_0) - (z - z_0)} \\
&= \int_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0)(1 - \frac{z - z_0}{\xi - z_0})} - \int_{C_1} \frac{f(\xi)d\xi}{(z - z_0)(1 - \frac{\xi - z_0}{z - z_0})} \\
&= \int_{C_2} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n d\xi - \int_{C_1} \frac{f(\xi)}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0}\right)^n d\xi \\
&= \int_{C_2} f(\xi) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^{n+1}} d\xi - \int_{C_1} f(\xi) \sum_{n=1}^{\infty} \frac{(\xi - z_0)^{n-1}}{(z - z_0)^n} d\xi.
\end{aligned}$$

Since for all $\xi \in C_2, |z - z_0| < |\xi - z_0|$, and for all $\xi \in C_1, |\xi - z_0| < |z - z_0|$, the geometric series above are absolutely convergent. We can therefore interchange the order of summation and integration. Thus

$$2\pi i f(z) \;=\; \sum_{n=0}^{\infty}\left((z-z_0)^n\int_{C_2}\frac{f(\xi)d\xi}{(\xi-z_0)^{n+1}}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{(z-z_0)^n}\int_{C_1}\frac{f(\xi)d\xi}{(\xi-z_0)^{-n+1}}\right).$$

Since $-C_1$ and C_2 are both homotopic to C, we can replace each of them by C and we have

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,$$

where each

$$a_n=\frac{1}{2\pi i}\int_C\frac{f(\xi)d\xi}{(\xi-z_0)^{n+1}}.$$

The proof is complete. □

DEFINITION 3.3. *If for all z in an annulus D,*

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n,$$

we call this series, the <u>Laurent series</u> *of f on D.*

In many instances we will obtain a Laurent series for a function and need to know that it is the same series given in Laurent's Theorem. We will see definition (3.3) designates just the one series.

LEMMA 3.4. *If there exists $r>0$, such that $\sum_{n=-\infty}^{\infty}\xi_n(z-a)^n=0$, for every $z\in B(a,r)$, then $\xi_n=0$ for all $n\in\mathbb{Z}$.*

Proof. Let C be any simple positively oriented closed contour around a and inside $B(a,R)$. First note that by each of Cauchy's Integral Formulas (2.28) applied to any constant function, $f(z)=\xi$, we have that

$$\int_C \xi(z-a)^n \, dz = \begin{cases} 2\pi i, & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2 \end{cases}.$$

Moreover by the Cauchy-Goursat Theorem 2.26,

$$if \quad n \geqslant 0, \quad then \quad \int_C \xi(z-a)^n \, dz \; = \; 0.$$

Thus for each $n \in \mathbb{N}$,

$$\begin{aligned}
0 \;&=\; \int_C (z-a)^{k+1} \sum_{n=-\infty}^{\infty} \xi_n (z-a)^n \, dz \\
&=\; \int_C \sum_{n=-\infty}^{\infty} \xi_n (z-a)^{n+(k+1)} \, dz \\
&=\; \int_C \sum_{n=-(k+1)}^{\infty} \xi_n (z-a)^{n+(k+1)} \, dz + \int_C \sum_{n=(k+2)}^{\infty} \frac{\xi_{-n}}{(z-a)^{n-(k+1)}} \, dz \\
&=\; \int_C \sum_{n=0}^{\infty} \xi_{n-(k+1)} (z-a)^n \, dz + \int_C \sum_{n=1}^{\infty} \frac{\xi_{-n+(k+1)}}{(z-a)^n} \, dz \\
&=\; \sum_{n=0}^{\infty} \int_C \xi_{n-(k+1)} (z-a)^n \, dz \; + \; \int_C \frac{\xi_k}{z-a} \, dz \; + \; \sum_{n=2}^{\infty} \int_C \frac{\xi_{-n+(k+1)}}{(z-a)^n} \, dz \\
&=\; 0 \; + \; 2\pi i \, \xi_k \; + \; 0 \\
&=\; 2\pi i \, \xi_k \\
&=\; \xi_k.
\end{aligned}$$

The proof is complete. □

THEOREM 3.5. *(The Uniqueness Theorem)*[2] *If there exists* $r > 0$, *such that*

$$\sum_{n=-\infty}^{\infty} \alpha_n (z-a)^n = \sum_{n=-\infty}^{\infty} \beta_n (z-a)^n,$$

for every $z \in B(a, r)$, *then* $\alpha_n = \beta_n \quad$ *for all* $n \in \mathbb{Z}$.

Proof. For all $z \in B(a, r)$, we have

$$\sum_{n=-\infty}^{\infty} (\alpha_n - \beta_n)(z-a)^n = \sum_{n=-\infty}^{\infty} \alpha_n (z-a)^n - \sum_{n=-\infty}^{\infty} \beta_n (z-a)^n = 0.$$

Thus by LEMMA 3.4, we have $\alpha_n = \beta_n$ for all $n \in \mathbb{Z}$. □

3.2 Singular Points

DEFINITION 3.6. *A function* f *has an* <u>isolated singularity</u> *at* z_0, *if there exists* $R > 0$ *such that* f *is analytic on the punctured disc* $\{z : 0 < |z - z_0| < R\}$ *but not at* z_0.

DEFINITION 3.7. *An isolated singularity,* z_0, *of* f *is* <u>removable</u>, *if there exists a function* g *and* $R > 0$ *such that* g *is analytic on* $B(z_0, R)$ *and* $f(z) = g(z)$ *on the punctured disc* $\{z : 0 < |z - z_0| < R\}$.

DEFINITION 3.8. *Let* z_0 *be an isolated singular point of* $f(z)$. *Then* z_0 *is a* <u>pole of order m</u> *of* f, *if there exists a natural number* m *and* $r > 0$ *such that*

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

for a function ϕ, that is analytic on $\{z : |z - z_0| < r\}$, with $\phi(z_0) \neq 0$. z_0 is a <u>simple pole</u> *when $m = 1$.*

Definition 3.9. *An isolated singularity, z_0, of f is* <u>essential</u>*, if it is neither removable nor a pole.*

3.3 Definition of Residue

Let z_0 be an isolated singularity of a function f, which is analytic on $D = \{z \in \mathbb{C} : 0 < |z - z_0| < R\}$. Then f has a Laurent series representation

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n,$$

where each

$$A_n = \frac{1}{2\pi i} \int_C \frac{f(\xi) d\xi}{(\xi - z_0)^{(n+1)}},$$

for any positively oriented simple closed curve C in the interior of D.

Definition 3.10. *When f has a Laurent series representation as in (3.3), and z_0 is an isolated singular point of f, the* <u>residue</u> *of f at z_0 is*

$$\text{Res}(f, z_0) = A_{-1} = \frac{1}{2\pi i} \int_C f(z) dz.$$

3.4 Residue at Infinity

DEFINITION 3.11. *If f is analytic on $\{z : |z| > R\}$, for some $R > 0$, then we say f has an* isolated singularity at ∞.

DEFINITION 3.12. *Let f be analytic on $\{z : |z| > R\}$ and let C be the positively oriented circle $\{z : |z| = R\}$. When all the singularities of f, except ∞, are inside C, we define*

$$\mathrm{Res}(f, \infty) = \frac{1}{2\pi i} \int_{-C} f(z)dz.$$

THEOREM 3.13. *If f is analytic on $\{z : |z| > R\}$, for some $R > 0$, with all the singularities of f, except ∞, inside $\{z : |z| < R\}$ then*

$$\mathrm{Res}(f, \infty) = -\mathrm{Res}\left[\frac{f(1/z)}{z^2}, 0\right].$$

Proof. let C be the positively oriented circle $\{z : |z| = R\}$. By Laurent's Theorem f has a valid Laurent series representation outside C we denote by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

For all z such that $0 \neq |1/z| < 1/R$, we have that $|z| > R$, hence $f(1/z)/z^2$ is analytic at z. Moreover,

$$\frac{f(1/z)}{z^2} = \sum_{n=-\infty}^{\infty} a_n z^{-n-2} = \sum_{n=-\infty}^{\infty} a_{-n} z^{n-2} = \sum_{n=-\infty}^{\infty} a_{-n-2} z^n,$$

which must be the Laurent series of $f(1/z)/z^2$, valid on $\{z : 0 < |z| < R\}$, by the uniqueness theorem. From this, we see

$$\operatorname{Res}\left[\frac{f(1/z)}{z^2},0\right] = a_{-(-1)-2} = a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz = -\operatorname{Res}(f,\infty),$$

completing the proof. □

3.5 The Cauchy Residue Theorem

The Residue Theorem was discovered by Augustin-Louis Cauchy in 1814 and immediately became a powerful tool in complex analysis for computing line integrals. The Residue Theorem soon became very importance in real analysis as a tool for evaluating some difficult real integrals, and then, as we show, in finding infinite sums, as well as other applications.

THEOREM 3.14. [2] *Suppose that f is an analytic function on and inside a simple closed positively oriented curve C, except at finitely many isolated singularities $z_1, ..., z_n$ inside C. Then*

$$\int_C f(z)dz = 2\pi i \sum_{i=1}^{n} \operatorname{Res}(f, z_i).$$

Proof. Let C be a simple closed positively oriented curve, and suppose f is an analytic function inside and on C. Consider circles, $C_1, C_2, \ldots C_n$, centered at $z_1, ..., z_n$, where each circle, C_i, has radius r_i, sufficiently small, so that $C_1, ..., C_n$ are disjoint and in the interior of C. We construct a simple closed positively oriented curve C' that surrounds all the points z_i along each circle C_i and joins these small circles with segments.

Since the curve C' follows each segment two times with opposite orientation it is enough to sum the integrals of f around the small circles. By the definition of residue we have

$$\int_C f(z)dz = \int_{C'} f(z)dz = \sum_{i=1}^{n} \int_{C_i} f(z)dz = 2\pi i \sum_{i=1}^{n} \mathrm{Res}(f, z_i).$$

The proof is complete. $\square$

3.6 Zeros and Poles

Since the zeros of the denominator of a quotient function cause the function not to be analytic, there is an obvious relationship between zeros and poles. In this section we explore this relationship.

DEFINITION 3.15. *When f is analytic at z_0, f has a zero of order n at z_0, if*

$$f(z) = (z - z_0)^n q(z),$$

for some function g such that $q(z_0) \neq 0$ and q is analytic on $B(z_0, \varepsilon)$ for some $\varepsilon > 0$.

THEOREM 3.16. [2] *If z_0 is a pole of f, then $\lim_{z \to z_0} f(z) = \infty$.*

Proof. Let n be the order of z_0. Then there exists a function $\phi(z)$, such that $\phi(z_0) \neq 0$, ϕ is analytic near z_0 and

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}.$$

Therefore, $\lim_{z \to z_0} f(z) = \infty$. $\square$

THEOREM 3.17. [2] *Assume that $g(z)$ and $h(z)$ are analytic functions at $z = z_0$, $h(z)$ has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$. Then*

$$f(z) = \frac{g(z)}{h(z)}$$

has a pole of order n at $z = z_0$.

Proof. Since $h(z)$ has a zero of order n at z_0, $h(z) = (z - z_0)^n q(z)$, where $q(z_0) \neq 0$, and q is analytic near z_0. Thus

$$f(z) = \frac{g(z)}{(z - z_0)^n q(z)} = \frac{g(z)/q(z)}{(z - z_0)^n}.$$

We have that $g(z)/q(z)$ is analytic near z_0 and not zero at z_0. We conclude that $f(z)$ has a pole of order n. □

3.7 Residue at a Pole

In the previous section we saw that the residue of a function $f(z)$ with an isolated singularity at a point z_0 could be found within the Laurent expansion of f as the coefficient of the $(z - z_0)^{-1}$ term. That can often be difficult. This section contains theorems for finding residues with alternative techniques that are often more convenient to use.

3.7.1 Residue at a Pole of Order m

THEOREM 3.18. [2] *Let f be analytic on the punctured disc $\{z : 0 < |z - z_0| < r\}$ for some $r > 0$. Then if f has a pole of order m at z_0, then*

$$\mathrm{Res}(f, z_0) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}, \tag{3.1}$$

where ϕ is as given in definition (3.8).

Proof. Since f has a pole of order m, there exists a natural number m and $r > 0$ such that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

for a function ϕ, that is analytic on $B(z_0, r)$,with $\phi(z_0) \neq 0$. It follows that

$$\begin{aligned} f(z) &= \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \\ &= \sum_{n=-m}^{\infty} \frac{\phi^{(n+m)}(z_0)}{(n+m)!}(z-z_0)^n. \end{aligned}$$

Since Laurent series are unique, this is the Laurent series of f. Therefore,

$$\mathrm{Res}(f, z_0) = A_{-1} = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

The proof is complete. □

Corollary 3.19. [2] *If $f(z)$ has a pole of order m at z_0, then*

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[\frac{(z-z_0)^m f(z)}{(m-1)!} \right].$$

Proof. Since ϕ is analytic on $B(z_0, r)$, $\phi^{(m-1)}$ is continuous. Moreover, $\phi(z) = (z - z_0)^m f(z)$ on $0 < |z - z_0| < r$, hence

$$\begin{aligned}\operatorname{Res}(f, z_0) &= \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \\ &= \lim_{z \to z_0} \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \\ &= \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[\frac{(z - z_0)^m f(z)}{(m-1)!} \right].\end{aligned}$$

The proof is complete. □

3.7.2 Residues at Simple Poles

COROLLARY 3.20. [2] *If z_0 is a simple pole of $f(z)$, then*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Proof. Since a simple pole is of order $m = 1$, this is immediate from COROLLARY 3.19. □

THEOREM 3.21. [2] *Let $p(z)$ and $q(z)$ both be analytic at z_0 and suppose $q(z_0) = 0$, $p(z_0) \neq 0$, and $q'(z_0) \neq 0$. If $f(z) = \frac{p(z)}{q(z)}$, then z_0 is a simple pole of $f(z)$ and*

$$\operatorname{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Proof. First, we need to show that z_0 is a zero of order 1. Suppose that q has a zero of order $n \geqslant 2$ at z_0, then $q(z) = (z - z_0)^n \phi(z)$ for an analytic function ϕ. So,

$$\begin{aligned} q'(z) &= n(z - z_0)^{n-1}\phi(z) + (z - z_0)^n \phi'(z) \\ &= (z - z_0)[n(z - z_0)^{n-2}\phi(z) + (z - z_0)^{n-1}\phi'(z)]. \end{aligned}$$

Since $n \geqslant 2$, then $n - 2 \geqslant 0$. So, q' has a zero, and $q'(z_0) \neq 0$. Order of q's zero is 1. Now, by THEOREM 3.17, z_0 is a simple pole. Thus by COROLLARY 3.21 and because $q(z_0) = 0$, we have

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} \\ &= \lim_{z \to z_0} \frac{(z - z_0)p(z)}{q(z) - q(z_0)} \\ &= \lim_{z \to z_0} p(z) \lim_{z \to z_0} \frac{z - z_0}{q(z) - q(z_0)} \\ &= \frac{p(z_0)}{q'(z_0)}. \end{aligned}$$

The proof is complete. $\square$

THEOREM 3.22. [2] *If $g(z)$ is analytic at z_0 and $f(z)$ has a simple pole at z_0, then*

$$\operatorname{Res}(fg, z_0) = g(z_0)\operatorname{Res}(f, z_0).$$

Proof. Since g is analytic at z_0, it's easy to see fg also has a simple pole at z_0. Therefore, by COROLLARY 3.20, we have

$$\begin{aligned}
\operatorname{Res}(fg, z_0) &= \lim_{z\to z_0} [(z - z_0) f(z) g(z)] \\
&= \lim_{z\to z_0} [(z - z_0) f(z)] \lim_{z\to z_0} g(z) \\
&= g(z_0) \operatorname{Res}(f, z_0).
\end{aligned}$$

The proof is complete. □

LEMMA 3.23. *Suppose that f is analytic and not identically zero in a region G.*

i. If z_0 is a zero of f of order $k \geq 1$, then f'/f has a simple pole at z_0 and

$$\operatorname{Res}(f'/f, z_0) = k.$$

ii. If z_0 is a pole of f of order $k \geq 1$, then f'/f has a simple pole at z_0 and

$$\operatorname{Res}(f'/f, z_0) = -k.$$

Proof. (i) Since f has a zero of order k, there exist a function ϕ and $R > 0$ such that $f(z) = \phi(z)(z - z_0)^k$, $\phi(z_0) \neq 0$ and ϕ is analytic in $B(z_0, R)$. For all z in $B(z_0, R)$, we have

$$\frac{f'(z)}{f(z)} = \frac{k\phi(z)(z - z_0)^{k-1} + \phi(z)'(z - z_0)^k}{\phi(z)(z - z_0)^k} = \frac{k}{z - z_0} + \frac{\phi'(z)}{\phi(z)}.$$

However $\phi(z_0) \neq 0$ and ϕ'/ϕ is analytic at z_0, hence ϕ'/ϕ has a convergent Taylor series. Thus,

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Therefore, we conclude that f'/f has a simple pole at z_0, and

$$\operatorname{Res}(f'/f, z_0) = k.$$

(ii) Since f has a pole of order k, we have $f(z) = \phi(z)/(z - z_0)^k$ and $R > 0$ such that $\phi(z_0) \neq 0$ and ϕ is analytic in $B(z_0, R)$. For any z in $B(z_0, R)$,

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{\phi'(z)(z-z_0)^k - k\phi(z)(z-z_0)^{k-1}}{(z-z_0)^{2k}} \cdot \frac{(z-z_0)^k}{\phi(z)} \\
&= \frac{\phi'(z)(z-z_0)^k - k\phi(z)(z-z_0)^{k-1}}{\phi(z)(z-z_0)^k} \\
&= \frac{\phi'(z)}{\phi(z)} - \frac{k}{z-z_0}.
\end{aligned}$$

Since $\phi(z_0) \neq 0$ and ϕ'/ϕ is analytic at z_0, it has a convergent Taylor series. So,

$$\frac{f'(z)}{f(z)} = \frac{-k}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

Hence, f'/f has a simple pole at z_0, and

$$\operatorname{Res}(f'/f, z_0) = -k,$$

and the proof is complete. □

THEOREM 3.24. *If p is a polynomial of degree at least 2, and $z_1, z_2, \ldots, z_n$ are the zeros of p, then*

$$\sum_{j=1}^{n} \operatorname{Res}\left(\frac{1}{p(z)}, z_j\right) = 0.$$

Proof. Suppose that $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$, where $a_n \neq 0$ and $n \geq 2$. By the fundamental theorem of algebra, p has at most n different zeros. Let C be a circle centered at 0 with radius R, sufficiently large that every singularity of $1/p$ is inside C. We now consider

$$\frac{1}{z^2}\frac{1}{p(\frac{1}{z})} = \frac{1}{z^2}\frac{1}{\frac{a_n}{z^n}+\frac{a_{n-1}}{z^{n-1}}+\cdots+a_0} = \frac{1}{z^2}\frac{z^n}{a_n+a_{n-1}z+\cdots+a_0z^n}$$

Since $n \geq 2$, the singularity at $z = 0$ is removable. Therefore, by the Cauchy-Goursat theorem 2.26,

$$\int_C \frac{1}{z^2}\frac{1}{p(\frac{1}{z})}dz = 0.$$

Thus by Cauchy's Residue Theorem 3.14, the definition of residue at infinity 3.12, and Theorem 3.13, we have

$$\begin{aligned}
\sum_{j=1}^{n}\text{Res}\left(\frac{1}{p(z)}, z_j\right) &= \frac{1}{2\pi i}\int_C \frac{1}{p(z)}dz \\
&= -\text{Res}\left(\frac{1}{p(z)}, \infty\right) \\
&= \text{Res}\left(\frac{1}{z^2 p(\frac{1}{z})}, 0\right) \\
&= 0.
\end{aligned}$$

This finishes the proof. $\square$

CHAPTER 4

BERNOULLI NUMBERS

Bernoulli numbers have long been used in algebra and number theory. In this section we define and explore properties of Bernoulli numbers in the framework of complex analysis. In the next chapter, we use them to obtain some important infinite sums.

4.1 The Bernoulli Numbers

DEFINITION 4.1. *The Bernoulli numbers* $\{B_n\}_1^\infty$ *are defined recursively by,*

$$\begin{aligned} B_0 &= 1 \qquad and \\ B_n &= \frac{-1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k, \quad for \quad n \geq 1. \end{aligned}$$

LEMMA 4.2. [2] *Let* $F(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$ *and let* $f(z) = 1/F(z)$. *Then* $f(z)$ *is analytic on* $B(0, 2\pi)$ *and*

$$f(z) = \begin{cases} \frac{z}{e^z-1} & if \quad z \neq 0 \\ 1 & if \quad z = 0 \end{cases}. \tag{4.1}$$

Proof. Since $F(0) = 1$, for $z \neq 0$,

$$\frac{e^z - 1}{z} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{(n+1)!} = F(z).$$

Thus (4.1) follows, and from that, we see f is analytic when $e^z \neq 1$, that is, when $z \neq 2\pi ik$ for some $k \in \mathbb{Z} \setminus \{0\}$. Therefore, $f(z)$ is analytic for all z such that $|z| < 2\pi$. □

THEOREM 4.3. [2]*Let* $F(z) = \sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}$ *and let* $f(z) = 1/F(z)$*. Then for all* $z \in B(0, 2\pi)$

$$f(z) = \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n,$$

where $\{B_n\}_1^{\infty}$ *are the Bernoulli numbers.*

Proof. By LEMMA 4.2, f is analytic on $B(0, 2\pi)$. Therefore, f has a convergent Maclaurin Series on $B(0, 2\pi)$, say

$$f(z) = \sum_{n=0}^{\infty}\frac{a_n}{n!}z^n.$$

Since $f(z)$ and $F(z) = (e^z - 1)/z$ are reciprocals, we have

$$1 = F(z)f(z) = \sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n.$$

By the Cauchy product theorem [2], we then have for each $z \in B(0, 2\pi)$,

$$1 = \sum_{k=1}^{\infty}c_k z^k,$$

where for each n,

$$c_n = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{1}{(n-k)!} = \sum_{k=0}^{n} \frac{a_k}{k!(n-k+1)!}.$$

The Maclaurin series for 1 has all zero coefficients, except $c_0 = 1$, hence by the uniqueness of Taylor series, for all $n \neq 0, c_n = 0$. Thus for $n \geq 1$,

$$0 = c_n = \sum_{k=0}^{n} \frac{a_k}{k!(n-k)!} = \frac{a_n}{n!} + \sum_{k=0}^{n-1} \frac{a_k}{k!(n-k+1)!}.$$

It follows that,

$$a_n = -n! \sum_{k=0}^{n-1} \frac{a_k}{k!(n-k+1)!} = \frac{-1}{n+1} \sum_{k=0}^{n-1} \frac{(n+1)!}{k!(n-k+1)!} a_k = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} a_k.$$

Since $a_0 = B_0$ and $\{a_n\}_0^\infty$ and $\{B_n\}_0^\infty$ have the same recursion formula, we conclude they are the same sequence. □

DEFINITION 4.4. *In light of* LEMMA *4.2 and* THEOREM *4.3, we call $z/(e^z - 1)$ the generating function for the Bernoulli numbers and write*

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$

assuming the value 1 *at the removable singularity at zero.*

COROLLARY 4.5. [2] *The odd Bernoulli numbers are zero except B_1.*

Proof. It is easy to see the function $f(z) = \frac{z}{e^z-1} + \frac{z}{2} - 1$ is even. Since for $z \neq 0$,

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n,$$

we see

$$\sum_{n=2}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1} + \frac{z}{2} - 1$$

is even as well. Therefore,

$$\sum_{n=2}^{\infty} \frac{B_n}{n!} z^n \quad = \quad \sum_{n=2}^{\infty} (-1)^n \frac{B_n}{n!} z^n.$$

It follows that when $k > 2$ is odd $B_n = -B_n$, and therefore zero. □

4.2 Results

In this section we use the results of the previous section to find several of our main results. These sums, found using Bernoulli numbers, will also become powerful tools in the evaluation of other series in the next chapter.

Result 4.6. *When* $0 < |z| < \pi$,

$$z \coth z = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

Proof. First note that when $0 < |z| < \pi$,

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2}\frac{e^z + 1}{e^z - 1} = \frac{z}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2}.$$

When $0 < |z| < \pi$, by DEFINITION 4.4, we have

$$z\coth z = \frac{2z}{e^{2z} - 1} + z = \sum_{n=0}^{\infty}\frac{B_n}{n!}(2z)^n + \frac{2z}{2} = 1 + \sum_{n=2}^{\infty}\frac{B_n}{n!}(2z)^n.$$

Now, since $B_{2n+1} = 0$, for all $n \geq 1$, by COROLLARY 4.5, this simplifies to

$$z\coth z = \sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}z^{2n}.$$

This finishes the proof. $\square$

RESULT 4.7. *When* $|z| < \pi$,

$$z\cot z = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}B_{2n}}{(2n)!}z^{2n}.$$

Proof. In RESULT 4.7 replace z by iz, and since $iz\coth(iz) = z\cot z$, we have,

$$z\cot z = \sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}(i)^{2n}}{(2n)!}z^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n}.$$

This finishes the proof. $\square$

RESULT 4.8. *When $|z| < \frac{\pi}{2}$,*

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1}.$$

Proof. Since $\tan z = \frac{1}{\cot z} = \cot z - \frac{\cot^2 z - 1}{\cot z} = \cot z - 2\cot(2z)$, we have by RESULT 4.7, that

$$\begin{aligned} \tan z &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1} - 2\sum_{n=0}^{\infty} (-1)^n \frac{2^{4n-1} B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}(1-2^{2n})B_{2n}}{(2n)!} z^{2n-1} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1}. \end{aligned}$$

This finishes the proof. ☐

RESULT 4.9. *When $|z| < \pi$,*

$$\csc z = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(2^{2n}-2)B_{2n}}{(2n)!} z^{2n-1}.$$

Proof. Since $\csc z = 1/\sin z$, we have for all z such that $|z| < \pi$,

$$\csc 2z = \frac{1}{2\sin z \cos z} = \frac{\csc^2 z}{2\cot z} = \cot z - \frac{\cot^2 z - 1}{2\cot z} = \cot z - \cot 2z.$$

So,

$$
\begin{aligned}
\csc z &= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}B_{2n}}{2^{2n-1}(2n)!}z^{2n-1} - \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}B_{2n}}{(2n)!}z^{2n-1} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{(2-2^{2n})B_{2n}}{(2n)!}z^{2n-1} \\
&= \sum_{n=0}^{\infty}(-1)^{n-1} \frac{(2^{2n}-2)B_{2n}}{(2n)!}z^{2n-1}.
\end{aligned}
$$

$\square$

CHAPTER 5

SUMMING SERIES BY RESIDUES

This chapter contains more of our main results, making use of the tools presented in the previous chapters. We will use the theory of residues to develop a powerful technique to find sums of the form $\sum_k f(k)$, where $f(z) = \frac{q(z)}{p(z)}$ is a rational function with degree $p(z)$ - degree $q(z) \geqslant 2$.

5.1 Foundations

In this section we develop the technique that will produce more of our main results. The concept is of capturing an ever widening set of singularities inside contours, obtaining corresponding finite sums, and then deducing the desired infinite sum.

Definition 5.1. *For convenience we will refer to a contour C_n as a* <u>*basic contour,*</u> *provided*

1. *C_n is positively oriented.*
2. *C_n is simple.*
3. *C_n is piecewise smooth.*
4. *C_n is centered at the origin.*
5. *C_n is on a square of side $2n + 1$ or on a circle of radius $n + 1/2$ for any $n \in \mathbb{N}$.*

LEMMA 5.2. *Let $r_n = n + 1/2$ for any $n \in \mathbb{N}$ and let $0 < \varepsilon < 1$. If (x, y) is on the intersection of*

$$(x - r_n)^2 + y^2 = \varepsilon^2 \qquad \text{and} \qquad x^2 + y^2 = r_n^2$$

for any $n \in \mathbb{N}$, then $|y| \geq \varepsilon/2$.

Proof. Solving this system of equations yields

$$y = \pm \varepsilon \sqrt{1 - \frac{\varepsilon^2}{(2r_n)^2}}.$$

Since $r_n \geq 3/2$ for all n and $\varepsilon < 1$,

$$\sqrt{1 - \frac{\varepsilon^2}{(2r_n)^2}} \geq \sqrt{1 - \frac{1}{9}} = \sqrt{8/9} > \frac{1}{2}.$$

Therefore, $|y| \geq \varepsilon/2$. □

LEMMA 5.3. [2] *There exists $B > 0$ such that whenever z is on any basic contour C_n,*

$$|\cot(\pi z)| < B \qquad \text{and} \qquad |\csc(\pi z)| < B.$$

Proof. For any $z = x + iy$, since cot and csc are odd functions, we will, without loss of generality, assume $y \geq 0$. It follows that

$$\left|e^{2\pi i z}\right| = \left|e^{2\pi i x - 2\pi y}\right| = \left|e^{2\pi i x} e^{-2\pi y}\right| = e^{-2\pi y} \leqslant 1. \tag{5.1}$$

Let $z \in B(\frac{1}{2}, \frac{1}{4})$. We have $1/4 < x < 3/4$, hence $\cos(2\pi x) < 0$. It follows that

$$
\begin{aligned}
\left|e^{2\pi iz}-1\right| &\geq \left|Re(e^{2\pi ix-2\pi y}-1)\right| \\
&= \left|Re\left(e^{-\pi y}\left(\cos(2\pi x)+i\sin(2\pi x)\right)-1\right)\right| \\
&= |e^{-\pi y}\cos(2\pi x)-1| \qquad (5.2) \\
&= 1-e^{-\pi y}\cos(2\pi x) \\
&\geq 1.
\end{aligned}
$$

Combining (5.1) and (5.2), we have for all $z \in B(\frac{1}{2}, \frac{1}{4})$,

$$
|\cot(\pi z)| = \left|\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}}\right| = \left|\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1}\right| \leqslant \left|e^{2\pi iz}\right| + 1 \leqslant 2 \qquad (5.3)
$$

and

$$
|\csc(\pi z)| = \left|\frac{1}{e^{\pi iz}-e^{-\pi iz}}\right| = \left|\frac{e^{\pi iz}}{e^{2\pi iz}-1}\right| \leqslant 1. \qquad (5.4)
$$

Now, consider $z \in B(n+\frac{1}{2}, \frac{1}{4})$ for any integer $n \neq 0$. Since $z-n \in B(\frac{1}{2}, \frac{1}{4})$ and both $|\cot|$ and $|\csc|$ are π-periodic, we see

$$
|\cot(\pi z)| = |\cot(\pi(z-n))| \leqslant 2 \qquad (5.5)
$$

and

$$
|\csc(\pi z)| = |\csc(\pi(z-n))| \leqslant 1 \qquad (5.6)
$$

for all $z \in S = \bigcup_{n\neq 0} B(n+\dfrac{1}{2}, \dfrac{1}{4})$.

For z on a square with verticies at $\pm(n+\frac{1}{2}) \pm i(n+\frac{1}{2})$ but $z \notin S$, we see $|y| \geq 1/4$.

If z is on a circle of radius $r_n = n + 1/2$ but $z \notin S$, then $|y| \geq |b|$, where (a, b) is a point of intersection of

$$(x - r_n)^2 + y^2 = 1/16 \qquad \text{and} \qquad x^2 + y^2 = r_n^2.$$

By LEMMA 5.2, $|y| \geq |a| \geq 1/8$. Noting that $e^{-2\pi y} < 1$, it follows that

$$|\cot(\pi z)| = \left| \frac{e^{2\pi i x} e^{-2\pi y} + 1}{e^{2\pi i x} e^{-2\pi y} - 1} \right| \leqslant \frac{e^{-2\pi y} + 1}{||e^{-2\pi y}| - 1|} \leqslant \frac{e^{-2\pi y} + 1}{1 - e^{-2\pi y}} \leqslant \frac{e^{-2\pi/8} + 1}{1 - e^{-\pi/4}}. \tag{5.7}$$

and

$$|\csc(\pi z)| = \left| \frac{e^{\pi i z}}{e^{2\pi i z} - 1} \right| = \frac{e^{\pi i x} e^{-\pi y}}{e^{2\pi i x} e^{-2\pi y} - 1} \leqslant \frac{e^{-\pi y} + 1}{||e^{-2\pi y}| - 1|} \leqslant \frac{e^{-\pi/8} + 1}{1 - e^{-\pi/4}}. \tag{5.8}$$

Taking B to be the minimum of the bounds given in lines (5.5), (5.6), (5.7) and (5.8) yields the desired bound for all z on all basic contours. $\square$

LEMMA 5.4. [2] *Let n be a positive integer, and let C_n be a basic contour. If $f(z) = \frac{p(z)}{q(z)}$ is a rational function with degree $q(z)$ - degree $p(z) \geqslant 2$, then,*

$$\lim_{n \to \infty} \int_{C_n} \frac{p(z)}{q(z)} \cot(\pi z) dz = 0 \tag{5.9}$$

and

$$\lim_{n \to \infty} \int_{C_n} \frac{p(z)}{q(z)} \csc(\pi z) dz = 0. \tag{5.10}$$

Proof. By LEMMA 5.3 $|\cot(\pi z)|$ and $|\csc(\pi z)|$ are bounded by some $B > 0$ on C_n. The function $12zf(z)$ is a rational function whose numerator is of degree at least one less than the degree of its denominator. Thus given any $\varepsilon > 0$, there exists a number N such that when $|z| \geq N$,

$$|2\pi z f(z)| < |12 z f(z)| < \frac{\varepsilon}{B}.$$

We also assume that when $n \geq N$, each pole of f is inside C_n. Let $n \geq N$. In the case C_n is a square, given any z on C_n, we have $1 \leqslant n < |z|$. It follows that $12|z| = 8|z| + 4|z| > 8n + 4$. Thus since $n \geq N$, $\quad |z| > N$ and hence

$$\left| \int_{C_n} f(z) \cot(\pi z) dz \right| \leqslant \int_{C_n} \frac{12|z|}{8n+4} |f(z)| B \, dz \leqslant \frac{L(C_n)}{8n+4} \frac{\varepsilon}{B} B = \varepsilon,$$

proving (5.9) in the case C_n is a square. For the case when C_n is a circle, $|z| = n + 1/2$, hence

$$\left| \int_{C_n} f(z) \cot(\pi z) dz \right| \leqslant \int_{C_n} \frac{2\pi|z|}{2\pi(n+1/2)} |f(z)| B \, dz \leqslant \frac{L(C_n)}{2\pi(n+1/2)} \frac{\varepsilon}{B} B = \varepsilon,$$

finishing (5.9).

The proof for both cases of (5.10) are the same as for (5.9), and are omitted. $\square$

THEOREM 5.5. [2] *Suppose that f is analytic at an integer k, then*

i. $\quad \operatorname{Res}(f(z)\cot(\pi z), k) = \frac{1}{\pi} f(k)$.
ii. $\quad \operatorname{Res}(f(z)\csc(\pi z), k) = \frac{(-1)^n}{\pi} f(k)$.
iii. $\quad \operatorname{Res}\left(f(z)\tan(\pi z), \frac{2k+1}{2}\right) = \frac{1}{\pi} f\left(\frac{2k+1}{2}\right)$.
iv. $\quad \operatorname{Res}\left(f(z)\sec(\pi z), \frac{2k+1}{2}\right) = \frac{(-1)^n}{\pi} f\left(\frac{2k+1}{2}\right)$.

Proof. (i) Since $\sin(\pi z) = 0$ if and only if z is an integer k and $\cos(\pi k) \neq 0$. We see by THEOREM 3.21, $\cot(\pi z) = \cos(\pi z)/\sin(\pi z)$ has a simple pole at each integer k and

$$\operatorname{Res}(\cot(\pi z), k) = \frac{\cos(\pi k)}{\frac{d}{dz}\sin(\pi k)} = \frac{\cos(\pi k)}{\pi \cos(\pi k)} = \frac{1}{\pi}.$$

Therefore, by THEOREMS 3.21, 3.22,

$$\operatorname{Res}(f(z)\cot(\pi z), k) = f(k)\operatorname{Res}(\cot(\pi z), k) = \frac{f(k)}{\pi}.$$

(ii) Recall that $\csc(\pi z) = 1/\sin(\pi z)$, and as in part (i) above, $\csc(\pi z) = 1/\sin(\pi z)$ has a simple pole at each integer k. So, by THEOREM 3.21,

$$\operatorname{Res}(\csc(\pi z), k) = \operatorname{Res}\left(\frac{1}{\sin(\pi z)}, k\right) = \frac{1}{\pi\cos(\pi k)} = \frac{(-1)^n}{\pi}.$$

Now, by THEOREMS 3.21, 3.22,

$$\operatorname{Res}(f(z)\csc(\pi z), k) = f(k)\operatorname{Res}(\csc(\pi z), k) = \frac{(-1)^n}{\pi} f(k).$$

(iii) Recall that $\tan(\pi z) = \sin(\pi z)/\cos(\pi z)$. Note that the zeros of $\cos(\pi z)$ are $\frac{2k+1}{2}$ where k is an integer. By THEOREM 3.21, those zeros are simple poles of tan and

$$\operatorname{Res}\left(\tan(\pi z), \frac{2k+1}{2}\right) = \frac{\sin\left(\frac{2\pi k+\pi}{2}\right)}{\pi\sin\left(\frac{2\pi k+\pi}{2}\right)} = \frac{1}{\pi}.$$

Now, by THEOREM 3.22,

$$\begin{aligned}\operatorname{Res}\left(f(z)\tan(\pi z), \frac{2k+1}{2}\right) &= f\left(\frac{2k+1}{2}\right)\operatorname{Res}\left(\tan(\pi z), \frac{2k+1}{2}\right)\\ &= \frac{1}{\pi} f\left(\frac{2k+1}{2}\right).\end{aligned}$$

(iv) Recall that $\sec(\pi z) = 1/\cos(\pi z)$, and from the previous part

$\sec(\pi z) = 1/\cos(\pi z)$ has a simple pole at each $\frac{2k+1}{2}$. So, by THEOREM 3.21,

$$\operatorname{Res}\left(\sec(\pi z), \frac{2k+1}{2}\right) = \operatorname{Res}\left(\frac{1}{\cos(\pi z)}, \frac{2k+1}{2}\right) = \frac{1}{\sin(\frac{2\pi k+\pi}{2})} = \frac{(-1)^n}{\pi}.$$

Now, by THEOREM 3.22,

$$\begin{aligned} \operatorname{Res}\left(f(z)\sec(\pi z), \frac{2k+1}{2}\right) &= f\left(\frac{2k+1}{2}\right)\operatorname{Res}\left(\sec(\pi z), \frac{2k+1}{2}\right) \\ &= \frac{(-1)^n}{\pi} f\left(\frac{2k+1}{2}\right). \end{aligned}$$

The proof is complete. □

THEOREM 5.6. *For every integer k, $\pi\coth(\pi z)$ has a simple pole at $z = ik$ and*

$$\operatorname{Res}(\pi\coth(\pi z), ik) = 1.$$

Proof. If we let $\sinh(\pi z) = 0$, then $e^{\pi z} - e^{-\pi z} = 0$, hence $e^{2\pi z} = 1$. Therefore, the zeros of $\sinh(\pi z)$ are $z = ik$ for every integer k. Since $\cosh \pi ik \neq 0$, by theorem 3.21 the poles of $\pi\coth(\pi z) = \pi\cosh(\pi z)/\sinh(\pi z)$ are simple and

$$\operatorname{Res}(\pi\coth(\pi z), ik) = \frac{\pi\cosh(\pi z)}{\frac{d}{dz}\sinh(\pi z)} = \frac{\pi\cosh(\pi k)}{\pi\cosh(\pi k)} = 1.$$

The proof is complete. □

5.2 Finite Sums

LEMMA 5.7. [3] *Let C_n be a basic contour. If f is analytic on C_n, except at finitely many singularities $z_1, \ldots, z_m$, all inside C_n none of which are integers, then*

$$\sum_{k=-n}^{n} f(k) = \frac{1}{2i}\int_{C_n} f(z)\cot(\pi z)dz - \pi\sum_{j=1}^{m}\mathrm{Res}(f(z)\cot(\pi z), z_j)$$

and

$$\sum_{k=-n}^{n} (-1)^k f(k) = \frac{1}{2i}\int_{C_n} f(z)\csc(\pi z)dz - \pi\sum_{j=1}^{m}\mathrm{Res}(f(z)\csc(\pi z), z_j).$$

Proof.

By Theorem 5.5,

$$\mathrm{Res}(f(z)\cot(\pi z), k) = \frac{f(k)\cos(\pi k)}{\pi\cos(\pi k)} = \frac{1}{\pi}f(k).$$

Moreover, since each z_j is inside C_n, we have, by the Cauchy Residue theorem, 3.14, that

$$\begin{aligned}\frac{1}{2\pi i}\int_{C_n} f(z)\cot(\pi z)dz &= \sum_{k=-n}^{n}\mathrm{Res}(f(z)\cot(\pi z), k) + \sum_{j=1}^{m}\mathrm{Res}(f(z)\cot(\pi z), z_j)\\ &= \frac{1}{\pi}\sum_{k=-n}^{n} f(k) + \sum_{j=1}^{m}\mathrm{Res}(f(z)\cot(\pi z), z_j).\end{aligned}$$

Therefore, we conclude that

$$\sum_{k=m}^{n} f(k) = \frac{1}{2i}\int_{C_n} f(z)\cot(\pi z)dz - \pi\sum_{j=1}^{m} Res(f(z)\cot(\pi z), z_j).$$

The proof of the second assertion is almost the same as the proof of the first. The only difference is that by THEOREM 5.5,

$$Res(f(z)\csc(\pi z), k) = \frac{f(z)}{\pi\cos(\pi z)} = \frac{(-1)^k}{\pi} f(k).$$

The rest of the proof is exactly the same. $\square$

DEFINITION 5.8. *Let $\delta > 0$ and suppose $\alpha < \beta$. We define,*

$$\begin{aligned} E_{\alpha,\beta} &= \lim_{\delta\to\infty}\left(\int_\alpha^{\alpha+i\delta} \frac{f(z)}{e^{-2\pi iz}-1}dz + \int_\alpha^{\alpha-i\delta} \frac{f(z)}{e^{2\pi iz}-1}dz \right. \\ &\qquad \left. - \int_\beta^{\beta+i\delta} \frac{f(z)}{e^{-2\pi iz}-1}dz - \int_\beta^{\beta-i\delta} \frac{f(z)}{e^{2\pi iz}-1}dz\right). \end{aligned}$$

THEOREM 5.9. *Suppose that f is analytic in the region $G = \{z : \alpha \leqslant Re\, z \leqslant \beta\}$. Also, for $z = z + iy$ suppose*

$$\lim_{|z|\to\infty} e^{-2\pi|z|} f(x+iy) = 0, \tag{5.11}$$

uniformly in G. If $m-1 < \alpha < m$, $n < \beta < n+1$, $(m, n \in \mathbb{Z})$, then

$$\sum_{k=m}^{n} f(k) = \int_\alpha^\beta f(x)dx + E_{\alpha,\beta}. \tag{5.12}$$

Proof. Let $\delta > 0$, and define $C = C_1 + C_2$, where

$$C_1 = [\alpha,\beta] + [\beta,\beta+i\delta] + [\beta+i\delta,\alpha+i\delta] + [\alpha+i\delta,\alpha], \text{ and}$$

$$C_2 = [\alpha,\beta] + [\beta,\beta-i\delta] + [\beta-i\delta,\alpha-i\delta] + [\alpha-i\delta,\alpha].$$

Let $C_1 = C \cap \{z : Im\, z > 0\}$ and $C_2 = C \cap \{z : Im\, z < 0\}$. Now, since f has no singularites in G, By THEOREM 5.7, we have,

$$\sum_{k=m}^{n} f(k) = \frac{1}{2i} \int_C f(z) \cot \pi z \, dz.$$

Hence,
$$\sum_{k=m}^{n} f(k) = \frac{1}{2i} \int_{C_1} f(z) \cot \pi z \, dz + \frac{1}{2i} \int_{C_2} f(z) \cot \pi z \, dz. \tag{5.13}$$

It is easy to verify these identities,

$$\frac{1}{2i} \cot \pi z = \frac{1}{2} + \frac{1}{e^{2\pi i z} - 1}$$

and

$$\frac{1}{2i} \cot \pi z = \frac{-1}{2} - \frac{1}{e^{-2\pi i z} - 1}.$$

Aplying these identities to equation (5.13), we have,

$$\begin{aligned}
\sum_{k=m}^{n} f(k) &= \int_{C_1} f(z) \left(\frac{-1}{2} - \frac{1}{e^{-2\pi i z} - 1} \right) dz + \int_{C_2} f(z) \left(\frac{1}{2} + \frac{1}{e^{2\pi i z} - 1} \right) dz \\
&= \int_{\alpha}^{\beta} f(x) dx + \int_{\alpha}^{\alpha + i\delta} \frac{f(z)}{e^{-2\pi i z} - 1} dz + \int_{\alpha}^{\alpha - i\delta} \frac{f(z)}{e^{2\pi i z} - 1} dz \\
&\quad - \int_{\beta}^{\beta + i\delta} \frac{f(z)}{e^{-2\pi i z} - 1} dz - \int_{\beta}^{\beta - i\delta} \frac{f(z)}{e^{2\pi i z} - 1} dz + \int_{\alpha}^{\beta} \frac{f(x + i\delta)}{e^{-2\pi i (x + i\delta)} - 1} dx \\
&\quad + \int_{\alpha}^{\beta} \frac{f(x - i\delta)}{e^{2\pi i (x - i\delta)} - 1} dx.
\end{aligned}$$

Let $\delta \longrightarrow \infty$. In light of hypothesis (5.11), we have

$$\begin{aligned}\sum_{k=m}^{n} f(k) &= \int_{\alpha}^{\beta} f(x)dx + \lim_{\delta\to\infty}\left(\int_{\alpha}^{\alpha+i\delta} \frac{f(z)}{e^{-2\pi iz}-1}dz + \int_{\alpha}^{\alpha-i\delta} \frac{f(z)}{e^{2\pi iz}-1}dz\right.\\ &\qquad \left. - \int_{\beta}^{\beta+i\delta} \frac{f(z)}{e^{-2\pi iz}-1}dz - \int_{\beta}^{\beta-i\delta} \frac{f(z)}{e^{2\pi iz}-1}dz\right)\\ &= \int_{\alpha}^{\beta} f(x)dx + E_{\alpha,\beta}.\end{aligned}$$

The proof is complete. □

5.3 Infinite Series

5.3.1 Non-integer Singularities

THEOREM 5.10. [2] *Suppose that* $f(z) = \frac{q(z)}{p(z)}$ *is a rational function with degree* $p(z)$ *- degree* $q(z) \geqslant 2$*. Also, suppose that* f *has poles at* $z_1, \ldots, z_m$*, none of which are integers. Then*

(i) $$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{j=1}^{m} \operatorname{Res}(f(z)\cot(\pi z), z_j)$$

and

(ii) $$\sum_{k=-\infty}^{\infty} (-1)^k f(k) = -\pi \sum_{j=1}^{m} \operatorname{Res}(f(z)\csc(\pi z), z_j).$$

Proof. (i) Let C_n be a basic contour and assume n is sufficiently large so that each z_j is inside C_n. Thus by LEMMA 5.7,

$$\int_{C_n} f(z)\cot(\pi z)dz = 2\pi i \sum_{k=-n}^{n} \frac{1}{\pi} f(k) + 2\pi i \sum_{j=1}^{m} \text{Res}(f(z)\cot(\pi z), z_j).$$

Now, let $n \longrightarrow \infty$. By LEMMA 5.4, $\lim_{n\to\infty} \int_{C_n} f(z)\cot(\pi z)dz = 0$, hence

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{j=1}^{m} \text{Res}(f(z)\cot(\pi z), z_j).$$

(ii) The proof of the second assertion can be proved in the same way as the first, with csc in place of cot and $(-1)^k f(k)$ in place of $f(k)$.

□

EXAMPLE 5.11. *If ia is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi}{a} \coth(a\pi).$$

Proof. Let

$$f(z) = \frac{1}{z^2 + a^2} = \frac{1}{(z - z_1)(z - z_2)},$$

where $z_1 = ai$ and $z_2 = -ai$. By DEFINITION 3.8, $f(z)\cot(\pi z)$ has a simple pole at z_1 and at z_2. Now, by THEOREM 3.22

$$\text{Res}\left(\frac{\cot(\pi z)}{(z - ia)(z + ia)}, ai\right) = \left[\frac{\cot(\pi ia)}{(ia + ia)}\right] \text{Res}\left(\frac{1}{(z - ia)}, ai\right) = \frac{\cot(\pi ia)}{2ia}$$

$$= \frac{1}{2ia}\frac{\cos(\pi ia)}{\sin(\pi ia)} = \frac{1}{2ia}\frac{\cosh(\pi a)}{i\sinh(\pi a)} = -\frac{1}{2a}\coth(\pi a).$$

We can calculate the residue at $z_2 = -ia$ in the same way, obtaining

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^2+a^2}, -ai\right) \quad = \quad -\frac{1}{2a}\coth(\pi a).$$

Therefore, by THEOREM 5.10, we have

$$\sum_{k=-\infty}^{\infty}\frac{1}{k^2+a^2} = -\pi\sum_{j=1}^{2}\operatorname{Res}\left(\frac{\cot(\pi z)}{z^2+a^2}, z_j\right) = \frac{\pi}{a}\coth(\pi a).$$

The proof is complete. □

EXAMPLE 5.12. *If ia is not an integer, then*

$$\sum_{k=1}^{\infty}\frac{1}{k^2+a^2} = \frac{\pi}{2a}\coth(\pi a) - \frac{1}{2a^2}.$$

Proof. From the previous example we have $\displaystyle\sum_{k=-\infty}^{\infty}\frac{1}{(k^2+a^2)} = \frac{\pi}{a}\coth(a\pi)$. Since $f(k) = 1/(k^2+a^2)$ is an even function, we have

$$\sum_{k=1}^{\infty}\frac{1}{k^2+a^2} = \frac{1}{2}\left(\sum_{k=-\infty}^{\infty}\frac{1}{k^2+a^2} - \frac{1}{a^2}\right) = \frac{\pi}{2a}\coth(a\pi) - \frac{1}{2a^2}.$$

The proof is complete. □

EXAMPLE 5.13. *If ia is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+a^2)^2} = \frac{\pi}{2a^3}\coth(a\pi) + \frac{\pi^2}{2a^2}\text{csch}^2(a\pi).$$

Proof. Let

$$f(z) = \frac{1}{(z^2+a^2)^2} = \frac{1}{(z-ia)^2(z+ia)^2},$$

which has poles of order 2 at $z_1 = ia$ and $z_2 = -ia$. By DEFINITION 3.8 , $f(z)\cot(\pi z)$ has a pole of order 2 at ia and at $-ia$. Hence, by THEOREM 3.19, we have

$$\begin{aligned}
\text{Res}\left(\frac{\cot(\pi z)}{(k^2+a^2)^2}, ia\right) &= \lim_{z\to ia} \frac{d}{dz}\left(\frac{(z-ia)^2\cot(\pi z)}{(z-ia)^2(z+ia)^2}\right) \\
&= \lim_{z\to ia} \frac{d}{dz}\left(\frac{\cot(\pi z)}{(z+ia)^2}\right) \\
&= \lim_{z\to ia} \frac{-\pi(z+ia)\csc^2(\pi z) - 2\cot(\pi z)}{(z+ia)^3} \\
&= \frac{-2\pi ia\csc^2(\pi ia) - 2\cot(\pi ia)}{(2ia)^3} \\
&= -\frac{\pi i^2\csc^2(\pi ia)}{4a^2} - \frac{i\cot(\pi ia)}{4a^3} \\
&= -\frac{\pi csh^2(\pi a)}{4a^2} - \frac{\coth(\pi a)}{4a^3}.
\end{aligned}$$

An almost identical calculation yields

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{(k^2+a^2)^2}, -ia\right) = -\frac{\pi csh^2(\pi a)}{4a^2} - \frac{\coth(\pi a)}{4a^3}.$$

Hence by THEOREM 5.10, we have

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+a^2)^2} = -\pi \sum_{j=1}^{2} \operatorname{Res}\left(\frac{\cot(\pi z)}{(k^2+a^2)^2}, z_j\right) = \frac{\pi^2 csh^2(\pi a)}{2a^2} + \frac{\pi \coth(\pi a)}{2a^3}.$$

The proof is complete. $\square$

EXAMPLE 5.14. *If $a > 0$ is not an integer, then*

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^2} = \pi^2 \csc(\pi a)\cot(\pi a).$$

Proof. Let $f(z) = (z+a)^{-2}$. Since $-a$ is not an integer, by DEFINITION 3.8, $f(z)\csc(\pi z)$ has a pole of order 2 at $z = -a$. Hence, by THEOREM 3.19

$$\begin{aligned}
\operatorname{Res}\left(\frac{1}{(z+a)^2}\csc(\pi z), -a\right) &= \lim_{z\to -a} \frac{d}{dz}\left[\frac{(z+a)^2}{(z+a)^2}\csc(\pi z)\right] \\
&= \lim_{z\to -a}\left[-\pi\cot(\pi z)\csc(\pi z)\right] \\
&= -\pi\cot(-\pi a)\csc(-\pi a) \\
&= -\pi\cot(\pi a)\csc(\pi a).
\end{aligned}$$

It then follows by THEOREM 5.10, that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+a)^2} = -\pi\left(-\pi\cot(\pi a)\csc(\pi a)\right) = \pi^2\csc(\pi a)\cot(\pi a).$$

The proof is complete. □

THEOREM 5.15. [3] *Suppose that a, b, and t are real numbers, and $|b| < |a|$, then*

$$\sum_{k=-\infty}^{\infty} (-1)^k \frac{a\,t}{\pi^2 k^2 + a^2 t^2} e^{\frac{i\pi bk}{a}} = \frac{\cosh(bt)}{\sinh(at)}.$$

Proof. Let

$$f(z) = \frac{a\,t}{\pi^2 z^2 + a^2 t^2} e^{\frac{i\pi bz}{a}} = \frac{a\,t\,e^{\frac{i\pi bz}{a}}}{\pi^2(z-z_1)(z-z_2)},$$

where $z_1 = ait/\pi$, and $z_2 = -ait/\pi$. Since these poles are simple, by THEOREM 3.22,

$$\begin{aligned} \operatorname{Res}\left(\frac{a\,t\,e^{\frac{i\pi bz}{a}}}{\pi^2 z^2 + a^2 t^2}\csc(\pi z), z_1\right) &= \left(\frac{a\,t\,e^{\frac{i\pi bz_1}{a}}}{\pi^2(z_1-z_2)}\csc(\pi z_1)\right)\operatorname{Res}\left(\frac{1}{z-z_1}, z_1\right) \\ &= \frac{1}{2\pi i}e^{-bt}\csc(iat). \end{aligned}$$

An almost identical calculation yields

$$\operatorname{Res}\left(\frac{at}{(z-z_1)(z-z_2)}e^{\frac{i\pi bz}{a}}\csc(\pi z), z_2\right) = \frac{1}{2\pi i}e^{b\,t}\csc(iat),$$

as well. Hence by THEOREM 5.10, we have

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} (-1)^k \frac{at}{\pi^2 k^2 + a^2 t^2} e^{\frac{i\pi bk}{a}} &= -\pi \sum_{j=1}^{2} \operatorname{Res}\left(\frac{a\, t e^{i\pi bz/a}}{\pi^2 z^2 + a^2 t^2} \csc(\pi z), z_j \right) \\
&= -\pi \left[\frac{1}{2\pi i} e^{-bt} \csc(iat) + \frac{1}{2\pi i} e^{bt} \csc(iat) \right] \\
&= \frac{e^{bt} + e^{-bt}}{-2i \sin(iat)} \\
&= \frac{\cosh(bt)}{\sinh(at)}.
\end{aligned}
$$

This finishes the proof. ∎

5.3.2 Integer Singularities

THEOREM 5.16. *Suppose that $f(z) = \frac{p(z)}{q(z)}$ is a rational function, with poles $\{z_1, z_2 \ldots, z_n\}$, some of which may be integers, and let $S = \mathbb{Z} \setminus \{z_1, z_2 \ldots, z_n\}$. Then,*

$$
\sum_{k \in S} f(k) = -\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z) \cot(\pi z), z_j \right).
$$

Proof. If $k \in S$, then by DEFINITION 3.8, $f(z)\cot(\pi z)$ has simple pole at k and $\operatorname{Res}(f(z)\cot(\pi z), k) = \frac{1}{\pi} f(k)$. Now, consider n such that all singularities of f are on the inside of C_n. Then, by LEMMA 5.7, we have

$$\begin{aligned}\int_{C_n} f(z)\cot(\pi z)dz &= 2\pi i \sum \{\textit{all the residues in } C_n\} \\ &= 2\pi i \sum_{k\in S, |k|\leqslant N} \frac{1}{\pi} f(k) + 2\pi i \sum_{z_j\in R} \operatorname{Res}(f(z)\cot(\pi z), z_j),\end{aligned}$$

Let $n \to \infty$. Then by LEMMA 5.4

$$2\pi i \sum_{k\in S} \frac{1}{\pi} f(k) + 2\pi i \sum_{z_j\in R} \operatorname{Res}(f(z)\cot(\pi z), z_j) = 0.$$

We conclude,

$$\sum_{k\in S} f(k) = -\pi \sum_{z_j\in R} \operatorname{Res}(f(z)\cot(\pi z), z_j).$$

The proof is complete. □

EXAMPLE 5.17. *Euler's famous sum:*

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof. Let $f(z) = 1/z^2$, then by DEFINITION 3.8 the function $f(z)\cot(\pi z)$ has a pole at $z = 0$ of order 3. By THEOREM 3.19 and L'Hôpitol's rule we have

$$\begin{aligned}\operatorname{Res}\left(\frac{1}{z^2}\cot(\pi z), 0\right) &= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left[\frac{z^3\cot(\pi z)}{z^2}\right] \\ &= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[-\pi z\csc^2(\pi z) + \cot(\pi z)\right] \\ &= \frac{1}{2}\lim_{z\to 0}\left[2\pi^2 z\cot(\pi z)\csc^2(\pi z) - 2\pi\csc^2(\pi z)\right] \\ &= \lim_{z\to 0}\left[\frac{\pi^2 z\cos(\pi z)}{\sin^3(\pi z)} - \frac{\pi}{\sin^2(\pi z)}\right]\end{aligned}$$

$$
\begin{aligned}
&= \lim_{z\to 0}\left[\frac{\pi^2 z\cos(\pi z)-\pi\sin(\pi z)}{\sin^3(\pi z)}\right] \\
&= \lim_{z\to 0}\left[\frac{-\pi^3 z\sin(\pi z)+\pi^2\cos(\pi z)-\pi^2\cos(\pi z)}{3\pi\sin^2(\pi z)\cos(\pi z)}\right] \\
&= \lim_{z\to 0}\left[\frac{-\pi^2 z}{3\sin(\pi z)\cos(\pi z)}\right] \\
&= -\frac{\pi}{3}.
\end{aligned}
$$

Now, taking $S=\mathbb{Z}\setminus\{0\}$ in THEOREM 5.16, we have,

$$\sum_{k\neq 0}\frac{1}{k^2} \;=\; -\pi\mathrm{Res}\left(\frac{1}{z^2}\cot(\pi z),0\right)=-\pi\left(\frac{-\pi}{3}\right)=\frac{\pi^2}{3}.$$

Since $1/z^2$ is an even function, we see

$$2\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{k\neq 0}\frac{1}{k^2}=\frac{\pi^2}{3}.$$

Therefore,

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

The proof is complete. $\square$

EXAMPLE 5.18. *If ia is not an integer, then*

$$\sum_{k=1}^{\infty}\frac{1}{k^2(k^2+a^2)}=\frac{3+a^2\pi^2-3\pi a\coth(\pi a)}{6a^4}.$$

Proof. Let

$$f(z) = \frac{1}{z^2(z^2+a^2)}.$$

By DEFINITION 3.8 the function $f(z)\cot(\pi z)$ has a pole of order 3 at $z_1 = 0$ and simple poles at each of $z_2 = ia$ and $z_3 = -ia$. Note that z_1 is an integer, whereas z_2 and z_3 are not. Thus by THEOREM 3.22, we have

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^2(z-ia)(z+ia)}, ai\right) = \frac{-i\coth(a\pi)}{(-a^2)(2ia)}\operatorname{Res}\left(\frac{1}{z-ia}, ai\right) = \frac{\coth(a\pi)}{2a^3}.$$

In the same way we found the previous residue,

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^2(z-ia)(z+ia)}, -ai\right) = \frac{\coth(a\pi)}{2a^3}.$$

In finding the residue of the function $f(z)\cot(\pi z)$ at the pole $z_1 = 0$, we will make use of the Bernoulli form of the Taylor series for $z\cot z$, RESULT 4.7, obtaining

$$\pi z\cot(\pi z) = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}(\pi z)^{2n}.$$

From this we obtain the Laurent series for $\cot(\pi z)$:

$$\begin{aligned}\cot(\pi z) &= \sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k} B_{2k}\pi^{2k-1}}{(2k)!}z^{2k-1}\\ &= \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \dots\end{aligned}$$

Moreover, as a geometric series,

$$\frac{1}{z^2(z^2+a^2)} = \frac{1}{a^2z^2}\,\frac{1}{1+(z/a)^2} = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k-2}}{a^{2k+2}} = \frac{1}{a^2z^2} - \frac{1}{a^4} + \frac{z^2}{a^6} + \dots.$$

It follows that,

$$\begin{aligned}
\frac{\cot(\pi z)}{z^2(z^2+a^2)} &= \left(\frac{1}{a^2z^2}-\frac{1}{a^4}+\frac{z^2}{a^6}+\dots\right)\times\left(\frac{1}{\pi z}-\frac{\pi z}{3}-\frac{\pi^3z^3}{45}-\dots\right)\\
&= \left(\frac{1}{a^2\pi z^3}-\frac{\pi}{3a^2z}+\frac{\pi^3 z}{45a^2}+\dots\right)\\
&\quad+\left(\frac{-1}{a^4\pi z}+\frac{\pi z}{3a^4}-\frac{\pi^3z^3}{45a^4}-\dots\right)\\
&\quad+\left(\frac{z}{\pi a^6}-\frac{\pi z^3}{3a^6}+\frac{\pi^3z^5}{45a^6}+\dots\right)\\
&\quad+\dots\\
&= \frac{z^{-3}}{a^2\pi}-\frac{3+a^2\pi^2}{3a^4\pi}z^{-1}+\frac{-\pi^4a^4+15\pi^2a^2+45}{45\pi a^6}z+\dots
\end{aligned}$$

Therefore, by definition,

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^2(z^2+a^2)},0\right)=-\frac{3+a^2\pi^2}{3a^4\pi}.$$

Taking $S=\mathbb{Z}\setminus\{0\}$ in Theorem 5.16, we have,

$$\begin{aligned}
\sum_{k\neq 0}\frac{1}{k^2(k^2+a^2)} &= -\pi\sum_{j=1}^{3}\operatorname{Res}\left(\frac{1}{z^2(z^2+a^2)}\cot(\pi z),z_j\right)\\
&= -\pi\left(\frac{\coth(a\pi)}{2a^3}+\frac{\coth(a\pi)}{2a^3}-\frac{3+a^2\pi^2}{3a^4\pi}\right)\\
&= \frac{3+a^2\pi^2-3\pi a\coth(\pi a)}{3a^4}.
\end{aligned}$$

Since $f(z)$ is an even function,

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k^2+a^2)} = \frac{1}{2} \sum_{k \neq 0} \frac{1}{k^2(k^2+a^2)} = \frac{3 + a^2\pi^2 - 3\pi a \coth(\pi a)}{6a^4}.$$

The proof is complete. □

THEOREM 5.19. *Suppose that $f(z) = \frac{p(z)}{q(z)}$ is a rational function, with poles $\{z_1, z_2 \ldots, z_n\}$, some of which may be integers, and let $S = \mathbb{Z} \setminus \{z_1, z_2 \ldots, z_n\}$. Then,*

$$\sum_{k \in S} (-1)^k f(k) = -\pi \sum_{j=1}^{n} \operatorname{Res}\left(f(z)\csc(\pi z), z_j\right).$$

Proof. Since $\csc(\pi z)$ and $\cot(\pi z)$ have the same denominator and the theorem has the same hypotheses otherwise, the proof for this theorem is similar to the the previous one and will be omitted. □

EXAMPLE 5.20. *If a is an integer, and $a \neq 0$. Then*

$$\sum_{k \in \mathbb{Z} \setminus \{0,a\}} \frac{(-1)^k}{k^2(k-a)} = \frac{6 + a^2\pi^2 - 12(-1)^{a+1}}{6a^3}.$$

Proof. Let

$$f(z) = \frac{1}{z^2(z-a)},$$

then by DEFINITION 3.8 the function $f(z)\csc(\pi z)$ has a pole of order 3 at $z_1 = 0$ and a pole of order 2 at $z_2 = a$. Note that z_1 and z_2 are integers.

To find the residues of the function $f(z)\csc(\pi z)$ at the pole $z_2 = a$, by Theorem 3.19 we have,

$$\operatorname{Res}\left(\frac{\csc(\pi z)}{z^2(z-a)}, a\right) = \lim_{z \to a} \frac{d}{dz}\left[\frac{(z-a)^2\csc(\pi z)}{z^2(z-a)}\right] = \frac{2(-1)^{a+1}}{a^3\pi}.$$

Now, to find the residue of the function $f(z)\csc(\pi z)$ at the pole $z_1 = 0$, we will use the sum identity for the cosecant 4.9

$$\begin{aligned}\csc(\pi z) &= \sum_{k=0}^{\infty}(-1)^{k-1}\frac{(2^{2k}-2)B_{2k}\pi^{2k-1}}{(2k)!}z^{2k-1}\\ &= \frac{1}{\pi z}+\frac{\pi z}{6}-\frac{7\pi^3 z^3}{360}-\dots.\end{aligned}$$

Also, we need the Taylor expansion for $z^{-2}(z-a)^{-1}$,

$$\frac{1}{z^2}\frac{1}{z-a}=\frac{1}{az^2}\frac{-1}{1-\left(\frac{z}{a}\right)}=-\sum_{k=0}^{\infty}\frac{z^{k-2}}{a^{k+1}}.$$

Hence,
$$-\sum_{k=0}^{\infty}\frac{z^{k-2}}{a^{k+1}}=-\frac{1}{az^2}-\frac{1}{a^2 z}-\frac{1}{a^3}+\dots.$$

Now, we will find the product of these two summations,

$$\begin{aligned}\csc(\pi z)\left(\frac{1}{z^2}\frac{1}{z-a}\right) &= \left(\frac{1}{\pi z}+\frac{\pi z}{6}-\frac{7\pi^3 z^3}{360}-\dots\right)\times\left(-\frac{1}{az^2}-\frac{1}{a^2 z}-\frac{1}{a^3}+\dots\right)\\ &= \left(-\frac{1}{a\pi z^3}-\frac{1}{a^2\pi z^2}-\frac{1}{a^3\pi z}+\dots\right)\\ &\quad+\left(-\frac{\pi}{6az}-\frac{\pi}{6a^2}-\frac{\pi z}{6a^3}-\dots\right)\\ &\quad+\left(-\frac{7\pi^3 z}{360a}-\frac{7\pi^3 z^2}{360a^2}-\frac{7\pi^3 z^3}{360a^3}-\dots\right)\\ &\quad+\dots.\end{aligned}$$

Hence, we see that the coefficient of $\frac{1}{z}$ is $\left(-\frac{6+a^2\pi^2}{6a^3\pi}\right)$, wchich is the residue of the function $f(z)\csc(\pi z)$ at the pole $z_1 = 0$. Hence,

$$\operatorname{Res}\left(\frac{1}{z^2(z-a)}\csc(\pi z), 0\right) = -\frac{6+a^2\pi^2}{6a^3\pi}.$$

Now, let $S = \mathbb{Z} \setminus \{0, a\}$. By THEOREM 5.19 we have,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}\setminus\{0,a\}} \frac{(-1)^k}{k^2(k-a)} &= -\pi \sum_{j=1,2} \operatorname{Res}\left(\frac{1}{z^2(z-a)}\csc(\pi z), z_j\right) \\
&= -\pi\left(\frac{2(-1)^{a+1}}{a^3\pi} - \frac{6+a^2\pi^2}{6a^3\pi}\right) \\
&= \frac{6+a^2\pi^2-12(-1)^{a+1}}{6a^3}.
\end{aligned}$$

The proof is complete. $\square$

5.3.3 Singularities at Zero

THEOREM 5.21. [2] *If n is a positive integer, and $\{B_k\}$ are the Bernoulli numbers, then*

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1}\frac{2^{2n-1}B_{2n}\pi^{2n}}{(2n)!}.$$

Proof. Let $f(z) = 1/z^{2n}$, then by DEFINITION 3.8 the function $f(z)\cot(\pi z)$ has a pole of order $2n+1$ at the singularity $z = 0$. By RESULT 4.7, we have the Laurent series,

$$\frac{\cot(\pi z)}{z^{2n}} = \frac{1}{\pi z^{2n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}\pi^{2k}}{(2k)!} z^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}\pi^{2k-1}}{(2k)!} z^{2k-2n-1}.$$

When $2k-2n-1=-1,\ k=n$, hence

$$\text{Res}\left(\frac{\cot(\pi z)}{z^{2n}},0\right)=\frac{(-1)^n2^{2n}B_{2n}\pi^{2n-1}}{(2n)!}.$$

Considering $S=\mathbb{Z}\backslash\{0\}$ in THEOREM 5.16, we have

$$\sum_{k\neq 0}\frac{1}{k^{2n}}=-\pi\text{Res}\left(\frac{\cot(\pi z)}{z^{2n}},0\right)=\frac{(-1)^{n-1}2^{2n}B_{2n}\pi^{2n}}{(2n)!}.$$

Since $f(k)=1/k^{2n}$ is an even function,

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}}=\frac{1}{2}\sum_{k\ \neq 0}\frac{1}{k^{2n}}=\frac{(-1)^{n-1}2^{2n-1}B_{2n}\pi^{2n}}{(2n)!},$$

completing the proof. □

EXAMPLE 5.22.

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{2\pi^2B_2}{2!}=\frac{\pi^2}{6}\qquad \textit{and}\qquad \sum_{k=1}^{\infty}\frac{1}{k^4}=-\frac{2^3\pi^4B_4}{4!}=\frac{\pi^4}{90}.$$

THEOREM 5.23. [2] *If n is a positive integer, and $\{B_k\}$ are the Bernoulli numbers, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2n}}=(-1)^n\frac{(2^{2n-1}-1)B_{2n}\pi^{2n}}{(2n)!}.$$

Proof. Let $f(z)=1/z^{2n}$, then by DEFINITION 3.8 the function $f(z)\csc(\pi z)$ has a pole of order $2n+1$ at the singularity $z=0$. By RESULT 4.9, we have the Laurent series,

$$\csc(\pi z) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2^{2k}-2)B_{2k}}{(2k)!} \pi^{2k-1} z^{2k-1}.$$

It follows that

$$\frac{\csc(\pi z)}{z^{2n}} = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2^{2k}-2)B_{2k}}{(2k)!} \pi^{2k-1} z^{2k-2n-1}.$$

We obtain the residue from the previous series, for when $2k - 2n - 1 = -1$, we see $n = k$. Hence,

$$\operatorname{Res}\left(\frac{\cot(\pi z)}{z^{2n}}, 0\right) = (-1)^{n-1} \frac{(2^{2n}-2)B_{2n}}{(2n)!} \pi^{2n-1}.$$

Considering $S = \mathbb{Z}\backslash\{0\}$ in THEOREM 5.16, we have

$$\sum_{k \neq 0} \frac{1}{k^{2n}} = -\pi \operatorname{Res}\left(\frac{\csc(\pi z)}{z^{2n}}, 0\right) = (-1)^{n} \frac{(2^{2n}-2)B_{2n}}{(2n)!} \pi^{2n}.$$

But, since $f(k) = \frac{(-1)^k}{k^{2n}}$ is an even function,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{2n}} = \frac{1}{2} \sum_{k \neq 0} \frac{(-1)^k}{k^{2n}} = (-1)^n \frac{2(2^{2n-1}-1)B_{2n}\pi^{2n}}{(2n)!}.$$

The proof is complete. $\square$

Example 5.24.

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}=\frac{\pi^2}{12}.$$

Proof. Taking $n=1$ in Theorem 5.23, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}=-\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^2}=\frac{B_2\pi^2}{2}.$$

Since $B_2=1/6$, the proof is complete. $\square$

CHAPTER 6

MITTAG-LEFFLER EXPANSION THEOREM

The Mittag-Leffner theorem seems unique in its concept, finding an infinite sum form of functions in terms of its singularities and the corresponding residues.

THEOREM 6.1 (Mittag-Leffler Expansion Theorem). [5] *Let $f(z)$ be analytic except at distinct simple poles $\{z_j\}_1^\infty$, for which $0 < |z_j| \leqslant |z_{j+1}|$ for all j. Denote $R_j = \operatorname{Res}(f, z_j)$ and let $\{C_n\}_1^\infty$ be circles of radius r_n, centered at 0, none of which pass through any z_j and such that $r_n \to \infty$. Moreover, assume there exists $B > 0$ such that when $z \in C_n$ for any n, $|f(z)| < B$. Then*

$$f(z) = f(0) + \sum_{j=1}^{\infty} R_j \left(\frac{1}{z - z_j} + \frac{1}{z_j} \right).$$

Proof. Let z_0 be any complex number except a pole of f. Define

$$F(z) = \frac{f(z)}{z - z_0},$$

then F has a simple pole at z_0, as well as at each z_j. By THEOREM 3.22, for all $j \in \mathbb{N}$,

$$\operatorname{Res}(F, z_j) = \frac{R_j}{z_j - z_0} \qquad \text{and} \qquad \operatorname{Res}(F, z_0) = f(z_0).$$

By the Cauchy Residue theorem 3.14, for any n,

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(z)}{z - z_0} dz = f(z_0) + \sum_{z_j \in C_n} \frac{R_n}{z_n - z_0}. \tag{6.1}$$

Letting $z_0 = 0$, in (6.1) yields

$$\frac{1}{2\pi i}\int_{C_n}\frac{f(z)}{z}dz = f(0) + \sum_{z_j\in C_n}\frac{R_n}{z_n}. \tag{6.2}$$

Subtracting (6.2) from (6.1), we obtain

$$\begin{aligned}\frac{z_0}{2\pi i}\int_{C_n}\frac{f(z)}{z(z-z_0)}dz &= \frac{1}{2\pi i}\int_{C_n} f(z)\left(\frac{1}{z-z_0}-\frac{1}{z}\right)dz \\ &= f(z_0) - f(0) + \sum_{z_j\in C_n} R_n\left(\frac{1}{z_n-z_0}-\frac{1}{z_n}\right). \qquad (6.3)\end{aligned}$$

Since $|z-z_0| \geqslant |z| - |z_0| = r_n - |z_0|$, for all z on C_n, we have

$$\left|\int_{C_n}\frac{f(z)}{z(z-z_0)}dz\right| \leqslant \frac{2\pi r_n B}{r_n(r_n-|z_0|)} = \frac{2\pi B}{r_n-|z_0|},$$

which shows that

$$\lim_{n\to\infty}\int_{C_n}\frac{f(z)}{z(z-z_0)}dz = 0,$$

as $n\to\infty$, and therefore as $r_n\to\infty$. It then follows from line (6.3) that

$$\begin{aligned}f(z_0) &= f(0) - \lim_{n\to\infty}\sum_{z_j\in C_n} R_j\left(\frac{1}{z_j-z_0}-\frac{1}{z_j}\right) \\ &= f(0) + \sum_{j=1}^{\infty} R_j\left(\frac{1}{z_0-z_j}+\frac{1}{z_j}\right).\end{aligned}$$

The proof is complete. $\square$

EXAMPLE 6.2. *If $z \neq k\pi$ for any $k\in\mathbb{Z}$, then*

$$\cot z = \frac{1}{z} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right).$$

Proof. Let $f(z) = \cot z - 1/z$. By THEOREM 5.5 we have that $\cot z$ has simple poles at $z = k\pi$, when k is an integer and that the residues at these poles are 1. It follows that the Laurent series is

$$\cot z = \sum_{k=-1}^{\infty} a_k z^k, \text{ where } a_{-1} = 1.$$

Therefore,

$$\cot z - \frac{1}{z} = \sum_{k=0}^{\infty} a_k z^k,$$

hence $z = 0$ is a removable singularity. By L'Hospital's rule we have

$$\lim_{z \to 0} \left(\cot z - \frac{1}{z} \right) = 0,$$

so, without loss of generality, $f(0) = 0$. Moreover, by LEMMA 5.3 we have $\cot z$ is bounded on basic contours C_n. Hence by Mittag-Leffler Expansion Theorem we have,

$$\cot z = \frac{1}{z} + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{z - k\pi} + \frac{1}{k\pi} \right).$$

The proof is complete. $\square$

LEMMA 6.3. *If* $z_j = \frac{\pi}{2}(2j+1)$ *for all* $j \in \mathbb{Z}$, *then*

i. $z_{-j-1} = -z_j$.

ii. $\dfrac{1}{z-z_j}+\dfrac{1}{z-z_{-j-1}}=\dfrac{2z}{z^2-z_j^2}$.

Proof. Given $z_j=\frac{\pi}{2}(2j+1)$,

i. $z_{-j-1}=\dfrac{\pi}{2}[2(-j-1)+1]=\dfrac{\pi}{2}(-2j-2+1)=-\dfrac{\pi}{2}(2j+1)=-z_j$.

Then from (i), we see

ii. $\dfrac{1}{z-z_j}+\dfrac{1}{z-z_{-j-1}}=\dfrac{1}{z-z_j}+\dfrac{1}{z+z_j}=\dfrac{z+z_j+z-z_j}{(z-z_j)(z+z_j)}=\dfrac{2z}{z^2-z_j^2}$.

The proof is complete. $\square$

EXAMPLE 6.4. *For all $z\neq\pi k$ for some $k\in\mathbb{Z}$,*

$$\tan z=\sum_{k=0}^{\infty}\frac{2z}{\left((2k+1)\frac{\pi}{2}\right)^2-z^2}.$$

Proof. Let $\{C_n\}_1^\infty$ be circles of radius πn, centered at 0. Using the methods of LEMMA 5.3, it can be shown that there exists $B>0$ such that $|tan(z)|<B$, when $z\in C_n$ for any n. Denote the singularities of tan as

$$\omega_j=(2j+1)\frac{\pi}{2}\ \text{ for all }\ j\in\mathbb{Z},$$

noting tan has simple poles with residues of 1 at each ω_j and none of them are on any C_n. We renumber these singularities in such a way to satisfy the remainding hypothesis of the Mittag-Leffler Expansion Theorem 6.1. Denote

$$z_k = \begin{cases} \omega_{k/2}, & \text{if } k \text{ is even} \\ \omega_{(1-k)/2}, & \text{if k is odd} \end{cases}$$

Therefore,

$$\begin{aligned}
\tan z &= \sum_{k=1}^{\infty} \left(\frac{1}{z - z_k} + \frac{1}{z_k} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{z - z_{2k}} + \frac{1}{z_{2k}} + \frac{1}{z - z_{2k-1}} + \frac{1}{z_{2k-1}} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{z - \omega_k} + \frac{1}{\omega_k} + \frac{1}{z - \omega_{1-k}} + \frac{1}{\omega_{1-k}} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{z - \omega_k} + \frac{1}{\omega_k} \right) + \sum_{k=1}^{\infty} \left(\frac{1}{z - \omega_{1-k}} + \frac{1}{\omega_{1-k}} \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{z - \omega_k} + \frac{1}{\omega_k} \right) + \sum_{k=0}^{\infty} \left(\frac{1}{z - \omega_{-1-k}} + \frac{1}{\omega_{-1-k}} \right)
\end{aligned}$$

By part (i) of LEMMA 6.3, we have $\omega_{-1-k} = -\omega_k$, hence

$$\tan z = \sum_{k=0}^{\infty} \left(\frac{1}{z - \omega_k} + \frac{1}{z - \omega_{-1-k}} \right).$$

Then by the (ii) part of LEMMA 6.3, we have

$$\tan z = \sum_{k=0}^{\infty} \frac{2z}{\left((2k+1)\frac{\pi}{2}\right)^2 - z^2}.$$

The proof is complete. $\square$

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