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On the Existence of Bogdanov-Takens Bifurcations

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ON THE EXISTENCE OF BOGDANOV – TAKENS BIFURCATIONS

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In Partial Fulfillment

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Master of Science, Mathematics

By

Zachary Deskin

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ABSTRACT

In bifurcation theory, there is a theorem (called Sotomayor's Theorem) which proves the existence of one of three possible bifurcations of a given system, provided that certain conditions of the system are satisfied. It turns out that there is a "similar" theorem for proving the existence of what is referred to as a Bogdanov-Takens bifurcation. The author is only aware of one reference that has the proof of this theorem. However, most of the details were left out of the proof. The contribution of this thesis is to provide the details of the proof on the existence of Bogdanov-Takens bifurcations.

KEYWORDS: Bogdanov-Takens bifurcation, bifurcation, dynamical systems, Hopf bifurcation, Saddle-Node bifurcation

This abstract is approved as to form and content

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Dr. Jorge Rebaza
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INTRODUCTION

In bifurcation theory, there is a theorem (called Sotomayor's theorem) which proves the existence of one of three types of bifurcations based on the conditions that the system satisfies. The bifurcations that Sotomayor's theorem proves the existence of (Saddle-Node, Transcritical, or Pitchfork bifurcations) are what are called codimension-1 bifurcations. It turns out that there is a "similar" result for proving the existence of one type of codimension-2 bifurcation (called the Bogdanov-Takens bifurcation) based on the conditions that the system satisfies.

In this thesis, we first provide some basic results from calculus, and linear algebra that will be used in this thesis. We also provide an introduction to dynamical systems, where we define equilibrium points of a system, periodic orbits of a system, and their stabilities. Furthermore, we introduce a few theorems (Stable Manifold, Hartman-Grobman, etc.) which can determine the behavior of the system near an equilibrium point/periodic orbit. Once we have a basic idea of what is going on with dynamical systems, we provide an introduction to bifurcation theory. We will look at some more common bifurcations, and see how the behavior of the system changes as the parameters of the system changes. We also define the codimension of a bifurcation, and provide a couple of examples to prove what the codimension of a bifurcation is. We also discuss the Center Manifold Theorem, and its importance for this thesis. We then discuss the Bogdanov-Takens bifurcation, and provide the statement of a theorem (and the details of its proof) which proves the existence of the Bogdanov-Takens bifurcation (under certain conditions of the system). We also provide an example of a system which undergoes a Bogdanov-Takens bifurcation.

The main goal of this thesis is to provide the details of the proof of the existence of a Bogdanov-Takens bifurcation, provided that certain conditions of the system satisfies. To the author's knowledge, we could only find one place where the proof of this theorem could be found (in [2]). However, there are little details provided in the proof from [2]. For this reason, my main contribution is to provide the details of the proof of this theorem.

CHAPTER 1: PRELIMINARIES

In this chapter, we present some topics from calculus and functional analysis that will be used in this thesis.

1.1. Taylor Series Expansion in 2-Dimensions

We introduce Taylor series here since we will need this idea later. We will also provide an example to demonstrate how to compute the Taylor series.

THEOREM 1. (Taylor Series Expansion) Suppose a function $f : \mathbb{R}^2 \to \mathbb{R}$ and its partial derivatives through $n+1$ are continuous throughout an open region R centered at a point (a,b). Then, throughout R,

$f(a+h,b+k) = f(a,b) + (hf_x + kf_y)\mid_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\mid_{(a,b)} + \ldots + \frac{1}{n!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^n f\mid_{(a,b)} + r(a+h,b+k)$, where r is the error term of order $n+1$. □

This theorem basically says that in the region R, f can be approximated by its Taylor polynomial. We provide an example to demonstrate how we apply Theorem 1.

EXAMPLE 2. Suppose we want to approximate the function $f(x,y) = xe^y$ near the origin. Then we have $f_x(x,y) = e^y$, $f_y(x,y) = xe^y$, $f_{xx}(x,y) = 0$, $f_{xy}(x,y) = e^y$, and $f_{yy}(x,y) = xe^y$. Also, we have $f(0,0) = 0e^0 = 0$, $f_x(0,0) = e^0 = 1$, $f_y(0,0) = 0e^0 = 0$, $f_{xx}(0,0) = 0$, $f_{xy}(0,0) = e^0 = 1$, and $f_{yy}(0,0) = 0e^0 = 0$. Hence, we have

$$f(x,y) \approx f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}(x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0))$$

$$f(x,y) \approx 0 + x(1) + y(0) + \frac{1}{2!}(x^2(0) + 2xy(1) + y^2(0))$$

$$f(x,y) \approx x + xy.$$

Hence, $f(x,y) \approx x + xy$. That is, $f(x,y)$ can be approximated by its Taylor polynomial $P(x,y) = x + xy$ of degree two, around $(0,0)$. $\triangledown$

We say that a function $f: E \to \mathbb{R}^n$ is ***analytic*** in the open set $E \subset \mathbb{R}^n$ if f has a Taylor series which converges to f in some neighborhood of a point x_0 in E.

1.2 The Fredholm Alternative Theorem

In this section, we introduce the Fredholm Alternative Theorem. This theorem basically says that either a vector is a solution to a linear system, or it is not. In which case, it gives a characterization that will be useful when we study the proof of Theorem 41, which will be discussed later.

THEOREM 3. [2], [4] (The Fredholm Alternative Theorem) Let A be a $m \times n$ matrix, and let $b \in \mathbb{R}^m$. Then either $x \in \mathbb{R}^n$ is a solution of the linear system $Ax = b$ or $A^T y = 0$ has a solution with $y \in \mathbb{R}^m$ satisfying $b^T y \neq 0$. $\square$

1.3 The Implicit Function Theorem

Here, we discuss the Implicit Function Theorem, which we will also use in the proof of Theorem 41, which will be discussed later.

THEOREM 4. [2], [4] (Implicit Function Theorem) Let $F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function defined in a neighborhood of $(x,y) = (0,0)$ such that $F(0,0) = 0$, and let $A = DF(0,0)$ be the Jacobian of the function F evaluated at $(0,0)$. If A is nonsingular, then there exists a smooth locally defined function $f: \mathbb{R}^n \to \mathbb{R}^m$ such that $F(x, f(x)) = 0$ for all x in some neighborhood of the origin of $\mathbb{R}^n$. $\square$

This theorem basically suggests that under certain conditions that if the origin is a solution to the equation $F(x,y) = 0$, then there are more possible solutions to the equation $F(x, f(x)) = 0$ for all x in some neighborhood of the origin in $\mathbb{R}^n$.

1.4 Generalized Eigenvectors

Here we introduce the idea of a generalized eigenvector v of a matrix A corresponding to an eigenvalue λ. We say that v is a ***generalized eigenvector*** of a $n \times n$ matrix A corresponding to an eigenvalue λ with algebraic multiplicity $m \leq n$ if

$$(A - \lambda I)^k v = 0,$$

for some $k = 1, 2, ..., m$. So we have that an eigenvector is a generalized eigenvector (where $k = 1$).

It turns out that there is an equivalent way to write the generalized eigenvectors of a matrix corresponding to an eigenvalue. To see this, let $v_1, v_2, ..., v_m$ be the generalized eigenvectors of a $n \times n$ matrix A corresponding to the eigenvalue λ. Then we have

$$(A - \lambda I)^m v_m = 0$$

$$(A - \lambda I)^{m-1}(A - \lambda I)v_m = 0.$$

Since v_{m-1} is a (generalized) eigenvector, and from the above system, we can define $(A - \lambda I)v_m = v_{m-1}$. Using a similar argument, we can write one of the generalized eigenvectors in terms of another. It turns out that we can find linearly independent generalized eigenvectors of a matrix A corresponding to an eigenvalue λ. This fact will be stated in Chapter 2.

CHAPTER 2: INTRODUCTION TO DYNAMICAL SYSTEMS

The main idea of dynamical systems is to study the qualitative behavior of solutions of a system of ordinary differential equations. In other words, the area of dynamical systems tries to answer the question: as the independent variable of the system (usually time t) increases, and parameters vary, what does the solution to a given system of ordinary differential equations tend to do? In particular, we look to see if the system is stable (solutions stay near a certain region of the system) or unstable (solutions go away from certain regions of the system). We also look for special solutions, attractors, etc. With this in mind, we can study the stability of a system locally or globally. To study local stability is to study what the system tends to do in a neighborhood of an equilibrium point, or around periodic orbits. In this thesis, we will be considering systems of the form

$$\dot{x} = f(x, \mu) \tag{0.1}$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$. So, we will be working with systems that depends not only on x, but also on a vector of parameters $\mu = [\mu_1, \mu_2, ..., \mu_m]^T$. Since μ is allowed to change, the stability of the vector field f can also change. This will be discussed in further detail in Chapter 4.

2.1 The Basic Background

Before we start looking at bifurcations of system (0.1) (which is the main subject studied here), we first need to define some terms that will be used throughout this thesis. We also need to discuss the local and global stability of systems of the form

$$\dot{x} = f(x) \tag{0.2}$$

where $f: \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Once we are familiar with these ideas, we will introduce some bifurcations and state Sotomayor's Theorem, which helps us prove the existence of certain bifurcations.

We begin by introducing some terminology we will be using throughout this thesis. We define an ***equilibrium point of*** (0.2) to be a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$. In other words, an equilibrium point of is a point where there is no change in the system (0.2). Now, before we go further, we need to define linearization of (0.2) about an equilibrium point. We define the ***linearization of*** (0.2) ***about an equilibrium point*** x_0 as the linear system

$$\dot{x} = Ax,$$

where $A = Df(x_0)$ is the Jacobian of the system (0.2) evaluated at x_0. Furthermore, we say an equilibrium point x_0 is ***hyperbolic*** if the real parts of all the eigenvalues of the Jacobian of (0.2) evaluated at x_0 are nonzero (i.e., if $\lambda_j = a_j \pm ib_j,\ j = 1,2,...,n$ are the eigenvalues of $Df(x_0)$, then for x_0 to be a hyperbolic equilibrium point of (0.2), we must have that $a_j \neq 0$ for all $j = 1,2,...,n$). If an equilibrium point x_0 of (0.2) is not hyperbolic, we say x_0 is ***nonhyperbolic*** (i.e., if $\lambda_j = a_j \pm ib_j$, where $j = 1,2,...,n$, are the eigenvalues of $Df(x_0)$, then for x_0 to be a nonhyperbolic equilibrium point of (0.2), we must have $a_j = 0$ for some $j = 1,2,...,n$).

Now, before we look at the behavior of the system (0.2), we first need to define some behavior we get in a linear system

$$\dot{x} = Ax \tag{0.3}$$

around the origin, where A is a nonsingular 2×2 matrix.[1] Let $\lambda_{1,2} = a \pm bi$ be the eigenvalues of A. We say that the origin is a ***saddle*** if λ_1 and λ_2 are real numbers such that either

[1]While we are considering the 2×2 case, it turns out we can generalize the matrix A to be $n \times n$ (where $n \geq 3$).

$\lambda_1 < 0 < \lambda_2$ or $\lambda_2 < 0 < \lambda_1$. We say the origin is a ***node*** if λ_1 and λ_2 are real numbers such that either $\lambda_1 < \lambda_2 < 0$ or $0 < \lambda_1 < \lambda_2$. Note that the order of λ_1 and λ_2 does not matter. Furthermore, if we see that a node has $\lambda_1 < \lambda_2 < 0$, we say that the origin is a ***(asymptotically) stable node***. Also, if we see that a node has $0 < \lambda_1 < \lambda_2$, we say that the origin is a ***unstable node***. The origin is a ***focus*** if λ_1 and λ_2 are both complex numbers such that either $a > 0$ or $a < 0$. If we see that a focus has $a < 0$, we say the origin is a ***(asymptotically) stable focus***. Also, if we see that a focus has $a > 0$, we say the origin is an ***unstable focus***. The origin is a ***center*** if λ_1 and λ_2 are both pure imaginary numbers (i.e., $a = 0$).

There is a result to study the behavior of a linear system based on the trace and determinant of a matrix (which is referred to as the Trace-Determinant Analysis). This analysis is quite useful in determining the stability of the origin of a linear system.

THEOREM 5. [1] (Trace-Determinant Analysis) Let A be a 2×2 matrix with trace T and determinant $D \neq 0$. Consider the system $\dot{x} = Ax$.

(1) If $D < 0$, then the origin is a saddle.

(2) If $D > 0$ and $T^2 - 4D \geq 0$, then the origin is a node (stable if $T < 0$ and unstable if $T > 0$).

(3) If $D > 0$ and $T^2 - 4D < 0$, then the origin is a focus (stable if $T < 0$ and unstable if $T > 0$).

(4) If $D > 0$, $T^2 - 4D < 0$, and $T = 0$, then the origin is a center. □

REMARK 6. The Trace-Determinant Analysis only works in two dimensions. △

The types of behavior we get around the origin that was mentioned above are what we see when we have a linear system. There are other types we can have in a nonlinear system around an equilibrium point (which may or may not be the origin). To define these other behavior types, we need to define a few terms first.[2] Assume the origin is a

[2]See [1] for details.

nonhyperbolic equilibrium point of a system that can be written in the form

$$\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases} \tag{0.4}$$

where P and Q are analytic in some neighborhood of the origin. The solution curves of (0.4) which approach the origin along tangent lines which divide a neighborhood of the origin into a finite number of open regions are called ***sectors***. A sector which is topologically equivalent to the left side of F1[3] below is called a ***hyperbolic sector***. Furthermore, the trajectories that lie on the boundary of a hyperbolic sector are called ***separatrices***. A sector that is topologically equivalent to the right side of F1 below is called a ***parabolic sector***. A sector which is topologically equivalent to F2[4] below is called an ***elliptic sector***.

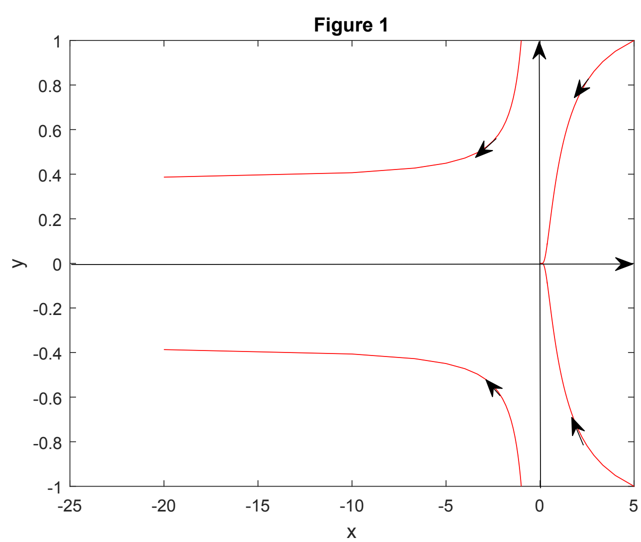


Figure 1. Phase portrait to describe hyperbolic and parabolic sectors.

[3]This figure came from Example 25 in Chapter 4.
[4]This figure came from an example from [1], in section 2.11.

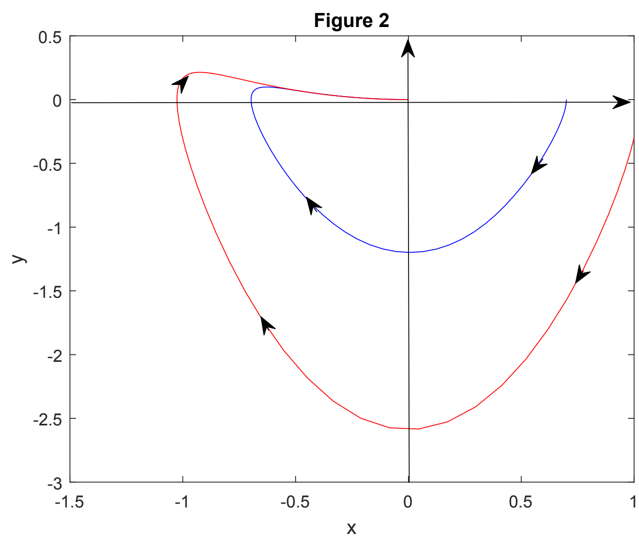


Figure 2. Phase portrait to describe an elliptic sector.

Now that we have introduced the idea of a sector and the different types of them, we are ready to define the new types of nonhyperbolic equilibrium point. A ***saddle-node*** is a type of nonhyperbolic equilibrium point of a system where there are two hyperbolic sectors and one parabolic sector. A ***cusp*** is a type of nonhyperbolic equilibrium point of a system where there are two hyperbolic sectors. A ***critical point with an elliptic domain*** is a type of nonhyperbolic equilibrium point where there are one hyperbolic, one elliptic, and two hyperbolic sectors.

Now, we give another definition. Let E be an open subset of $\mathbb{R}^n$. Let $\phi(t,x_0)$ be the solution of the differential equation

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \tag{0.5}$$

We define the ***flow of the differential equation*** (0.5) as a mapping $\phi_t : E \to E$ defined by $\phi_t(x_0) = \phi(t,x_0)$.

In Chapter 1, we discussed the idea of a generalized eigenvector. It turns out that we have two theorems that tells us that we can find a collection of generalized eigenvectors that forms a basis for $\mathbb{R}^n$ (or $\mathbb{R}^{2n}$ if the eigenvalues are complex).

THEOREM 7. [1] Let A be a real $n \times n$ matrix with real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ repeated according to their multiplicities. Then there is a basis of generalized eigenvectors for $\mathbb{R}^n$. $\square$

The next theorem is for the case where the eigenvalues are complex.

THEOREM 8. [1] Let A be a real $2n \times 2n$ matrix with complex eigenvalues $\lambda_j = a_j \pm ib_j$, for $j = 1, 2, ..., n$. Then there is a basis of generalized eigenvectors for $\mathbb{R}^{2n}$. $\square$

REMARK 9. We cannot guarantee the existence of a basis of eigenvectors, because some eigenvalues may be deficient (that is, for a repeated eigenvalue λ, there might be smaller number of linearly independent eigenvectors associated to λ). $\triangle$

We say that a point p is a ***ω-limit point of*** (0.2) if there is a sequence $t_n \to \infty$ such that

$$lim_{n\to\infty}\phi(t_n, x) = p.$$

In other words, the trajectory moves toward p as $t \to \infty$. The set of all ω-limit points of a trajectory Γ is called the ***ω-limit set of*** Γ. We also say that a point p is an ***α-limit point of*** (0.2) if there is a sequence $t_n \to -\infty$ such that

$$lim_{n\to\infty}\phi(t_n, x) = p.$$

In other words, the trajectory moves away from p as $t \to \infty$. The set of all α-limit points of a trajectory Γ is called the ***α-limit set of*** Γ.

Now, we define a ***periodic orbit of*** (0.2) as any closed solution curve of (0.2) that is not an equilibrium point of (0.2). A ***limit cycle*** Γ is a periodic orbit which is the ω or

α-limit set of some trajectory of (0.2). If a cycle Γ is the ω-limit set of every trajectory in a neighborhood of Γ, then it is called a ***stable limit cycle***. If a cycle Γ is the α-limit set of every trajectory in a neighborhood of Γ, then it is called a ***unstable limit cycle***.

Now, as with equilibrium points, we can define hyperbolic periodic orbits. However, in order to define a hyperolic periodic orbit, we need to define some other terms first. Let $\Gamma: \ x = \gamma(t)$, be a periodic orbit of (0.2) of period T that is contained in an open subset E of $\mathbb{R}^n$, where $0 \leq t \leq T$. We define the ***linearization of*** (0.2) ***about*** Γ as the nonautonomous linear system

$$\dot{x} = A(t)x \tag{0.6}$$

where

$$A(t) = Df(\gamma(t))$$

is a continuous, T-periodic function of t for all $t \in \mathbb{R}$.

The fact that the linear system (0.6) is nonautonomous is a problem, since the matrix given above changes as t changes. If we can, we would like to change the linear system (0.6) so that the matrix is constant. To help us with this problem, we will introduce a theorem that will be helpful. Now before we see this theorem, we need to introduce another term. We define a ***fundamental matrix solution of*** (0.6) as a nonsingular $n \times n$ matrix $\Phi(t)$ which satisfies the matrix differential equation

$$\dot{\Phi}(t) = A(t)\Phi(t),$$

for all $t \in \mathbb{R}$. Now that we have this definition, we introduce a theorem (which is referred to as *Floquet's Theorem*) to help us overcome the problem mentioned above.

THEOREM 10. [1] (Floquet's Theorem) Let $A(t)$ be a continuous, T-periodic matrix.

Then for all $t \in \mathbb{R}$, any fundamental matrix solution of (0.6) can be written in the form

$$\Phi(t) = Q(t)e^{Bt} \tag{0.7}$$

where $Q(t)$ is a nonsingular, differentiable, T-periodic matrix and B is a constant matrix. Furthermore, if $\Phi(0) = I$, then $Q(0) = I$. □

So, why is this theorem useful? While the matrix $Q(t)$ from (0.6) still depends on t, but we now have reduced the system so that $Q(t)$ is multiplied to a constant matrix e^{Bt}. However, we have a corollary of this theorem that will solve the problem we were having above.

COROLLARY 11. [1] Under the hypothesis of Floquet's Theorem, $x(t)$ is a solution of $\dot{x} = A(t)x$ if and only if $y(t)$ is a solution of $\dot{y} = By$, where $y = Q^{-1}(t)x$. □

The idea behind this corollary is that to study the nonlinear system (0.2) around a periodic orbit is equivalent to studying the linear system given in the previous corollary. From this, we give another definition that will help us define a hyperbolic periodic orbit. We define a ***monodromy matrix M*** to be the matrix $M = \Phi(T)$, where Φ is any fundamental matrix solution of (0.3) such that $\Phi(0) = I$. In particular, we have $\Phi(T) = Q(T)e^{BT} = Q(0)e^{BT} = Ie^{BT} = e^{BT}$ (i.e., $M = e^{BT}$). The eigenvalues of $M = e^{BT}$ are referred to as ***Floquet multipliers***, which are of the form $\mu_j = e^{\lambda_j T}$, where $j = 1, 2, ..., n$ and $\lambda_j = a_j \pm ib_j$ are the eigenvalues of the constant matrix B. The eigenvalues of the constant matrix B are called the ***Floquet exponents***. Note that for every periodic orbit, $\mu_j = 1$ (or $\lambda_j = 0$) is always an eigenvalue of M, for some $j = 1, 2, ..., n$. Also, we have that $Re(\lambda_j) < 0$ if and only if $||\mu_j|| < 1$ and $Re(\lambda_j) > 0$ if and only if $||\mu_j|| > 1$, for some $j = 1, 2, ..., n$.

We are now ready to define a hyperbolic periodic orbit. We say a periodic orbit is ***hyperbolic*** if it has exactly one eigenvalue μ_j with $||\mu_j|| = 1$. If a periodic orbit is not hyperbolic, we say that periodic orbit is ***nonhyperbolic***.

CHAPTER 3: LOCAL AND GLOBAL STABILITY ANALYSIS OF DYNAMICAL SYSTEMS

Now that we have a basic idea of dynamical systems, we will look at the local and global stability analysis of dynamical systems. Once again, consider the system

$$\dot{x} = f(x) \tag{0.8}$$

3.1 Local Stability Analysis

In the previous chapter, we discussed the concept of hyperbolic and nonhyperbolic equilibrium points of the system (0.8). So, why do we care if an equilibrium point is hyperbolic or not? Here are two theorems that might give some insight to this question. In the following theorems, let

$$\dot{y} = Ay,\ A = Df(0) \tag{0.9}$$

where 0 is the equilibrium point of (0.8).

Before we state these theorems, we need a definition. We say a set $E \subset \mathbb{R}^n$ is ***invariant with respect to the flow*** ϕ_t ***of*** $\dot{x} = f(x)$ if $\phi_t(E) \subset E$. In other words, if x_0 is a point in E and $t \geq 0$, then the point $\phi_t(x_0)$ is also a point in E.

THEOREM 12. [1] (Stable Manifold Theorem) Let E be an open subset of $\mathbb{R}^n$ containing the origin; let $f \in C^1(E)$, and let φ_t be the flow of the nonlinear system (0.8). Suppose $f(0) = 0$ and that $A = Df(0)$ has k eigenvalues with negative real part and $(n-k)$ eigenvalues with positive real part (i.e., 0 is a hyperbolic equilibrium point of (0.8)). Then there exists a k-dimensional (stable) differentiable manifold S tangent to the stable subspace E^S of (0.9) such that for all $t \geq 0$, $\varphi_t(S) \subset S$, and for all $x_0 \in S$:

$$lim_{t\to\infty}\ \varphi_t(x_0) = 0$$

and there exists an $(n-k)$ dimensional (unstable) differentiable manifold U tangent to the unstable subspace E^U of (0.9) such that for all $t \geq 0$, $\varphi_t(U) \subset U$, and for all $x_0 \in U$:

$$lim_{t\to-\infty}\ \varphi_t(x_0) = 0. \ \square$$

THEOREM 13. [1] (Hartman-Grobman Theorem) Let E be an open subset of $\mathbb{R}^n$ containing the origin; let $f \in C^1(E)$, and let φ_t be the flow of the nonlinear system (0.8). Suppose $f(0) = 0$ and that $A = Df(0)$ has no eigenvalues with zero real part (i.e., 0 is a hyperbolic equilibrium point of (0.8)). Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $x_0 \in U$ and for all $t \in \mathbb{R}$, we have

$$H \circ \varphi_t(x_0) = e^{At} H(x_0). \ \square$$

The main idea behind these two theorems (when combined) is that we can determine the behavior around an equilibrium point by seeing what happens in the corresponding linear system (0.9). Now, the main assumption of these two theorems is that the equilibrium point is *hyperbolic*. So, if an equilibrium point is not hyperbolic, then these two theorems do not apply. So, we have to turn to some other results to determine the behavior around nonhyperbolic equilibrium points.

This is the beginning of what we call Local Stability Analysis of Equilibrium Points, that is, we analyze the system to determine the behavior around equilibrium points. So, if an equilibrium point is hyperbolic, then the Stable Manifold and Hartman-Grobman Theorems apply and we can determine what happens in a neighborhood around that equilibrium point.

EXAMPLE 14. (Application)

Consider the system

$$\begin{cases} \dot{S} = \mu(H-S) - aSf(B) \\ \dot{I} = aSf(B) - rI \\ \dot{B} = -\gamma B + eI \end{cases}$$

where $S(t)$ and $I(t)$ represent populations of susceptible and infected individuals respectively (total population is H), and $B(t)$ represents the concentration of bacteria in water reservoirs. all the parameters are positive.

μ is the rate of growth and death of susceptibles,

a is the rate at which susceptibles are exposed to bacteria,

$f(B) = \frac{B}{K+B}$ is the probability of susceptibles to get infected,

K is the half-saturation constant,

r is the loss rate of infected (due to recovery or death),

γ is the death rate of bacteria,

e is the rate at which individuals release bacteria into water reservoirs.

Then the equilibrium points of this system are:

$P_1 = (H,0,0)$ and $P_2 = (\frac{eH\mu+\gamma Kr}{e(a+\mu)}, \frac{\mu(aeH-\gamma Kr)}{er(a+\mu)}, \frac{\mu(aeH-\gamma Kr)}{\gamma r(a+\mu)}) = (S^*, I^*, B^*)$.

So, these are the two points in the system where the rates of S, I, B do not change. P_1 is called the disease-free equilibrium point (i.e., at P_1, there is no disease). P_2 is called the endemic equilibrium point. Also, the Jacobian of the system is

$$Df(S,I,B) = \begin{bmatrix} -\mu - \frac{aB}{K+B} & 0 & -\frac{aSK}{(K+B)^2} \\ \frac{aB}{K+B} & -r & \frac{aSK}{(K+B)^2} \\ 0 & e & -\gamma \end{bmatrix}.$$ So the Jacobian at P_1 is

$$Df(P_1) = \begin{bmatrix} -\mu - \frac{a(0)}{K+(0)} & 0 & -\frac{a(H)K}{(K+(0))^2} \\ \frac{a(0)}{(K+(0))^2} & -r & \frac{a(H)K}{(K+(0))^2} \\ 0 & e & -\gamma \end{bmatrix} = \begin{bmatrix} -\mu & 0 & -\frac{aH}{K} \\ 0 & -r & \frac{aH}{K} \\ 0 & e & -\gamma \end{bmatrix}.$$ The eigenvalues of this Jacobian are $\lambda_1 = -\mu$, $\lambda_2 = \frac{-(\gamma+r)-\sqrt{(\gamma+r)^2-\frac{4(\gamma rK-eaH)}{K}}}{2}$, and $\lambda_3 = \frac{-(\gamma+r)+\sqrt{(\gamma+r)^2-\frac{4(\gamma rK-eaH)}{K}}}{2}$.

Note that all of the eigenvalues have nonzero real part. Hence, P_1 is a hyperbolic equilibrium point of this system. So, we can apply the Stable Manifold Theorem and the Hartman-Grobman Theorem. Assume $(\gamma+r)^2 - \frac{4(\gamma rK-eaH)}{K} \geq 0$. Note that we have $\lambda_1 < 0$ and $\lambda_2 < 0$. If $\lambda_3 < 0$, then P_1 is a stable node. If $\lambda_3 > 0$, then P_1 is an unstable saddle. Now, assume $(\gamma+r)^2 - \frac{4(\gamma rK-eaH)}{K} < 0$. Note that P_1 cannot be a center. For if it were, then we would have $-(\gamma+r) = 0$ which gives $\gamma = -r < 0$ or $r = -\gamma < 0$, a contradiction either way. Hence, we have P_1 is a focus (stable if $-(\gamma+r) < 0$ and unstable if $-(\gamma+r) > 0$).

We claim that the endemic equilibrium point P_2 exists if and only if the disease-free equilibrium point is unstable. To show this, first we assume that P_2 exists. Then we have $\frac{\mu(aeH-\gamma Kr)}{er(a+\mu)} > 0$ and $\frac{\mu(aeH-\gamma Kr)}{\gamma r(a+\mu)} > 0 \implies \mu(aeH-\gamma Kr) > 0 \implies aeH - \gamma Kr > 0 \implies \lambda_3 > 0$. Hence, P_1 is unstable. Conversely, assume P_1 is unstable. Then we have $aeH - \gamma Kr > 0 \implies \mu(aeH-\gamma Kr) > 0 \implies \frac{\mu(aeH-\gamma Kr)}{er(a+\mu)} > 0$ and $\frac{\mu(aeH-\gamma Kr)}{\gamma r(a+\mu)} > 0$. Thus, P_2 exists. This proves the claim.

So, what does this mean in the context of the application here? Basically, if $\lambda_3 < 0$ (or $R_0 = \frac{aeH}{\gamma rK} < 1$, where R_0 is the so-called Basic Reproduction Number), the disease will eventually die out. On the other hand, if $R_0 > 1$, the disease will spread out throughout the community. $\triangledown$

The phase portraits for this system are given as follows. (See F3 - F4.)

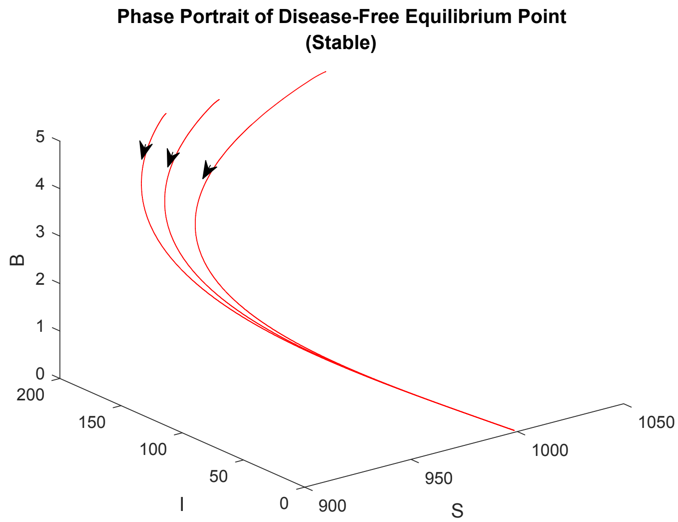


Figure 3. Phase portrait of a stable disease-free equilibrium point.

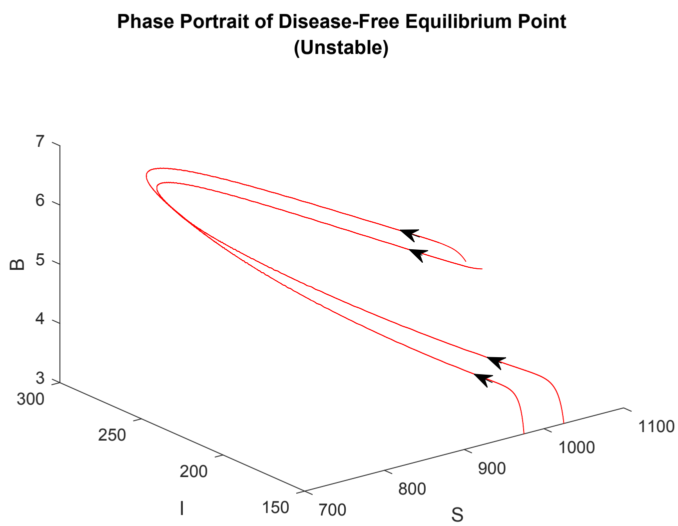


Figure 4. Phase portrait of an unstable disease-free equilibrium point and a stable endemic equilibrium point.

Now, what if an equilibrium point is not hyperbolic? There are two theorems that work for nonhyperbolic equilibrium points (in two dimensions). Now, the assumptions in these theorems depend on how many eigenvalues of the Jacobian at the nonhyperbolic equilibrium point are zero. The first theorem that will be mentioned here assumes there is exactly one eigenvalue equal to zero, while the second theorem assumes there are two eigenvalues equal to zero.

If the Jacobian at a nonhyperbolic equilibrium point has exactly one eigenvalue equal zero, then the system can be written as

$$\begin{cases} \dot{x} = p(x,y) \\ \dot{y} = y + q(x,y) \end{cases} \tag{0.10}$$

where p and q are analytic in a neighborhood of the origin and have expansions starting with second degree terms of x and y.

THEOREM 15. [1] Let the origin be an isolated equilibrium point for the analytic system (0.10). Let $y = \phi(x)$ be the solution of the equation $y + q(x,y) = 0$ in a neighborhood of the origin and let the function $\Psi(x) = p(x, \phi(x))$ be a series expansion in a neighborhood of the origin have the form $\Psi(x) = a_m x^m + a_{m+1} x^{m+1} + \ldots$, where $m \geq 2$ and $a_m \neq 0$. Then

(i) for m odd and $a_m > 0$, the origin is an unstable node,

(ii) for m odd and $a_m < 0$, the origin is a (topological) saddle and

(iii) for m even, the origin is a saddle-node. □

REMARK 16. The previous theorem is a particular case of the Center Manifold Theorem, which will be presented in Chapter 5. △

If the Jacobian at a nonhyperbolic equilibrium point has two eigenvalues equal to zero, then the system can be written in the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = a_k x^k [1 + h(x)] + b_n x^n y [1 + g(x)] + y^2 R(x,y) \end{cases} \tag{0.11}$$

where h, g, and R are analytic in a neighborhood of the origin, $h(0) = g(0) = 0$, $k \geq 2$, $a_k \neq 0$, and $n \geq 1$.

Now, let's look at the case for the analytic system (0.11).

THEOREM 17. [1] Let the origin be an isolated equilibrium point for the analytic system (0.11). Let $k = 2m+1$ with $m \geq 1$ in (6) and let $\lambda = b_n^2 + 4(m+1)a_k$. If $a_k > 0$, then the origin is a (topological) saddle. If $a_k < 0$, then the origin is

(i) a focus or a center if $b_n = 0$ and also if $b_n \neq 0$ and $n > m$ or if $n = m$ and $\lambda < 0$,

(ii) a node if $b_n \neq 0$, n is an even number and $n < m$ and also if $b_n \neq 0$, n is an even number, $n = m$ and $\lambda \geq 0$ and

(iii) a critical point with an elliptic domain if $b_n \neq 0$, n is an odd number and $n < m$ and also if $b_n \neq 0$, n is an odd number, $n = m$ and $\lambda \geq 0$.

Let $k = 2m$ with $m \geq 1$ in (0.11). Then the origin is

(i) a cusp if $b_n = 0$ and also if $b_n \neq 0$ and $n \geq m$ and

(ii) a saddle-node if $b_n \neq 0$ and $n < m$. $\square$

We now provide an example of how we can apply Theorem 17. We will provide examples of how to apply Theorem 15 later in this thesis.

EXAMPLE 18. Consider the following system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^4 + xy \end{cases} \tag{0.12}$$

We see that the Jacobian of the system (0.12) is

$$Df(x,y) = \begin{bmatrix} 0 & 1 \\ 4x^3 + y & x \end{bmatrix}$$

The Jacobian of (0.12) evaluated at the origin is

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 4(0)^3 + (0) & (0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It can be shown that the eigenvalues of $Df(0,0)$ are $\lambda_1 = \lambda_2 = 0$. From this, we can see that the system (0.12) is of the form (0.11) (with $k=4$, $m=2$, $n=1$, and $b_1 = 1 \neq 0$, $a_4 = 1$, $h(x) = g(x) = R(x,y) = 0$). Since we have that $k=4$ (so k is even), $b_1 \neq 0$, and $1 = n < m = 2$, then by Theorem 17, we see that the origin is a saddle-node. $\triangledown$

Now, we can ask the same question about the importance of hyperbolic periodic orbits versus nonhyperbolic periodic orbits. It turns out that there is a Stable Manifold Theorem for periodic orbits. Here, we will state the Stable Manifold Theorem for Periodic Orbits.

THEOREM 19. [1] (Stable Manifold Theorem for Periodic Orbits) Let $f \in C^1(E)$, where E is an open subset of $\mathbb{R}^n$ containing a periodic orbit

$$\Gamma: \ x = \gamma(t)$$

of period T. Let φ_t be the flow of a given system and $\gamma(t) = \varphi_t(x_0)$. If k of the Floquet exponents of $\gamma(t)$ have negative real part where $0 \leq k \leq n-1$ and $(n-k-1)$ of them have positive real part (i.e., Γ is hyperbolic), then there is a $\delta > 0$ such that the stable manifold of Γ,

$$S(\Gamma) = \{x \in N_\delta(\Gamma) \mid d(\varphi_t(x), \Gamma) \to 0 \ as\ t \to \infty \ and\ \varphi_t(x) \in N_\delta(\Gamma)\ for\ t \geq 0\}$$

is a $(k+1)$-dimensional, differentiable manifold which is positively invariant under the flow φ_t and the unstable manifold of Γ,

$$U(\Gamma) = \{x \in N_\delta(\Gamma) \mid d(\varphi_t(x), \Gamma) \to 0 \ as\ t \to -\infty \ and\ \varphi_t(x) \in N_\delta(\Gamma)\ for\ t \leq 0\}$$

is an $(n-k)$-dimensional, differentiable manifold which is negatively invariant under the flow φ_t. $\square$

3.2 Global Stability Analysis

Now, we're ready to talk about the global stability of the system

$$\dot{x} = f(x)$$

where $f:\ \mathbb{R}^n \to \mathbb{R}^n$ and $x \in \mathbb{R}^n$. There are two main results we have to study the global stability of the system (0.8): Lyapunov's Theorem and LaSalle's Invariance Principle.

THEOREM 20. [1] (Lyapunov's Theorem) Let $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$. Assume there is a real-valued function $V:\ \mathbb{R}^n \to \mathbb{R}$ such that

(1) $V(x_0) = 0$

(2) $V(x) > 0$, for all $x \neq x_0$.

If $\dot{V}(x) \leq 0$ for all $x \in \mathbb{R}^n$, then x_0 is stable. If $\dot{V}(x) < 0$ for all $x \in \mathbb{R}^n \sim \{x_0\}$, then x_0 is asymptotically stable. If $\dot{V}(x) > 0$ for all $x \in \mathbb{R}^n \sim \{x_0\}$, then x_0 is unstable, where $\dot{V}(x) = \frac{dV(t)}{dt}$ denotes the derivative of the function V with respect to time t. □

REMARK 21. For Lyapunov's theorem, the point x_0 does not have to be hyperbolic for the theorem to apply. △

Before we look at the next result, we first need to define another term. We say that a set $E \subset \mathbb{R}^n$ is ***invariant with respect to*** $\dot{x} = f(x)$ if for all $x_0 \in E$ and for all $t \geq 0$, we have $x_0(t) \in S$. In other words, if x_0 is a point in E and $t \geq 0$, then $x_0(t)$ will also be in the set E.

Now, we look at LaSalle's invariance principle.

THEOREM 22. [8] (LaSalle's Invariance Principle) Let $\Omega \subset D$ be a compact invariant set with respect to $\dot{x} = f(x)$. Let $Q:\ D \to \mathbb{R}$ be a C^1 function such that $Q'(x(t)) \leq 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $Q'(x) = 0$. Let $M \subset E$ be the largest invariant set in E. Then

$$lim_{t\to\infty}[inf\ ||x(t) - y||] = 0. \ \square$$

The main idea of LaSalle's invariance principle is that every solution starting in Ω approaches M as $t \to \infty$. So, M is globally stable.

At this time, we introduce a theorem that can be used to get a Lyapunov function to apply LaSalle's Invariance Principle. To make better sense of the notation, consider the system defined as follows:

$$\begin{cases} \dot{x} = \mathscr{F}(x,y) - \mathscr{H}(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{0.13}$$

with $g = [g_1, g_2, ..., g_m]^T$, $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$ represents the population in disease compartment, $y = [y_1, y_2, ..., y_m]^T \in \mathbb{R}^m$ represents the population in nondisease compartment, $\mathscr{F} = [\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_n]^T$, and $\mathscr{H} = [\mathscr{H}_1, \mathscr{H}_2, ..., \mathscr{H}_n]^T$ (where $\mathscr{F}_i$ represents the rate of new infections in the ith disease compartment, $\mathscr{H}_i$ represents the transition terms, for example, death and recovery in the ith disease compartment).

THEOREM 23. [7] Let F, V, and $f(x,y)$ be defined as follows:

$F = \left[\frac{\partial \mathscr{F}_i}{\partial x_j}(0, y_0)\right]$, $V = \left[\frac{\partial \mathscr{H}_i}{\partial x_j}(0, y_0)\right]$, and $f(x,y) = (F - V)x - \mathscr{F}(x,y) + \mathscr{H}(x,y)$.

If $f(x,y) \geq 0$ in $\Gamma \subset \mathbb{R}_+^{n+m}$, $F \geq 0$, $V^{-1} \geq 0$, and $R_0 \leq 1$ (where R_0 is an eigenvalue of the matrix $V^{-1}F$ associated to the left eigenvector w^T), then the function $Q = w^T V^{-1} x$ is a Lyapunov function of the model (0.13) on Γ. $\square$

We provide an example to show how to apply LaSalle's Invariance Principle to a particular model.

EXAMPLE 24. (Application)

Consider the system from the application from the previous section

$$\begin{cases} \dot{S} = \mu(H-S) - aSf(B) \\ \dot{I} = aSf(B) - rI \\ \dot{B} = -\gamma B + eI \end{cases} \tag{0.14}$$

First we define the system (0.14) in the form:

$$\begin{bmatrix} S \\ \begin{bmatrix} I \\ B \end{bmatrix} \end{bmatrix} := \begin{bmatrix} y \\ x \end{bmatrix}$$

Then we have that the system (0.14) in the compartment form

$$\begin{cases} \dot{x} = F^*(x,y) - V^*(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

where $x = \begin{bmatrix} \frac{aSB}{K+B} & eI \end{bmatrix}^T$ is the disease compartment, $y = S$, $F^*(x,y) = [\frac{\partial F_i^*}{\partial x_j}(0,y_0)]$, and $V^*(x,y) = [\frac{\partial V_i^*}{\partial x_j}(0,y_0)]$, for $1 \leq i \leq n$, and $1 \leq j \leq n$. So, the system (0.14) can be written into the form:

$F^*(x,y) = \begin{bmatrix} \frac{aSB}{K+B} & eI \end{bmatrix}^T$ and $V^*(x,y) = \begin{bmatrix} rI & \gamma B \end{bmatrix}^T \implies$

$$\begin{cases} \dot{x_1} = \frac{aSB}{K+B} - rI \\ \dot{x_2} = eI - \gamma B \\ \dot{y} = \mu(H-S) - \frac{aSB}{K+B}. \end{cases}$$

Now, we need to find $F = \frac{\partial F_i^*}{\partial x_i}(0,0,H)$ and $V = \frac{\partial V_i^*}{\partial x_i}(0,0,H)$.

$$F = \begin{bmatrix} 0 & \frac{(K+B)(aS)-aSB}{(K+B)^2} \\ e & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{aSK}{(K+B)^2} \\ e & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{aH}{K} \\ e & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} r & 0 \\ 0 & \gamma \end{bmatrix} \implies V^{-1} = \frac{1}{r\gamma}\begin{bmatrix} \gamma & 0 \\ 0 & r \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}.$$

So, we have $V^{-1}F = \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} 0 & \frac{aH}{K} \\ e & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{aH}{rK} \\ \frac{e}{\gamma} & 0 \end{bmatrix}$, which has eigenvalues $\lambda = \pm\sqrt{\frac{aeH}{\gamma rK}}$. Let $\lambda^* = \sqrt{\frac{aeH}{\gamma rK}}$, and assume that $\lambda^* \leq 1$. Now, we need to find w. To do this, we need to solve the system $(A-\lambda I)w = 0 \implies \begin{bmatrix} -\sqrt{\frac{aeH}{\gamma rK}} & \frac{e}{\gamma} \\ \frac{aH}{rK} & -\sqrt{\frac{aeH}{\gamma rK}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies w = \begin{bmatrix} w_1 \\ \frac{\gamma}{e}\lambda^* w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{\gamma}{e}\lambda^* \end{bmatrix}$ (by letting $w_1 = 1$).

Let $Q(I,B) = \begin{bmatrix} 1 & \frac{\gamma}{e}\lambda^* \end{bmatrix} \begin{bmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} I \\ B \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & \frac{\lambda^*}{e} \end{bmatrix} \begin{bmatrix} I \\ B \end{bmatrix} = \frac{1}{r}I + \frac{\lambda^*}{e}B$. Then we have $Q(0,0) = \frac{1}{r}(0) + \frac{\lambda^*}{\gamma}(0) = 0$ and $Q(I,B) > 0$ for all other $\begin{bmatrix} I & B \end{bmatrix}^T \in \mathbb{R}^2$ since $I > 0$ and $B > 0$. By definition, Q is a Lyapunov function for (0.14). Also, we have $Q'(I,B) = \frac{1}{r}\dot{I} + \frac{\lambda^*}{e}\dot{B} = \frac{1}{r}(\frac{aSB}{K+B} - rI) + \frac{\lambda^*}{e}(-\gamma B + eI) = \frac{aSB}{r(K+B)} - I - \frac{\gamma\lambda^*}{e}B + \lambda^* I = \frac{aSB}{r(K+B)} - \frac{\lambda^* SB}{e} + (\lambda^* - 1)I$. We claim that $Q' \leq 0$. To see this, we note that $Q' \leq 0$ iff $\frac{aSB}{r(K+B)} - \frac{\lambda^*\gamma B}{e} \leq 0$ iff $B(\frac{aS}{r(K+B)} - \frac{\lambda^*\gamma}{e}) \leq 0$ iff $\frac{aeSH - \lambda^* - \gamma rH(K+B)}{erH(K+B)} \leq 0$ iff $aeSH - \lambda^* - \gamma rH(K+B) \leq 0$. Note that we have $aeHS - \lambda^*\gamma rHK - \lambda^*\gamma rHB \leq aeHS - \lambda^*\gamma rHK \leq aeH - \gamma rKH \leq 0$ since $\lambda^* \leq 1$. This proves the claim.

We will now show that the disease-free equilibrium point is globally stable by using LaSalle's invariance principle. Let $Q(x) = w^T V^{-1} x$ be the Lyapunov function as defined above (where $x = \begin{bmatrix} I & B \end{bmatrix}^T$). Then we have that Q is a C^1 function, and as shown above, $Q'(x) \leq 0$. Now, consider $Q'(x) = 0$. Then from above, we have $Q'(x) = (\lambda^* - 1)I + (\frac{aS}{r(K+B)} - \frac{\lambda^*\gamma}{e})B = 0$. However, $Q'(x) = 0$ whenever $I = B = 0$. So, the set of all points such that $Q'(x) = 0$, which is $E = \{(I,B,S) \mid I = B = 0\}$. On E, the system reduces to $\dot{S} = \mu(H - S)$. Solving this differential equation using an integrating factor gives us the solution $S(t) = H + Ce^{-\mu t}$, for some $C \in \mathbb{R}$. Hence, the largest (and only) invariant set in

E is $M = (0,0,H)$, since $lim_{t\to\infty} S(t) = H$. But, M is the disease-free equilibrium point of (0.14). Thereore, by LaSalle's Invariance Principle, M is globally stable. $\triangledown$

CHAPTER 4: BIFURCATIONS

In Chapters 2 and 3, we only considered systems of the form $\dot{x} = f(x)$. That is, the only thing that the system depended on was the solution to the system itself. In this Chapter, we consider systems of the form

$$\dot{x} = f(x, \mu) \qquad (0.15)$$

That is, the system not only depends on the solution of the system, but also on the vector $\mu \in \mathbb{R}^m$. The components of the vector μ usually are parameters to the system that is allowed to change. Since μ is allowed to change, then so can the behavior of the system. If we see that the behavior of vector fields near f behaves "similarly" for any vector μ, we say that f is a structually stable vector field. On the other hand, if we see that there is a vector $\mu_0 \in \mathbb{R}^m$ so that the behavior near f changes drastically (e.g. number of equilibrium points (or periodic orbits) changes, the stability of equilibrium points (or periodic orbits) changes, etc.), we say that f is a structually unstable vector field. It is the structually unstable vector field we are interested in studying (in fact, one can define a bifurcation at μ_0 when f is structually unstable at μ_0). Basically, bifurcation theory studies bifurcations (what properties do they have, when do they exist, etc.).

We will first introduce some basic bifurcations (using examples to help define the bifurcations). After we introduce these bifurcations, we will introduce a theorem that will mathematically prove the existence of these bifurcations called Sotomayor's Theorem. To have a better understanding of the bifurcations, we will look at an example to introduce them. For each of the bifurcations we look at here, we will provide some phase portraits (a geometric interpretation of what the qualitative behavior of the system looks like) and a bifurcation diagram (a geometric interpretation of the relationship between the parameters and a solution of the system). The value in which a bifurcation occurs is called the ***bifurcation value***.

4.1 Types of Bifurcations

EXAMPLE 25. (*Transcritical Bifurcation*)

Consider the system

$$\begin{cases} \dot{x} = \mu x - x^2 \\ \dot{y} = -y \end{cases} \tag{0.16}$$

where $\mu \in \mathbb{R}$. First, we can see that the equilibrium points of are $(0,0)$, and $(\mu,0)$. Notice that when $\mu = 0$, there is only one equilibrium point. However, if $\mu \neq 0$, then there are two equilibrium points.

Now, we have that the Jacobian of the system (0.16) is

$$Df(x,y) = \begin{bmatrix} \mu - 2x & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, we get that the Jacobian of (0.16) at $(0,0)$ is

$$Df(0,0) = \begin{bmatrix} \mu - 2(0) & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}.$$ The eigenvalues of $Df(0,0)$ are $\lambda_1 = \mu$ and $\lambda_2 = -1$. Since μ is allowed to change, the stability of $(0,0)$ could also change. We claim that $\mu = 0$ is the bifurcation value for this system, since the dynamics of the system changes as μ goes from negaive to positive. For instance, if $\mu < 0$, then we have that $(0,0)$ is hyperbolic and is a stable node. Now, if $\mu > 0$, then we have that $(0,0)$ is hyperbolic and is a (unstable) saddle.

We get that the Jacobian of (0.16) at $(\mu,0)$ is

$$Df(\mu,0) = \begin{bmatrix} \mu - 2(\mu) & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -\mu & 0 \\ 0 & -1 \end{bmatrix}.$$ The eigenvalues of $Df(\mu,0)$ are $\lambda_1 = -\mu$ and $\lambda_2 = -1$. If $\mu < 0$, then we have that $(\mu,0)$ is hyperbolic and is a (unstable) saddle. If $\mu > 0$, then we have that $(\mu,0)$ is hyperbolic and is a stable node.

Now, if $\mu = 0$, then $(0,0)$ is nonhyperbolic; so, we have to use either Theorem 15 or Theorem 17 to determine the behavior around $(0,0)$.

Now, we will look at some phase portraits of this system. (See F5 - F7.)

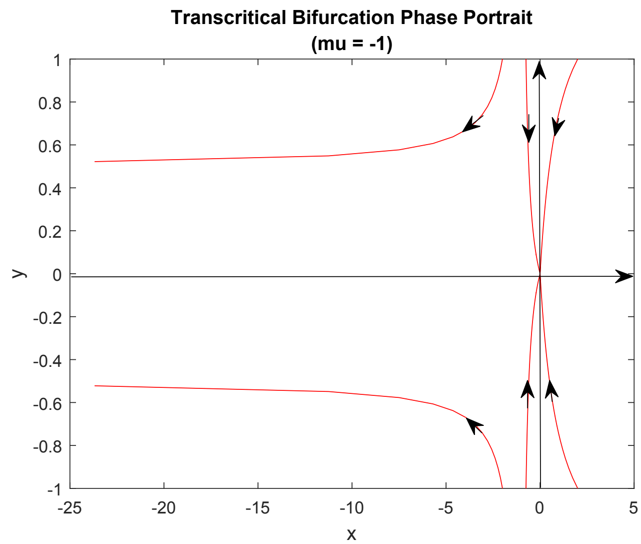


Figure 5. Phase portrait of transcritical bifurcation (when $\mu < 0$).

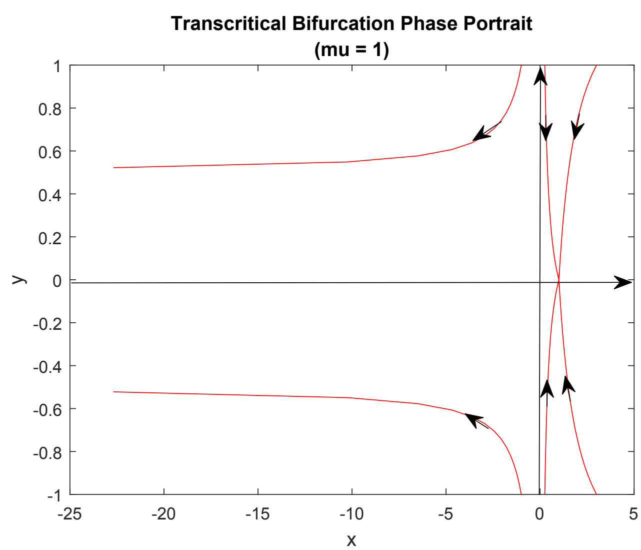


Figure 6. Phase portrait of transcritical bifurcation (when $\mu > 0$).

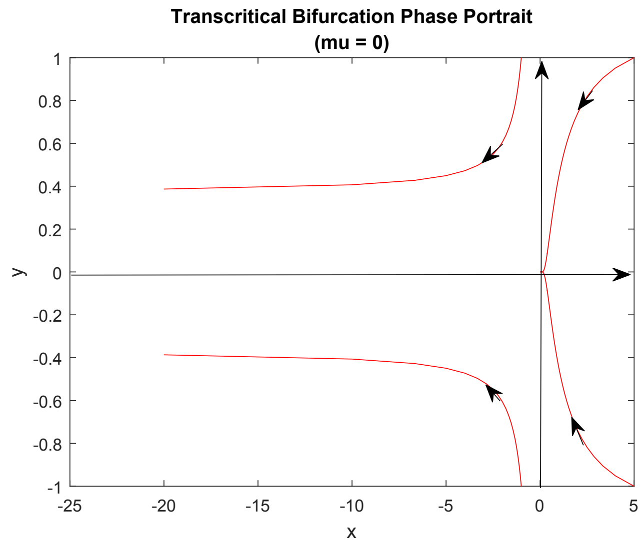


Figure 7. Phase portrait of transcritical bifurcation (when $\mu = 0$).

Here is the bifurcation diagram of the system. (See F8.)

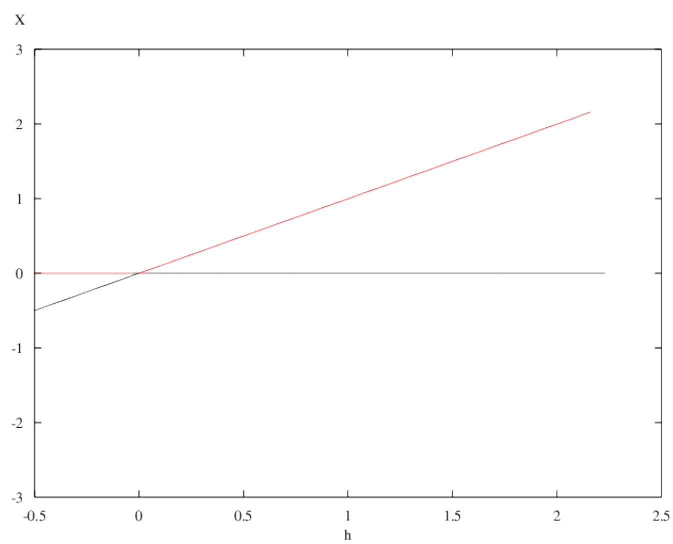


Figure 8. Bifurcation diagram of transcritical bifurcation.

Now, let us look at some of the characteristics of this bifurcation. First, we see that there were two equilibrium points, followed by one equilibrium point, followed by two equilibrium points again. Second, while the number of equilibrium points remains the same (so we did not create any new equilibrium points), the stability of these equilibrium points are interchanged at the bifurcation value $\mu_0 = 0$. This type of bifurcation is called a ***transcritical bifurcation***. $\triangledown$

EXAMPLE 26. (*Saddle-Node Bifurcation*)

Consider the system

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases} \tag{0.17}$$

where $\mu \in \mathbb{R}$. First, we can see that the equilibrium points of are $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. Again, observe that if $\mu < 0$, there are no equilibrium points. Also, if $\mu = 0$, there is one equilibrium point. However, if $\mu > 0$, there are two equilibrium points. So, assume that $\mu \geq 0$.

Now, we have that the Jacobian of (0.17) is

$$Df(x,y) = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, we have that the Jacobian of (0.17) at $(\sqrt{\mu}, 0)$ is

$$Df(\sqrt{\mu}, 0) = \begin{bmatrix} -2(\sqrt{\mu}) & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of $Df(\sqrt{\mu}, 0)$ are $\lambda_1 = -2\sqrt{\mu}$ and $\lambda_2 = -1$. We again claim that $\mu = 0$ is the bifurcation value for this system, since the dynamics of the system changes as μ goes from negative to positive. For instance, if $\mu > 0$, then we have that $(\sqrt{\mu}, 0)$ is hyperbolic and is a (stable) node.

We also have that the Jacobian of (0.17) at $(-\sqrt{\mu}, 0)$ is

$$Df(-\sqrt{\mu}, 0) = \begin{bmatrix} -2(-\sqrt{\mu}) & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of $Df(-\sqrt{\mu},0)$ are $\lambda_1 = 2\sqrt{\mu}$ and $\lambda_2 = -1$. If $\mu > 0$, then we have that $(-\sqrt{\mu},0)$ is hyperbolic and is a (unstable) saddle.

Note that if $\mu < 0$, then there are no equilibrium points since $\sqrt{\mu} \notin \mathbb{R}$. Also, if $\mu = 0$, then $(0,0)$ is nonhyperbolic and we have to use one of the theorems mentioned above to determine the behavior around $(0,0)$.

Now, let's look at some of the phase portraits of this bifurcation. (See F9 - F11.)

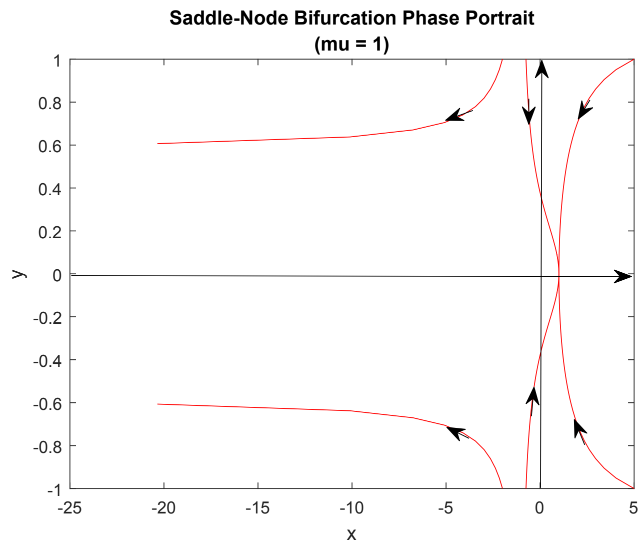


Figure 9. Phase portrait of saddle-node bifurcation (when $\mu > 0$).

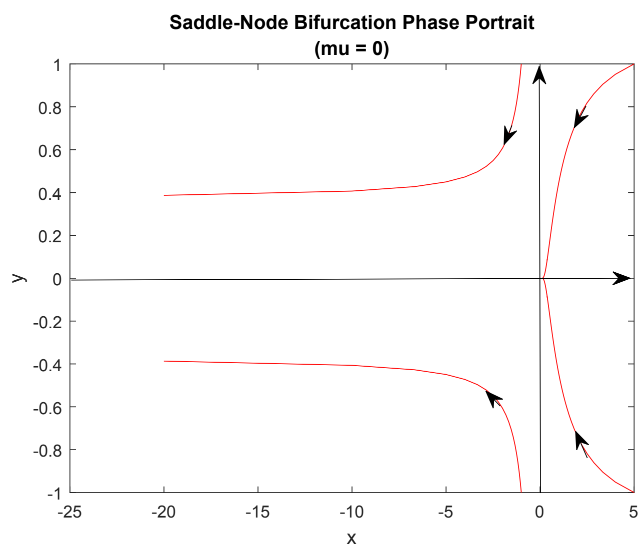


Figure 10. Phase portrait of saddle-node bifurcation (when $\mu = 0$).

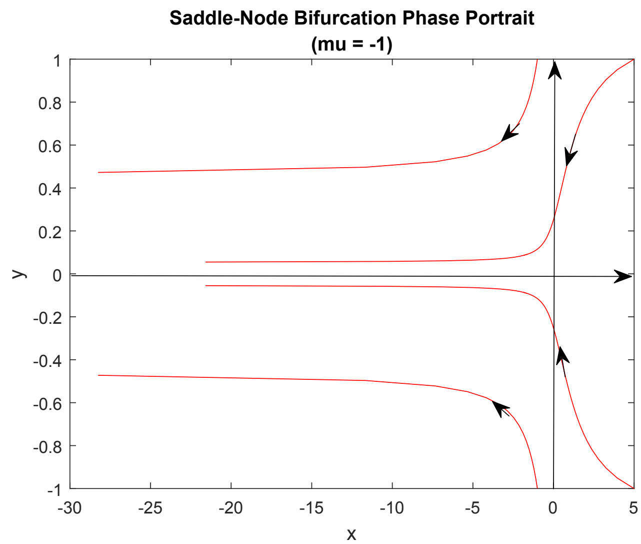


Figure 11. Phase portrait of saddle-node bifurcation (when $\mu < 0$).

Note that there are no equilibrium points in Figure 11.

Now, let's look at the bifurcation diagram of this bifurcation. (See F12.)

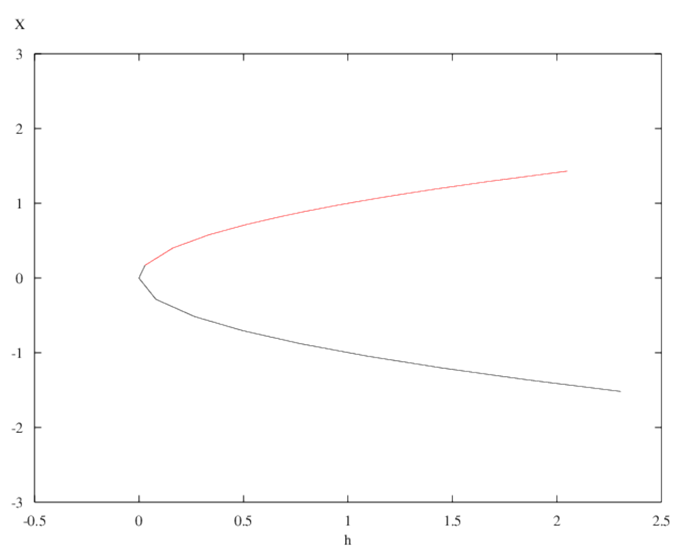


Figure 12. Bifurcation diagram of saddle-node bifurcation.

Now, let's look at some of the characteristics of this bifurcation. Mainly, we see that we have no equilibrium points before the bifurcation value $\mu_0 = 0$, followed by one equilibrium point, followed by two equilibrium points (one stable, one unstable). This type of bifurcation is called a ***saddle-node bifurcation***. $\triangledown$

EXAMPLE 27. (*Pitchfork Bifurcation*)

Consider the system

$$\begin{cases} \dot{x} = \mu x - x^3 \\ \dot{y} = -y \end{cases} \tag{0.18}$$

where $\mu \in \mathbb{R}$. First, we see that the equilibrium points of are $(0,0)$, $(\sqrt{\mu},0)$, and $(-\sqrt{\mu},0)$. Once more, we see that if $\mu \leq 0$, there is one equilibrium point. However, if $\mu > 0$, there are three equilibrium points.

We have that the Jacobian of (0.18) this system is

$$Df(x,y) = \begin{bmatrix} \mu - 3x^2 & 0 \\ 0 & -1 \end{bmatrix}.$$

From this, we have that the Jacobian of this system at $(0,0)$ is

$Df(0,0) = \begin{bmatrix} \mu & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvalues of $Df(0,0)$ are $\lambda_1 = \mu$ and $\lambda_2 = -1$. We claim that $\mu = 0$ is the bifurcation value for this system, since the dynamics of the system change as μ goes from negative to positive. For instance, if $\mu < 0$, then $(0,0)$ is hyperbolic and is a (stable) node. If $\mu > 0$, then $(0,0)$ is hyperbolic and is a (unstable) saddle.

Now, the Jacobian of (0.18) at $(\pm\sqrt{\mu},0)$ is

$$Df(\pm\sqrt{\mu},0) = \begin{bmatrix} \mu - 3(\pm\sqrt{\mu})^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2\mu & 0 \\ 0 & -1 \end{bmatrix}.$$ The eigenvalues of $Df(\pm\sqrt{\mu},0)$ are $\lambda_1 = -2\mu$ and $\lambda_2 = -1$. If $\mu > 0$, then $(\pm\sqrt{\mu},0)$ are hyperbolic and are a (stable) node.

Note that if $\mu = 0$, then $(0,0)$ is nonhyperbolic and we have to use either Theorem 15 or Theorem 17 to determine the behavior around $(0,0)$. Note also that if $\mu < 0$, then the equilibrium points $(\pm\sqrt{\mu},0)$ does not exist. In such a case, then the only equilibrium point to the system is $(0,0)$.

Now, let's look at some of the phase portraits of this bifurcation. (See F13 - F15.)

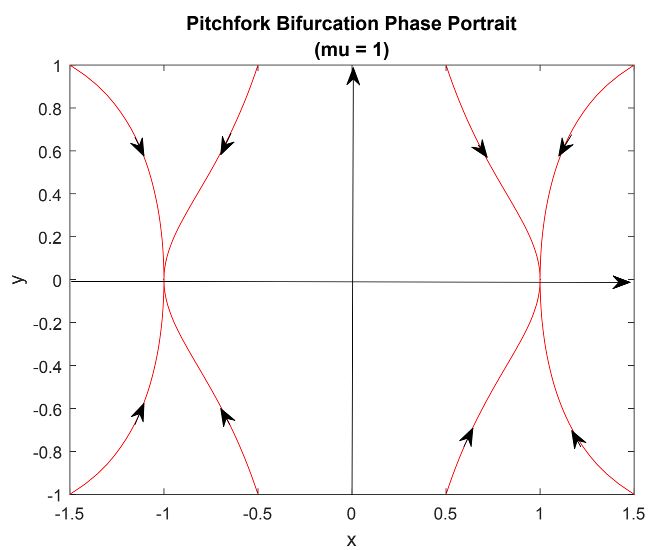


Figure 13. Phase portrait of pitchfork bifurcation (when $\mu > 0$).

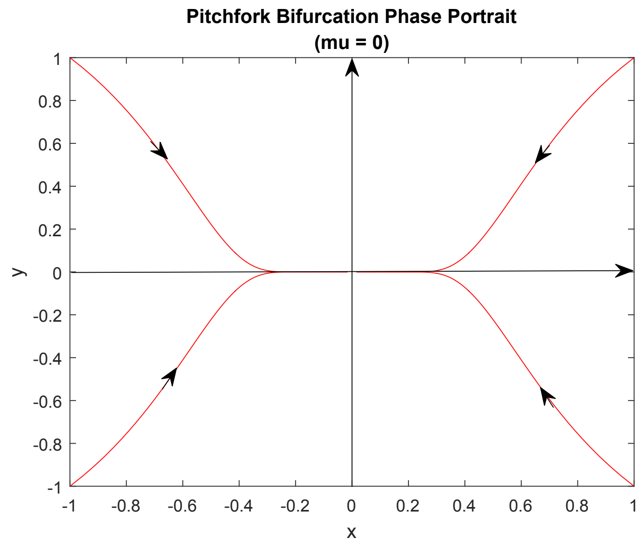


Figure 14. Phase portrait of pitchfork bifurcation (when $\mu = 0$).

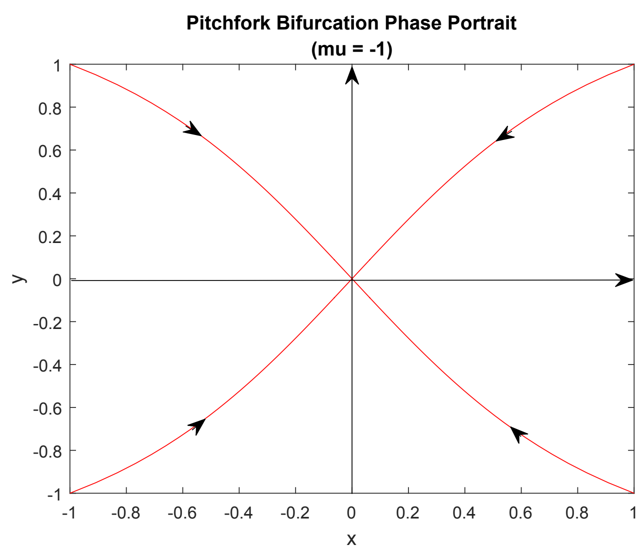


Figure 15. Phase portrait of pitchfork bifurcation (when $\mu < 0$).

Now, let's look at the bifurcation diagram of this bifurcation. (See F16.)

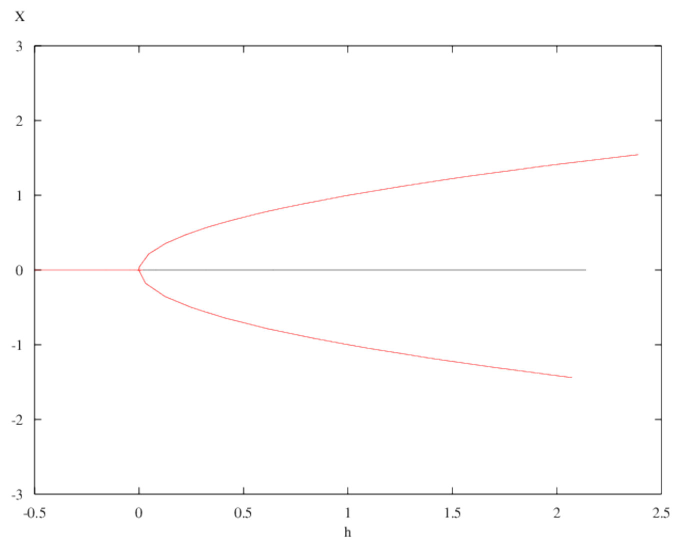


Figure 16. Bifurcation diagram of pitchfork bifurcation.

Now, let us look at some of the characteristics of this bifurcation. Mainly, we see that we have one equilibrium point before the bifurcation value $\mu_0 = 0$, followed by one equilibrium point, followed by three equilibrium points (two stable, one unstable). Now, one of these equilibrium points continues to exist for values below and above $\mu_0 = 0$, but it will change its stability (from unstable to stable or vice versa). This type of bifurcation is called a ***pitchfork bifurcation***. $\triangledown$

EXAMPLE 28. (*Hopf Bifurcation*)

Consider the system

$$\begin{cases} \dot{x} = -y + x(\mu - x^2 - y^2) \\ \dot{y} = x + y(\mu - x^2 - y^2) \end{cases} \tag{0.19}$$

where $\mu \in \mathbb{R}$. Then the only equilibrium point of the system is the origin $(0,0)$. The Jacobian of (0.19) is

$Df(x,y) = \begin{bmatrix} \mu - 3x^2 - y^2 & -1 - 2xy \\ 1 - 2xy & \mu - x^2 - 3y^2 \end{bmatrix}$. The Jacobian of (0.19) at $(0,0)$ is

$Df(0,0) = \begin{bmatrix} \mu - 3(0)^2 - (0)^2 & -1 - 2(0)(0) \\ 1 - 2(0)(0) & \mu - (0)^2 - 3(0)^2 \end{bmatrix} = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix}$. The eigenvalues of the Jacobian of (0.19) at $(0,0)$ are $\lambda_1 = \mu + i$ and $\lambda_2 = \mu - i$. We claim that $\mu = 0$ is the bifurcation value of this system, since the dynamics of the system changes as μ goes from negative to positive. For instance, if $\mu < 0$, then we have that $(0,0)$ is a (stable) focus. If $\mu > 0$, then we have that $(0,0)$ is an (unstable) focus. If $\mu = 0$, then $(0,0)$ is nonhyperbolic so linearization fails us.

Now, if we rewrite the system into polar coordinates, we get

$$\begin{cases} \dot{r} = r(\mu - r^2) \\ \dot{\theta} = 1 \end{cases}$$

If $\mu < 0$, then $\dot{r} < 0$ (i.e., r decreases to 0). If $\mu > 0$, then $\dot{r} = 0 \Leftrightarrow r = 0$ or $r = \sqrt{\mu}$. So, there is a periodic orbit of radius $r = \sqrt{\mu}$ (e.g., $\Gamma: \ \gamma(t) = \sqrt{\mu}(cos(t), sin(t))$). If $\mu = 0$, then $\dot{r} < 0$. So, if $\mu = 0$, then $(0,0)$ is a (stable) focus.

Let's look at some phase portraits of this bifurcation. (See F17 - F19.)

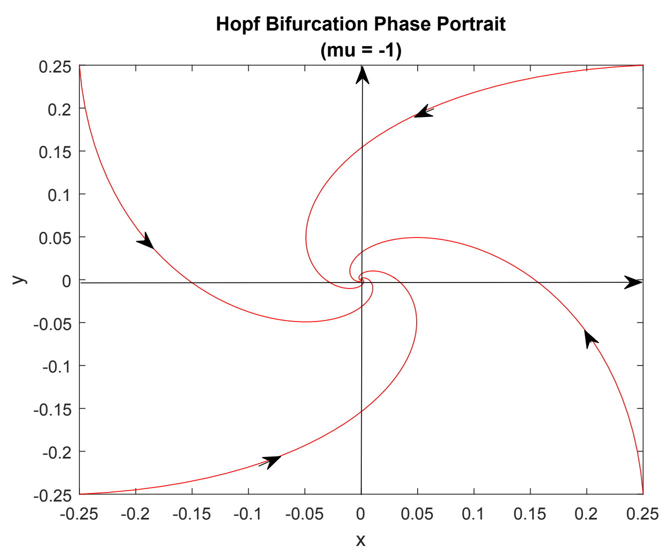


Figure 17. Phase portrait of Hopf bifurcation (when $\mu < 0$).

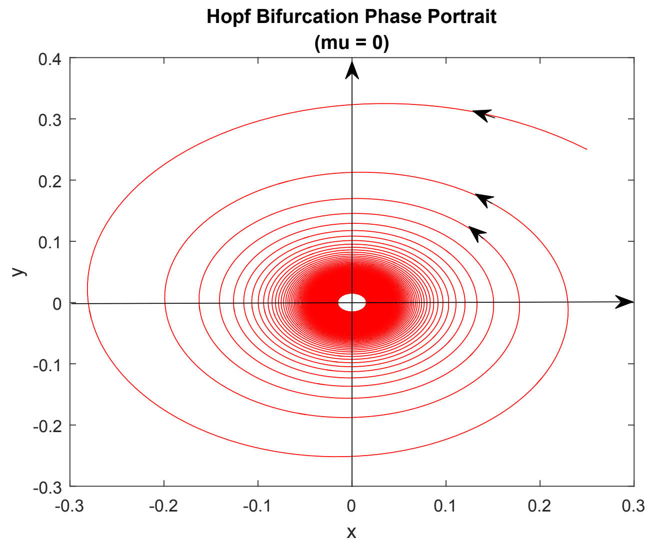


Figure 18. Phase portrait of Hopf bifurcation (when $\mu = 0$).

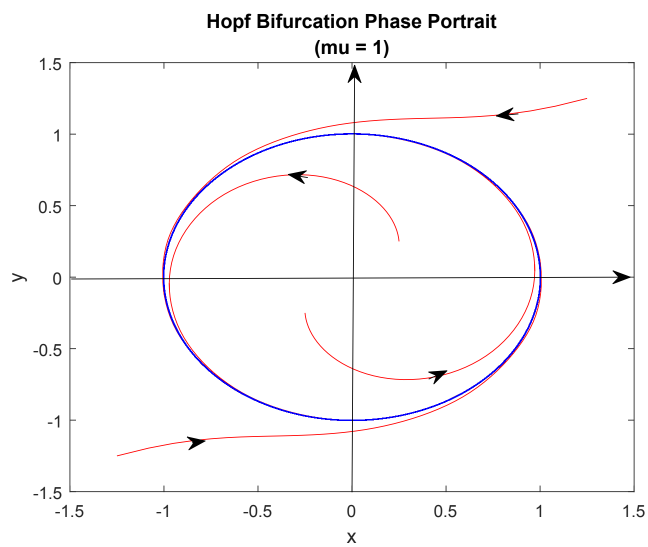


Figure 19. Phase portrait of Hopf bifurcation (when $\mu > 0$).

Note that the circle in the last phase portrait is to represent the periodic orbit that is formed as μ passes through $\mu_0 = 0$.

Here is the bifurcation diagram of this bifurcation. (See F20.)

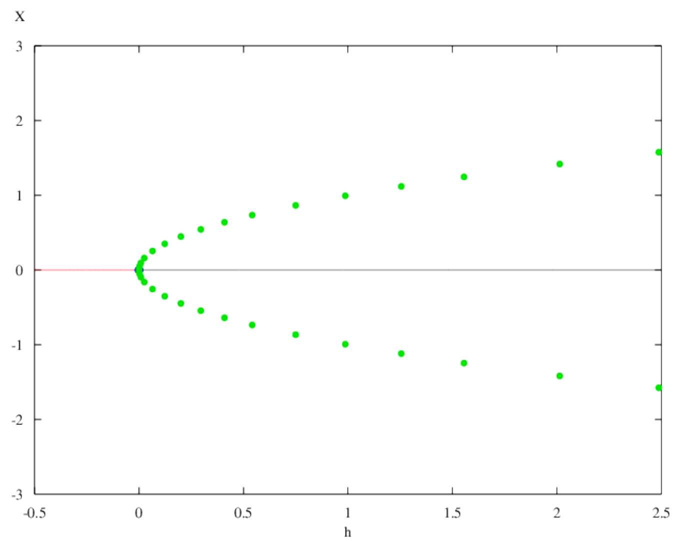


Figure 20. Bifurcation diagram of Hopf bifurcation.

Now, let's discuss some of the characteristics about this bifurcation. The main characteristic of this bifurcation is that some periodic orbits are formed as μ passes through the bifurcation value, and this happens when a focus point changes stability from stable to unstable. This type of bifurcation is referred to as a ***Hopf bifurcation***. Note that the circles in F20 represent periodic orbits. ▽

Before we introduce Sotomayor's Theorem, we first need a definition. Let $E \subset \mathbb{R}^n$ be open, $f: E \to \mathbb{R}^n$ be a function such that $f \in C^2(E)$. For $D^2 f(x_0): E \times E \to \mathbb{R}^n$ and for $(x, y) \in E \times E$, we define

$$D^2 f(x_0)(x,y) = \sum_{j_1, j_2 = 1}^{n} \frac{\partial^2 f(x_0)}{\partial x_{j_1} \partial x_{j_2}} x_{j_1} y_{j_2}.$$

We are now ready to introduce Sotomayor's Theorem.

THEOREM 29. [1] (Sotomayor's Theorem) Suppose $f(x_0, \mu_0) = 0$, and let $A = Df(x_0, \mu_0)$ have a simple eigenvalue $\lambda = 0$. Let v be an eigenvector of A, and w be a left eigenvector of A corresponding to $\lambda = 0$.

(1) If $w^T f_\mu(x_0,\mu_0) \neq 0$ and $w^T D^2 f(x_0,\mu_0)(v,v) \neq 0$, then there is a saddle - node bifurcation as μ passes through $\mu = \mu_0$.

(2) If $w^T f_\mu(x_0,\mu_0) = 0$, $w^T Df_\mu(x_0,\mu_0)v \neq 0$, and $w^T D^2 f(x_0,\mu_0)(v,v) \neq 0$, then there is a transcritical bifurcation as μ passes through $\mu = \mu_0$.

(3) If $w^T f_\mu(x_0,\mu_0) = 0$, $w^T Df_\mu(x_0,\mu_0)v \neq 0$, $w^T D^2 f(x_0,\mu_0)(v,v) = 0$, and

$w^T D^3 f(x_0,\mu_0)(v,v,v) \neq 0$, then there is a pitchfork bifurcation as passes through $\mu = \mu_0$. $\square$

Sotomayor's Theorem gives us sufficient conditions for the existence of either a saddle-node, transcritical, or pitchfork bifurcations (provided that the system meets certain conditions).

EXAMPLE 30. (Application)

Consider the system from a previous example

$$\begin{cases} \dot{S} = \mu(H-S) - aSf(B) \\ \dot{I} = aSf(B) - rI \\ \dot{B} = -\gamma B + eI \end{cases}$$

where the parameters $\mu > 0$, $a > 0$, $r > 0$, $\gamma > 0$, $e > 0$, $H > 0$, and $f(B) = \frac{B}{K+B}$, where $K > 0$. Here, we will consider e to be the bifurcation parameter for this system. Using Sotomayor's Theorem, we will prove that there is a transcritical bifurcation at the disease-free equilibrium point. Recall that the eigenvalues of the Jacobian of this system at the equilibrium point P_1 are $\lambda_1 = -\mu$, $\lambda_2 = \frac{-(\gamma+r)-\sqrt{(\gamma+r)^2 - \frac{4(\gamma rK - eaH)}{K}}}{2}$, and $\lambda_3 = \frac{-(\gamma+r)+\sqrt{(\gamma+r)^2 - \frac{4(\gamma rK - eaH)}{K}}}{2}$. Note that we have a simple eigenvalue, say $\lambda_2 = 0 \iff (\gamma+r) = \sqrt{(\gamma+r)^2 + \frac{4aeH}{K}}$. Solving this for e gives $e = \frac{\gamma rK}{aH} := e^*$. At this point, we would like to find the eigenvector v associated to $\lambda_2 = 0$. So, we have

$$Df(P_1)v = \begin{bmatrix} -\mu & 0 & -\frac{aH}{K} \\ 0 & -r & \frac{aH}{K} \\ 0 & e^* & -\gamma \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -\mu v_1 - \frac{aH}{K}v_3 = 0 \implies v_1 =$$

$-\frac{aH}{\mu K}v_3$. Also, we have $e^* v_2 - \gamma v_3 = 0 \implies v_2 = \frac{\gamma}{e^*}v_3$. Hence, we have $v = \begin{bmatrix} -\frac{aH}{\mu K}v_3 \\ \frac{\gamma}{e^*}v_3 \\ v_3 \end{bmatrix} =$

$\begin{bmatrix} -\frac{aH}{\mu K} \\ -\frac{\gamma}{e^*} \\ 1 \end{bmatrix}$. Now, we would like to find a left eigenvector w of $Df(P_1)$ associated to $\lambda_2 = 0$.

So, we get $w^T Df(P_1)^T = [w_1, w_2, w_3]^T \begin{bmatrix} -\mu & 0 & 0 \\ 0 & -r & e^* \\ -\frac{aH}{K} & \frac{aH}{K} & -\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -\mu w_1 = 0 \implies$

$w_1 = 0$. Also, $-rw_2 + e^* w_3 = 0 \implies w_2 = \frac{e^*}{r}w_3$. So, we have $w = \begin{bmatrix} 0 \\ \frac{e^*}{r}w_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^*}{r} \\ 1 \end{bmatrix}$. Now,

we need to find $f_{e^*}(S,I,B)$, which is $f_{e^*}(S,I,B) = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$. So, at P_1we have $f_{e^*}(P_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Hence, we have $w^T f_{e^*}(P_1) = [0, \frac{e^*}{r}, 1]^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$. Now, we need $Df_{e^*}(S,I,B)$, which is

$Df_{e^*}(S,I,B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. So, at the disease-free equilibrium point $(H,0,0)$, we have

$Df_{e^*}(H,0,0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Hence, we get

$$w^T Df_{e^*}(H,0,0)v = \begin{bmatrix} 0 & \frac{e^*}{r} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{aH}{\mu K} \\ \frac{\gamma}{e^*} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{e^*}{r} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{\gamma}{e^*} \end{bmatrix} = \frac{\gamma}{r} \neq 0.$$

Now, we need to find $D^2 f(H,0,0)(v,v)$. For this, we have

$\frac{\partial f_1}{\partial x_1} = -\mu - \frac{aB}{(K+B)}$, $\frac{\partial f_1}{\partial x_2} = 0$, $\frac{\partial f_1}{\partial x_3} = -\frac{aSK}{(K+B)^2}$, $\frac{\partial f_2}{\partial x_1} = \frac{aB}{(K+B)}$, $\frac{\partial f_2}{\partial x_2} = -r$, $\frac{\partial f_2}{\partial x_3} = \frac{aSK}{(K+B)^2}$, $\frac{\partial f_3}{\partial x_1} = 0$, $\frac{\partial f_3}{\partial x_2} = e$, and $\frac{\partial f_3}{\partial x_3} = -\gamma$, where $f_1 = \dot{S}$, $f_2 = \dot{I}$, $f_3 = \dot{B}$, $x_1 = S$, $x_2 = I$, and $x_3 = B$. Now, to find $D^2 f(H,0,0)(v,v)$, we need to compute $\sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} u_i v_j$. All we need is $\frac{\partial^2 f_i(x_0)}{\partial x_j \partial x_k}$ (where $i,j,k = 1,2,3$). From the information above, we have

$\frac{\partial^2 f_1}{\partial x_1^2} = 0$, $\frac{\partial^2 f_1}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f_1}{\partial x_1 \partial x_3} = -\frac{aK}{(K+B)^2}$, $\frac{\partial^2 f_1}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 f_1}{\partial x_2^2} = 0$, $\frac{\partial^2 f_1}{\partial x_2 \partial x_3} = 0$, $\frac{\partial^2 f_1}{\partial x_3 \partial x_1} = -\frac{aK}{(K+B)^2}$, $\frac{\partial^2 f_1}{\partial x_3 \partial x_2} = 0$, $\frac{\partial^2 f_1}{\partial x_3^2} = -\frac{2aSK}{(K+B)^3}$,

$\frac{\partial^2 f_2}{\partial x_1^2} = 0$, $\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{aK}{(K+B)^2}$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 f_2}{\partial x_2^2} = 0$, $\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = 0$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_1} = \frac{aK}{(K+B)^2}$, $\frac{\partial^2 f_2}{\partial x_3 \partial x_2} = 0$, $\frac{\partial^2 f_2}{\partial x_3^2} = -\frac{2aSK}{(K+B)^3}$,

$\frac{\partial^2 f_3}{\partial x_1^2} = 0$, $\frac{\partial^2 f_3}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f_3}{\partial x_1 \partial x_3} = 0$, $\frac{\partial^2 f_3}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 f_3}{\partial x_2^2} = 0$, $\frac{\partial^2 f_3}{\partial x_2 \partial x_3} = 0$, $\frac{\partial^2 f_3}{\partial x_3 \partial x_1} = 0$, $\frac{\partial^2 f_3}{\partial x_3 \partial x_2} = 0$, $\frac{\partial^2 f_3}{\partial x_3^2} = 0$. So the components of $D^2 f(H,0,0)(v,v)$ are as follows:

$\sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} u_i v_j = (-\frac{aK}{(K+B)^2})v_1 v_3 + (-\frac{aK}{(K+B)^2})v_3 v_1 + (\frac{2aSK}{(K+B)^3})v_3^2$
$= (-\frac{aK}{(K+B)^2})(\frac{-aH}{\mu K}) + (-\frac{aK}{(K+B)^2})(-\frac{aH}{\mu K}) + (\frac{2aSK}{(K+B)^3})(1)^2 = \frac{2a^2 H}{\mu (K+B)^2} + \frac{2aSK}{(K+B)^3}$,

$\sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} u_i v_j = (\frac{aK}{(K+B)^2})v_1 v_3 + (\frac{aK}{(K+B)^2})v_3 v_1 + (-\frac{2aSK}{(K+B)^3})v_3^2$
$= (\frac{aK}{(K+B)^2})(\frac{-aH}{\mu K}) + (\frac{aK}{(K+B)^2})(-\frac{aH}{\mu K}) + (-\frac{2aSK}{(K+B)^3})(1)^2 = -\frac{2a^2 H}{\mu (K+B)^2} - \frac{2aSK}{(K+B)^3}$, and

$\sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} u_i v_j = 0$. Hence, we have $D^2 f(H,0,0)(v,v) = \begin{bmatrix} \frac{2a^2 H}{\mu(K+B)^2} + \frac{2aSK}{(K+B)^3} \\ -\frac{2a^2 H}{\mu(K+B)^2} - \frac{2aSK}{(K+B)^3} \\ 0 \end{bmatrix}$.

Now, we have $w^T D^2 f(H,0,0)(v,v) = \begin{bmatrix} 0 & \frac{e^*}{r} & 1 \end{bmatrix} \begin{bmatrix} \frac{2a^2 H}{\mu(K+B)^2} + \frac{2aSK}{(K+B)^3} \\ -\frac{2a^2 H}{\mu(K+B)^2} - \frac{2aSK}{(K+B)^3} \\ 0 \end{bmatrix} = -\frac{2a^2 e^* H}{\mu r (K+B)^2} - \frac{2ae^* SK}{r(K+B)^3} \neq 0$, since all of the parameters are positive. Therefore, by Sotomayor's Theorem, there exists a transcritical bifurcation at $e = e^*$. ▽

As Sotomayor's Theorem can provide sufficient conditions for the existence of a bifurcation (either saddle-node, transcritical, or pitchfork), we now introduce a theorem that can prove the existence of a Hopf bifurcation. However, before we state this theorem, we first need to define a term. Consider the system

$$\begin{cases} \dot{x} = ax + by + p(x,y) \\ \dot{y} = cx + dy + q(x,y) \end{cases} \tag{0.20}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c \in \mathbb{R}$, $d \in \mathbb{R}$, $A = Df(0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $D = det(A) = ad - bc > 0$, $T = tr(A) = a + d = 0$, and $p(x,y)$, $q(x,y)$ are analytic functions defined as follows:

$p(x,y) = \sum_{i+j\geq 2} a_{ij}x^i y^j = (a_{20}x^2 + a_{11}xy + a_{02}y^2) + (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3)$, and

$q(x,y) = \sum_{i+j\geq 2} b_{ij}x^i y^j = (b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3)$.

We define the ***Lyapunov coefficient*** as follows:

$\sigma = \frac{-3\pi}{2bD^{3/2}}[ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) + c^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2ac(b_{02}^2 - a_{20}a_{02})$

$-2ab(a_{20}^2 - b_{20}b_{02}) - b^2(2a_{20}b_{20} + b_{11}b_{20}) + (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}) - (a^2 + bc)[3(cb_{03} - ba_{30})$

$+2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})]]$.

Now that we have defined the Lyapunov coefficient, we are now ready to state a theorem which proves the existence of a Hopf bifurcation.

THEOREM 31. [1] (Existence of Hopf bifurcation) Let σ be the Lyapunov coefficient. If $\sigma \neq 0$, then a Hopf bifurcation occurs at the origin of the planar system (0.20) at the bifurcation value $\alpha = 0$. Furthermore, if $\sigma < 0$, then there exists a unique stable limit cycle which bifurcates from the origin of (0.20) as α increases from zero and if $\sigma > 0$, then there exists a unique unstable limit cycle which bifurcates from the origin of (0.20) as α decreases. $\square$

4.2 The Codimension of a Bifurcation at a Nonhyperbolic Equilibrium Point

In this section, we introduce the concept of the codimension of a bifurcation at a nonhyperbolic equilibrium point. We will define the codimension of a bifurcation at a nonhyperbolic equilibrium point, and we will give examples to demonstrate how to determine the codimension of a bifurcation at a nonhyperbolic equilibrium point. Consider the system

$$\dot{x} = f(x, \mu), \ \mu \in \mathbb{R} \tag{0.21}$$

Before we can define the codimension of a bifurcation at a nonhyperbolic equilibrium point, we first introduce the idea of a structually stable vector field.

Let E be an open subset of $\mathbb{R}^n$. We say a vector field $f \in C^1(E)$ is ***structurally stable*** if there is an $\varepsilon > 0$ such that for all $g \in C^1(E)$ with $||f - g|| < \varepsilon$, f and g are topologically equivalent on E. In other words, f is structurally stable if for small changes in f, the qualitative behavior of solutions remains about the same. If a vector field f is not structrually stable, then we say f is ***structurally unstable***.

Let $f_0(x) = f(x, \mu_0)$ be a structurally unstable vector field. We define an ***unfolding of*** $f_0(x)$ to be a family of m-parameter vector fields that contains $f_0(x)$. We define a ***universal unfolding of*** $f_0(x)$ ***at a nonhyperbolic equilibrium point*** x_0 to be an unfolding of $f_0(x)$ with the additional condition that all the other unfoldings of $f_0(x)$ are homeomorphic (or topologically equivalent) to the family of m-parameter vector fields, in a neighborhood of x_0. The minimum number of parameters needed for (0.21) to be a universal unfolding of $f_0(x)$ at a nonhyperbolic equilibrium point x_0 is called the ***codimension of the bifurcation at*** x_0.

Before we give an example, we need to define the normal form of a system. The ***normal form of a system*** $\dot{x} = f(x)$ is basically a rewriting of the system, which is topologically equivalent to the original system, to help simplify the nonlinear part of the system. So, we may have a system that is complicated, and we would like to work with a simpler version of the system. The normal form of a system helps simplify the nonlinear part of a system, which is topologically equivalent to the system we started with.

EXAMPLE 32. Consider the saddle-node bifurcation, which has the normal form (up to time scaling)

$$\begin{cases} \dot{x} = -x^2 \\ \dot{y} = -y. \end{cases} \tag{0.22}$$

We first determine the behavior around $(0,0)$. First, we note that the Jacobian of (0.22) is

$$Df(x,y) = \begin{bmatrix} -2x & 0 \\ 0 & -1 \end{bmatrix}.$$

So, we have that the Jacobian of (0.22) at $(0,0)$ is

$$Df(0,0) = \begin{bmatrix} -2(0) & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of $Df(0,0)$ are

$\lambda_1 = 0$ and $\lambda_2 = -1$.

Since (at least) one of the eigenvalues of $Df(0,0)$ is zero, then by definition, $(0,0)$ is a nonhyperbolic equilibrium point. So, we cannot determine the behavior of (0.22) around $(0,0)$ using the Stable Manifold/Hartman-Grobman Theorems. However, the eigenvalues of $Df(0,0)$ tell some information about this system. Since there is exactly one eigenvalue of $Df(0,0)$ equal to zero, then Theorem 15 applies to (0.22). Let $\varphi(x)$ be the solution of the equation $-y = 0$ in a neighborhood of the origin (i.e., $\varphi(x) = 0$). Let $\Psi(x) = p(x, \varphi(x))$ be the expansion of $p(x,y)$ in a neighborhood of the origin (i.e., $\Psi(x) = -x^2$). Since $a_m = -1 < 0$ and $m = 2$ is even, then $(0,0)$ is a saddle-node.

Claim: Adding higher degree terms to the first equation of this system will not affect the behavior around $(0,0)$. In other words, $(0,0)$ is a saddle-node in the system

$$\begin{cases} \dot{x} = -x^2 + \mu_3 x^3 \\ \dot{y} = -y. \end{cases} \tag{0.23}$$

Before we prove this claim, we first note that as a consequence of adding higher degree terms, we get an extra equilibrium point. This can be seen as follows:

$-x^2 + \mu_3 x^3 = 0 \Longrightarrow x^2(1 - \mu_3 x) = 0 \Longrightarrow x = 0$ or $x = \frac{1}{\mu_3}$

$-y = 0 \Longrightarrow y = 0 \Longrightarrow$ The equilibrium points of the claimed system are $(0,0)$ and $(\frac{1}{\mu_3}, 0)$.

Now, the Jacobian of (0.23) is as follows:

$$Df(x,y) = \begin{bmatrix} -2x + 3\mu_3 x^2 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that the equilibrium point $(\frac{1}{\mu_3}, 0)$ is hyperbolic since

$$Df(\tfrac{1}{\mu_3}, 0) = \begin{bmatrix} -2(\frac{1}{\mu_3}) + 3\mu_3(\frac{1}{\mu_3})^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu_3} & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow \lambda_1 = \frac{1}{\mu_3} \text{ and } \lambda_2 = -1.$$

Since $\mu_3 \neq 0$, then the real part of both of the eigenvalues are nonzero. Therefore, by definition, $(\frac{1}{\mu_3}, 0)$ is a hyperbolic equilibrium point of (0.23).

We will now prove the claim. Notice that the Jacobian at $(0,0)$ is:

$$Df(0,0) = \begin{bmatrix} -2(0) + 3\mu_3(0)^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow \lambda_1 = 0 \text{ and } \lambda_2 = -1.$$

Since (at least) one of the eigenvalues is zero, then by definition, $(0,0)$ is a nonhyperbolic equilibrium point of (0.23). Notice that there is exactly one eigenvalue of $Df(0,0)$ is equal to zero . So Theorem 15 also applies to (0.23). Let $\varphi(x)$ be the solution of $-y=0$ (i.e., $\varphi(x)=0$). Let $\Psi(x)$ be the series expansion of $p(x,\varphi(x))$ (i.e., $\Psi(x)=-x^2+\mu_3 x^3$). Since it is the first term of $\Psi(x)$ that determines the behavior around $(0,0)$ and the first term of $\Psi(x)$ for (0.23) is identical to $\Psi(x)$ of (0.22), therefore, we reach the same conclusion. That is to say that $(0,0)$ is a saddle-node in (0.23). This proves the claim. □

Now consider the system

$$\begin{cases} \dot{x}=\mu_1+\mu_2 x-x^2 \\ \dot{y}=-y \end{cases} \tag{0.24}$$

If we translate the system so that $(\frac{\mu_2}{2},0)$ is the new origin by a change of variables defined by $\alpha=x-\frac{\mu_2}{2}$ and $\beta=y$. Then the system (0.24) becomes

$$\begin{cases} \dot{\alpha}=\mu-\alpha^2 \\ \dot{\beta}=-\beta \end{cases} \tag{0.25}$$

Hence, (0.25) is a universal unfolding of the original system of this example. Therefore, the saddle-node bifurcation is a codimension-1 bifurcation. ▽

EXAMPLE 33. Consider the pitch-fork bifurcation, which has a normal form (up to time scaling)

$$\begin{cases} \dot{x}=-x^3 \\ \dot{y}=-y \end{cases} \tag{0.26}$$

We first determine the behavior around $(0,0)$ in this system. First, we note that the Jacobian of (0.26) is

$$Df(x,y) = \begin{bmatrix} -3x^2 & 0 \\ 0 & -1 \end{bmatrix}.$$

So, we have that the Jacobian of (0.26) at $(0,0)$ is

$$Df(0,0) = \begin{bmatrix} -3(0)^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvalues of $Df(0,0)$ are

$\lambda_1 = 0$ and $\lambda_2 = -1$.

Since (at least) one of the eigenvalues of $Df(0,0)$ is zero, then by definition, $(0,0)$ is a nonhyperbolic equilibrium point. So, we cannot determine the behavior of (0.26) around $(0,0)$ using the Stable Manifold/Hartman-Grobman Theorems. However, the eigenvalues of $Df(0,0)$ tells some information about (0.26). Since there is exactly one eigenvalue of $Df(0,0)$ equal to zero, then Theorem 15 applies to (0.26). Let $\varphi(x)$ be the solution of the equation $-y = 0$ in a neighborhood of the origin (i.e., $\varphi(x) = 0$). Let $\Psi(x) = p(x, \varphi(x))$ be the expansion of $p(x,y)$ in a neighborhood of the origin (i.e., $\Psi(x) = -x^2$). Since $a_m = -1 < 0$ and $m = 2$ is even, then $(0,0)$ is a saddle-node.

Claim: Adding higher degree terms to the first equation of this system will not affect the behavior around $(0,0)$. In other words, $(0,0)$ is a saddle-node in the system

$$\begin{cases} \dot{x} = -x^3 + \mu_4 x^4 \\ \dot{y} = -y. \end{cases} \tag{0.27}$$

Before we prove this claim, we first note that as a consequence of adding higher degree terms, we get an extra equilibrium point. This can be seen as follows:

$-x^3 + \mu_4 x^4 = 0 \Longrightarrow -x^3(1 - \mu_4 x) = 0 \Longrightarrow x = 0$ or $x = \frac{1}{\mu_4}$

$-y = 0 \Longrightarrow y = 0 \Longrightarrow$ The equilibrium points of (0.27) are $(0,0)$ and $(\frac{1}{\mu_4}, 0)$.

Now, the Jacobian of (0.27) is as follows:

$$Df(x,y) = \begin{bmatrix} -3x^2+4\mu_4x^3 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that the equilibrium point $(\frac{1}{\mu_4},0)$ is hyperbolic since

$$Df(\tfrac{1}{\mu_4},0) = \begin{bmatrix} -2(\frac{1}{\mu_4})+3\mu_4(\frac{1}{\mu_4})^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\mu_4} & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow \lambda_1 = \frac{1}{\mu_4} \text{ and } \lambda_2 =$$

-1.

Since $\mu_4 \neq 0$, then the real part of both of the eigenvalues are nonzero. Therefore, by definition, $(\frac{1}{\mu_4},0)$ is a hyperbolic equilibrium point of (0.27).

We will now prove the claim. Notice that the Jacobian at $(0,0)$ is:

$$Df(0,0) = \begin{bmatrix} -2(0)+3\mu_4(0)^2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \Longrightarrow \lambda_1 = 0 \text{ and } \lambda_2 = -1.$$

Since (at least) one of the eigenvalues of $Df(0,0)$ is zero, then by definition, $(0,0)$ is a nonhyperbolic equilibrium point of (0.27). Notice that there is exactly one eigenvalue of $Df(0,0)$ that is equal to zero. So Theorem 15 applies to (0.27). Let $\varphi(x)$ be the solution of $-y=0$ (i.e., $\varphi(x)=0$). Let $\Psi(x)$ be the series expansion of $p(x,\varphi(x))$ (i.e., $\Psi(x) = -x^3+\mu_4x^4$). Since it is the first term of $\Psi(x)$ that determines the behavior around $(0,0)$ and the first term of $\Psi(x)$ for (0.27) is identical to $\Psi(x)$ of (0.26), therefore, we reach the same conclusion. That is to say that $(0,0)$ is a saddle-node in (0.27). This proves the claim. □

Now consider the system

$$\begin{cases} \dot{x} = \mu_1+\mu_2x+\mu_3x^2-x^3 \\ \dot{y} = -y \end{cases} \tag{0.28}$$

If we translate the system so that $(\frac{\mu_3}{3},0)$ is the new origin by change of coordinates defined by $\alpha = x-\frac{\mu_3}{3}$ and $\beta = y$. Then the system (0.28) become

$$\begin{cases} \dot{\alpha} = a + b\alpha - \alpha^3 \\ \dot{\beta} = -\beta \end{cases} \tag{0.29}$$

Hence, the above system is a universal unfolding of the original system of this example. Therefore, the pitchfork bifurcation is a codimension-2 bifurcation.

To determine the qualitative behavior of (0.29), note that for $b > 0$, the cubic equation $x^3 - bx - a = 0$ has three roots iff $a^2 < \frac{4b^3}{27}$, two roots iff $a^2 = \frac{4b^3}{27}$, and one root iff $a^2 > \frac{4b^3}{27}$. It turns out that the two curves of the bifurcation points intersect, giving what is called a cusp bifurcation. Here are some phase portraits of (0.29) to illustrate this idea. (See F21 - F23.)

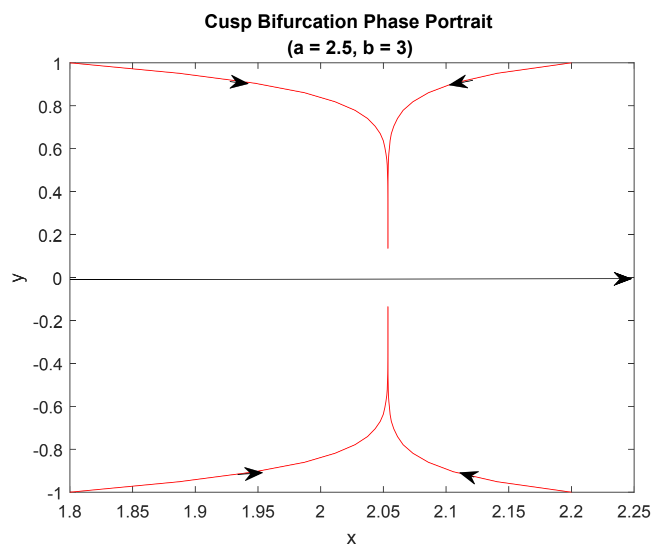


Figure 21. Phase portrait of cusp bifurcation (with one equilibrium point).

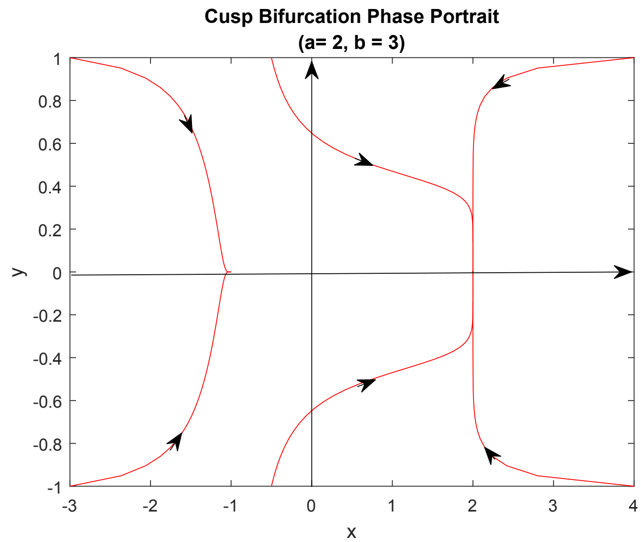


Figure 22. Phase portrait of cusp bifurcation (with two equilibrium points).

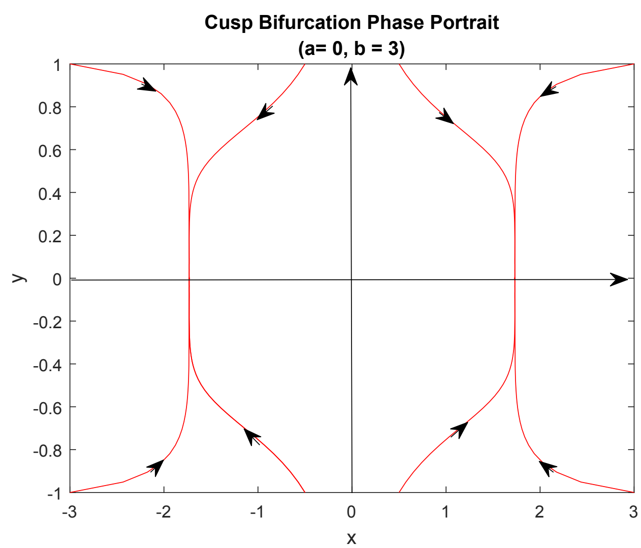


Figure 23. Phase portrait of cusp bifurcation (with three equilibrium points).

Now, here are some of the bifurcation diagrams of the cusp bifurcation (where $a = 1.2$, $b = 1$, $\alpha = 0.5$, and $\beta = 0.5$). (See F24 - F25.)

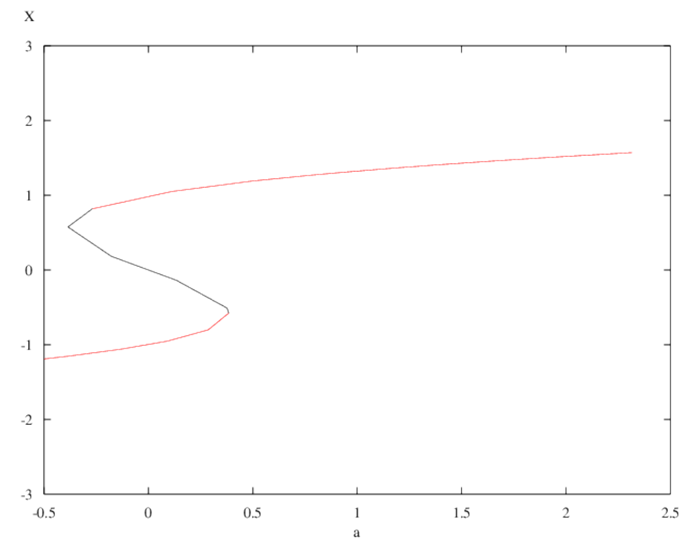


Figure 24. Bifurcation diagram of cusp bifurcation (on *ax*-axis).

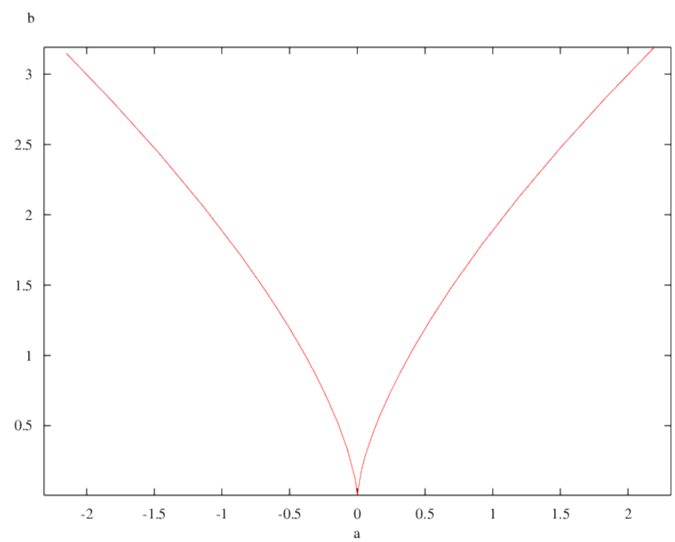


Figure 25. Two-parameter bifurcation diagram of cusp bifurcation (on *ab*-axis).

The first bifurcation diagram shows us that when $a < -0.3849...$ or $a > 0.3849...$ (roughly), there is only one equilibrium point (which is stable). It also shows that when $-0.3849... \leq a \leq 0.3849...$ (roughly), there are three equilibrium points (two of them are stable and one of them is unstable). So, we see that $(\mu_1, \mu_2) = (0, 0)$ is the bifurcation value.

Now, in F24, we let see a relationship between the solution $x(t)$ and the bifurcation parameter a.

In F25, we see that the two codimension-1 bifurcation curves collide at $(0,0)$, creating the codimension-2 cusp bifurcation. Also, we have that the region to the left side of the left curve of F25, there are no equilibrium points to the system (locally). Similarly, we have that the region to the right side of the right curve of F25, there are no equilibrium points (locally). Now, in the region between the two curves in F25, there are three equilibrium points (locally). $\triangledown$

REMARK 34. From the previous example, we see that two fold bifurcations (which are codimension-1 bifurcations) collide to give us a codimension-2 bifurcation at the origin. Later, we will see the case where two curves of *distinct* bifurcations will collide. $\triangle$

CHAPTER 5: CENTER MANIFOLD THEOREM

In Chapter 2, we discussed the Stable Manifold Theorem, which tells us that there exists a stable (and an unstable) manifold which behave similarly to the stable and unstable subspaces respectively of a linear system. The main assumption of the Stable Manifold Theorem is that the equilibrium point is hyperbolic. Note also that it does not mention about the existence of a center manifold.

Also, in Chapter 2, we mention the Hartman - Grobman Theorem, which tells us about the qualitative behavior of nonlinear dynamical systems around hyperbolic equilibrium points. Once again, the main assumption for the Hartman - Grobman Theorem is that the equilibrium point is hyperbolic.

In this chapter, we will look at what we need in order for a center manifold to exist, which will be called the Center Manifold Theorem. This theorem will be broken into two parts[5], since this theorem tells a lot about the center manifold. The equilibrium point will be arbitrary (hyperbolic or not).

5.1 The Center Manifold Theorem (Part I)

As mentioned above, the Center Manifold Theorem will be explained by breaking it into two parts. In this section, we will mention the first part of the Center Manifold Theorem.

THEOREM 35. (Center Manifold Theorem Part I) Let $E \subseteq \mathbb{R}^n$ be open, $x_0 = 0 \in E$, $f(0) = 0$, and $f \in C^r(E)$ (with $r \geq 1$). Assume $A = Df(0)$ has c eigenvalues with real part equal to zero (i.e., $Re(\lambda_j) = 0$, where λ_j $(j = 1, 2, ..., c)$ are the eigenvalues of A), s eigenvalues with $Re(\lambda_j) < 0$, and u eigenvalues with $Re(\lambda_j) > 0$ (i.e., $c + s + u = n$). Then there exists a c-dimensional invariant (center) manifold $W^c(0)$ of class C^r, which is tangent to the center subspace E^c at 0. □

[5]The Center Manifold Theorem (both parts combined) can be found in [1].

The main idea behind the first part of the Center Manifold Theorem is that the center manifold exists and is tangent to E^c. Now, before we move forward, we should make a few remarks regarding this part of the Center Manifold Theorem.

REMARK 36. The existence of the stable and unstable manifolds is also guaranteed with the Center Manifold Theorem. Also, the only thing this part of the center manifold theorem gives us is the existence of a center manifold, but no information about the dynamics of the system on the center manifold. $\triangle$

5.2 The Center Manifold Theorem (Part II)

Recall from the previous section that the only thing the first part of the center manifold theorem tells us is the existence of a center manifold. It does *not* tells us the behavior on the center manifold. So, how can we determine what happens on the center manifold? That is what the second part of the Center Manifold Theorem gives us. For simplicity, we assume that $u = 0$, and $x_0 = 0 \in \mathbb{R}^n$. We also assume that $Df(x_0)$ has no eigenvalues with $Re(\lambda_j) > 0$, for all $j = 1, 2, ..., n$. Note that the following theorem generalizes Theorem 15. We will denote the local center manifold with $W^c_{loc}(x_0)$.

THEOREM 37. (Center Manifold Theorem Part II) Let $E \subseteq \mathbb{R}^n$ be open, $x_0 = 0 \in E$, $f(0) = 0$, and $f \in C^r(E)$ (with $r \geq 1$). Assume $A = Df(0)$ has c eigenvalues with real part equal to zero (i.e., $Re(\lambda_j) = 0$, where λ_j $(j = 1, 2, ..., n)$ are the eigenvalues of A), and s eigenvalues with $Re(\lambda_j) < 0$, (i.e., $c + s = n$). Then the system $\dot{x} = f(x)$ can be written as

$$\begin{cases} \dot{x} = Cx + F(x,y) \\ \dot{y} = Py + G(x,y) \end{cases} \tag{0.30}$$

where $x \in \mathbb{R}^c$, $y \in \mathbb{R}^s$, $F: \mathbb{R}^n \to \mathbb{R}^c$, $G: \mathbb{R}^n \to \mathbb{R}^s$, $F(0,0) = 0$, $G(0,0) = 0$, $DF(0,0) = 0$, and $DG(0,0) = 0$. Furthermore, there is $\delta > 0$ and a function $h \in C^r(N_\delta(0))$ such that $W^c_{loc}(0) = \{(x,y) \in \mathbb{R}^c x \mathbb{R}^s \ : \ y = h(x), \ ||x|| < \delta\}$, with $h(0) = 0$, $Dh(0) = 0$, and satisfies

$$Dh(x)[Cx + F(x, h(x))] - Ph(x) - G(x, h(x)) = 0. \tag{0.31}$$

Also, the flow on the center manifold $W^c(0)$ can be defined by the system

$$\dot{x} = Cx + F(x, h(x)). \; \square \tag{0.32}$$

There are some remarks that should be mentioned here before we look at an example of how to use this theorem.

REMARK 38. We have that (0.31) helps us find the function h that approximates the center manifold, and (0.32) helps us determine the qualitative behavior on the center manifold. $\triangle$

We also have the following remark.

REMARK 39. While the statement of the second part of the theorem assumes that there are no eigenvalues with positive real part (i.e., $u = 0$), it turns out that basically we get the same result even when $u \neq 0$. $\triangle$

5.3 Example

Consider the system

$$\begin{cases} \dot{x_1} = x_1 y - x_1 x_2^2 \\ \dot{x_2} = x_2 y - x_1^2 x_2 \\ \dot{y} = -y + x_1^2 + x_2^2. \end{cases} \tag{0.33}$$

It can be shown that (0.30) can be written as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} + \begin{bmatrix} x_1y - x_1x_2^2 \\ x_2y - x_1^2x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

where $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $P = -1$, $F(x_1, x_2, y) = \begin{bmatrix} x_1y - x_1x_2^2 & x_2y - x_1^2x_2 \end{bmatrix}^T$, and $G(x_1, x_2, y) = x_1^2 + x_2^2$. Let $h(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + \dots$. Then $Dh(x_1, x_2) = \begin{bmatrix} 2ax_1 + bx_2 + \dots & bx_1 + 2cx_2 + \dots \end{bmatrix}^T$, and we have $h(0,0) = 0$, and $Dh(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. Substituting these into (0.31) gives us

$\begin{bmatrix} 2ax_1 + bx_2 + \dots & bx_1 + 2cx_2 + \dots \end{bmatrix} \begin{bmatrix} ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + \dots - x_1x_2^2 \\ ax_1^2x_2 + bx_1x_2^2 + cx_2^3 + \dots - x_1^2x_2 \end{bmatrix} + (ax_1^2 + bx_1x_2 + cx_2^2 + \dots) - x_1^2 - x_2^2 = 0$

Notice that when you multiply out the first part of the equation, the smallest power of the polynomial is 4, so we will overlook this part and focus on the last part of the equation. With this in mind, we have

$$(a-1)x_1^2 + bx_1x_2 + (c-1)x_2^2 + \dots = 0$$

This gives us $a = 1$, $b = 0$, $c = 1$, and so on. If we need more terms, we just go out a few more terms in the function h as defined above and follow the procedure we went through above. Hence, we get that $h(x_1, x_2) = x_1^2 + x_2^2 + O(||x||^3)$. So, the Center Manifold Theorem (Part II) tells us that the center manifold looks like the surface $h(x_1, x_2) = x_1^2 + x_2^2$ (locally).

Now, to determine the behavior on the center manifold, we substitute the function h into the system (0.32), which gives us (plus $O(||x||^3)$)

$$\begin{cases} \dot{x_1} = x_1^3 \\ \dot{x_2} = x_2^3. \end{cases} \tag{0.34}$$

To determine the behavior of the system (0.34), we can use polar coordinates to see that $r^2 = x_1^2 + x_2^2 \implies r\dot{r} = x_1\dot{x_1} + x_2\dot{x_2} \implies \dot{r} = \frac{x_1^4+x_2^4}{r} > 0$ (since $r > 0$). Therefore, if our initial value lands on the center manifold, then solutions would go *away* from the origin.

Here is the phase portrait of the center manifold in this example. So, we can see that the center manifold is approximated by the surface of the paraboloid $h(x_1, x_2) = x_1^2 + x_2^2$. Note that we did not give some solutions of what happens outside of the center manifold. This is because outside the center manifold, the origin is hyperbolic; so, we can use the Stable Manifold and Hartman-Grobman theorems. ▽

Now, let us look at a phase portrait to illustrate a center manifold from the example in this section. (See F26.)

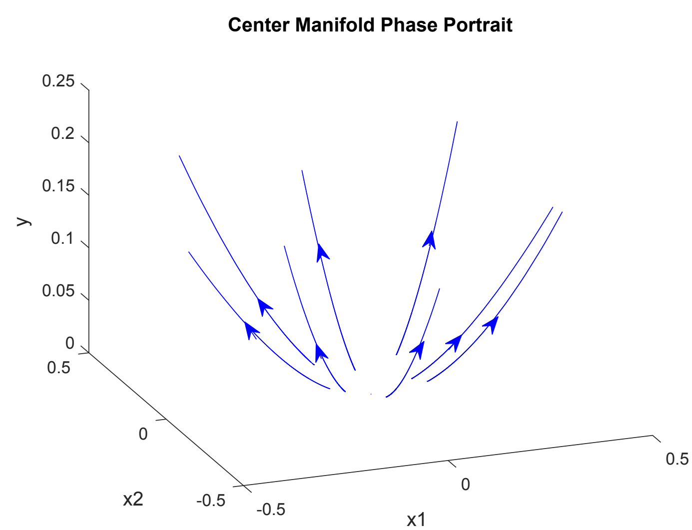


Figure 26. Phase portrait to describe the center manifold of (0.34).

REMARK 40. The Center Manifold Theorem tells us that if $Df(x_0)$ has c eigenvalues with $Re(\lambda_j) = 0$, we can restrict our study to a c-dimensional system to describe the behavior of solutions on a c-dimensional center manifold. This fact will be used in the next chapter. $\triangle$

CHAPTER 6: BOGDANOV - TAKENS BIFURCATION

In Chapter 4, we discussed several bifurcations. It turns out that most of the bifurcations we discussed are codimension-1 bifurcations. In this chapter, we discuss a special codimension-2 bifurcation: a Bogdanov-Takens bifurcation. We also state and provide the details of a proof which shows under certain conditions that there exists a Bogdanov-Takens bifurcation. We will provide some examples where a Bogdanov-Takens bifurcation occurs as well.

6.1 Bogdanov - Takens Bifurcation

A Bogdanov-Takens bifurcation is an example of a codimension-2 bifurcation, which is by far more complex than codimension-1 bifurcations, and it describes very rich dynamics of the given system. The basic idea of what happens with a Bogdanov-Takens bifurcation is that we have two codimension-1 bifurcation curves that collide at a single point. The point where the two bifurcation curves collide is where the Bogdanov-Takens bifurcation happens.

6.2 The Bogdanov - Takens Bifurcation Existence Theorem

Recall from a previous section that Sotomayor's Theorem proves the existence of a saddle-node, transcritical, or pitchfork bifurcations (provided that certain conditions hold for the given system). It turns out that there is a theorem that is "similar" to Sotomayor's Theorem to prove the existence of a bifurcation. The theorem is similar in the sense that provided certain conditions of the system hold implies the existence of a Bogdanov-Takens bifurcation. The difference is that Sotomayor's Theorem can prove the existence of one of three types of codimension-1 bifurcations, while the theorem in this section can prove the existence of only one type of codimension-2 bifurcation.

In this section, we will not only state this theorem, but we will also provide the details of the proof of this theorem; this is the main contribution of this thesis. Before we discuss this theorem, we mention that while we are assuming that we have a 2-dimensional system in the statement of the theorem, our system can be n-dimensional (for an example, see section 5.3). This is because one of the conditions for existence of a Bogdanov-Takens bifurcation is that the Jacobian of our system has exactly two eigenvalues with zero real part (i.e., a Bogdanov-Takens condition), and we do not worry what happens in the other $(n-2)$-dimensions of the system. We do not worry about what happens in the other $(n-2)$-dimensions since the origin (the equilibrium point) will be hyperbolic in those $(n-2)$-dimensions; so the Stable Manifold and Hartman-Grobman Theorems apply to the other $(n-2)$-dimensions. So, by the Center Manifold Theorem, it is enough to study a 2-dimensional system rather than a n-dimensional system. See also Remark 40.

The proof of the following theorem consists of applying a series of changes of variables as well as time and space rescalings so that a given system can be expressed in a simple polynomial form, locally.

THEOREM 41. [2] (Existence of Bogdanov-Takens Bifurcation) Suppose that a planar system $\dot{x} = f(x,\alpha)$, $x \in \mathbb{R}^2$, $\alpha \in \mathbb{R}^2$, with smooth f, at $\alpha = 0$ the equilibrium $x = (x_1, x_2) = (0,0)$ with a double zero eigenvalue $\lambda_{1,2}(0) = 0$. Assume that the following genericity conditions are satisfied:

the Jacobian matrix

$$A(0) = Df(0,0) \neq 0; \tag{0.35}$$

$$a_{20}(0) + b_{11}(0) \neq 0; \tag{0.36}$$

$$b_{20}(0) \neq 0; \tag{0.37}$$

the map

$$(x,\alpha) \mapsto (f(x,\alpha), tr(\frac{\partial f(x,\alpha)}{\partial x}), det(\frac{\partial f(x,\alpha)}{\partial x})). \tag{0.38}$$

is regular at point $(x,\alpha) = (0,0)$.

Then there exists a smooth invertible variable transformations smoothly depending on the parameters, a direction-preserving time reparametrization, and smooth invertible parameter changes, which together reduce the system to

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + A^2\eta_1^2 \pm B^2\eta_1\eta_2 + O(||\eta||^3). \end{cases} \qquad \square$$

PROOF. We will prove this theorem in steps. Consider the planar system

$$\dot{x} = f(x,\alpha),\ x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}^2 \tag{0.39}$$

where f is smooth. Suppose that (0.39) has, at $\alpha = 0$, the equilibrium $x = (x_1, x_2) = (0,0)$ with two zero eigenvalues, $\lambda_{1,2}(0) = 0$.

(***Step 0***)

By Taylor's Theorem, we can write (0.39) (at $\alpha = 0$) in the form

$$\dot{x} = A_0 x + F(x), \tag{0.40}$$

where $A_0 = Df(0,0)$ and $F(x) = f(x,0) - A_0 x$ is a smooth function, and $F(x) = O(||x||^2)$. We have $det(A_0) = 0$ since $det(A_0) = \lambda_1\lambda_2 = (0)(0) = 0$, and $tr(A_0) = 0$ since $tr(A_0) = \lambda_1 + \lambda_2 = 0 + 0 = 0$. Assume (0.35) holds. By Theorem 7, we can find two linearly independent vectors $v_{0,1} \in \mathbb{R}^2$ such that

$$A_0 v_0 = 0,\; A_0 v_1 = v_0. \tag{0.41}$$

In particular, v_0 is an eigenvector of A_0 corresponding to the eigenvalue 0, and v_1 is a generalized eigenvector of A_0 corresponding to the eigenvalue 0, respectively. Furthermore, again from Theorem 7, there exist two linearly independent left eigenvector (and generalized left eigenvector) $w_{0,1} \in \mathbb{R}^2$ of the matrix A_0 such that

$$A_0^T w_1 = 0,\; A_0^T w_0 = w_1. \tag{0.42}$$

Since v_0 and w_1 are generalized eigenvectors, then they are not uniquely determined. Furthermore, we have that the vectors v_1 and w_0 are not uniquely determined, even if the vectors v_0 and w_1 are fixed. (For example, if v_1 is a fixed solution of the second equation of (0.41), then the vector $v = v_1 + \gamma v_0$ is also a solution for any $\gamma \in \mathbb{R}$ since $A_0 v = A_0(v_1 + \gamma v_0) = A_0 v_1 + \gamma A_0 v_0 = v_0 + 0 = v_0$). However, we claim that we can find four vectors that satisfy (0.41), and (0.42) such that

$$v_0^T w_0 = v_1^T w_1 = 1. \tag{0.43}$$

where $x^T y$ is the standard dot product of two vectors $x, y \in \mathbb{R}^2$. To show this, we first show that $v_0^T w_0 = v_1^T w_1$. Using the relations from (0.41) and (0.42), we have $v_0^T w_0 = (A_0 v_1)^T w_0 = (v_1^T A_0^T) w_0 = v_1^T (A_0^T w_0) = v_1^T w_1$. Now, we use a property about the dot product of two vectors; in particular, for the vectors v_0, w_0: $v_0^T w_0 = ||v_0|| \times ||w_0|| \times cos(\theta)$, where θ is the angle between the vectors v_0 and w_0. Now, we can control the vectors v_0, w_0 so that $||v_0|| \times ||w_0|| = \frac{1}{cos(\theta)}$ so that $v_0^T w_0 = 1$ (and then using a similar argument to show that $v_1^T w_1 = 1$).

We also claim that we have

$$v_0^T w_1 = v_1^T w_0 = 0. \tag{0.44}$$

We first show that $v_0^T w_1 = 0$. Using the relations from (0.41) and (0.42), we have that $v_0^T w_1 = (A_0 v_1)^T w_1 = (v_1^T A_0^T) w_1 = v_1^T (A_0^T w_1) = v_1^T * 0 = 0$. To show that $v_1^T w_0 = 0$, consider the linear system

$$v_1 x = v_0, \tag{0.45}$$

where $x \in \mathbb{R}$. Since v_0 and v_1 are linearly independent, then x is not a solution of the system (0.45). By the Fredholm Alternative Theorem, we have that (in particular for $y = w_0$) $v_1^T w_0 = 0$ has a solution with $w_0^T v_0 = v_0^T w_0 = 1 \neq 0$. Hence, we have that (0.44) holds.

Select $\{v_0, v_1\}$ as a basis for $\mathbb{R}^2$, then any $x \in \mathbb{R}^2$ can be uniquely written as

$$x = y_1 v_0 + y_2 v_1 \tag{0.46}$$

by definition of $\{v_0, v_1\}$ being a basis for $\mathbb{R}^2$, for some $y_1, y_2 \in \mathbb{R}$. Using the relations (0.41), (0.42), and (0.46), we have $x^T w_0 = (y_1 v_0 + y_2 v_1)^T w_0 = y_1 v_0^T w_0 + y_2 v_1^T w_0 = y_1 * 1 + y_2 * 0 = y_1$. A similar calculation shows that $x^T w_1 = y_2$. Now, the new coordinates are given by

$$\begin{cases} y_1 = x^T w_0 \\ y_2 = x^T w_1 \end{cases} \tag{0.47}$$

Again, using the relations (0.41) and (0.42) (along with (0.40)) gives us (for $\alpha = 0$)

$\dot{y_1} = \dot{x}^T w_0$

$$\dot{y_1} = (A_0 x + F(x))^T w_0$$

$$\dot{y_1} = (x^T A_0^T + F(x)^T) w_0$$

$$\dot{y_1} = x^T A_0^T w_0 + F(x)^T w_0$$

$$\dot{y_1} = x^T w_1 + F(x)^T w_0$$

$$\dot{y_1} = y_2 + F(y_1 v_0 + y_2 v_1)^T w_0$$

A similar argument gives $\dot{y_2} = F(y_1 v_0 + y_2 v_1)^T w_1$. Hence, our new system takes the form (for $\alpha = 0$)

$$\begin{cases} \dot{y_1} = y_2 + F(y_1 v_0 + y_2 v_1)^T w_0 \\ \dot{y_2} = F(y_1 v_0 + y_2 v_1)^T w_1. \end{cases} \tag{0.48}$$

For $\alpha \neq 0$, we can write the system (0.39) in terms of the system (0.47). This would gives us

$$\begin{cases} \dot{y_1} = f(y_1 v_0 + y_2 v_1, \alpha)^T w_0 \\ \dot{y_2} = f(y_1 v_0 + y_2 v_1, \alpha)^T w_1 \end{cases} \tag{0.49}$$

At this point, we can represent the right hand side of (0.49) as a Taylor series at $y = (y_1, y_2) = (0,0)$. Doing this gives us:

$$\begin{cases} \dot{y_1} = y_2 + a_{00}(\alpha) + a_{10}(\alpha) y_1 + a_{01}(\alpha) y_2 + \frac{1}{2} a_{20}(\alpha) y_1^2 + a_{11}(\alpha) y_1 y_2 + \frac{1}{2} a_{02}(\alpha) y_2^2 + P_1(y, \alpha) \\ \dot{y_2} = b_{00}(\alpha) + b_{10}(\alpha) y_1 + b_{01}(\alpha) y_2 + \frac{1}{2} b_{20}(\alpha) y_1^2 + b_{11}(\alpha) y_1 y_2 + \frac{1}{2} b_{02}(\alpha) y_2^2 + P_2(y, \alpha) \end{cases}, \tag{0.50}$$

where $a_{kl}(\alpha)$, $b_{kl}(\alpha)$ (with $k = 0,1,2,\ l = 0,1,2$), and $P_{1,2}(y, \alpha) = O(||y||^3)$ are smooth functions of their arguments, and (for example)

$$a_{11}(\alpha) = \frac{\partial^2}{\partial y_1 \partial y_2}[f(y_1 v_0 + y_2 v_1, \alpha)^T w_0] \mid_{y=0}.$$

We claim that

$$a_{00}(0) = a_{10}(0) = a_{01}(0) = b_{00}(0) = b_{10}(0) = b_{01}(0) = 0. \tag{0.51}$$

We first prove that $a_{00}(0) = b_{00}(0) = 0$. Observe that

$$a_{00}(\alpha) = [f(y_1 v_0 + y_2 v_1, \alpha)^T w_0] \mid_{y=0} = f(0, \alpha)^T w_0$$

$$b_{00}(\alpha) = [f(y_1 v_0 + y_2 v_1, \alpha)^T w_1] \mid_{y=0} = f(0, \alpha)^T w_1.$$

For $\alpha = 0$, we have that $a_{00}(0) = f(0,0)^T w_0$. Since $x = 0$ is an equilibrium point of (0.39) (at $\alpha = 0$), then, by definition of an equilibrium point, we have that $f(0,0) = 0$. Hence, we have that $a_{00}(0) = f(0,0)^T w_0 = 0^T w_0 = 0$. A similar argument shows that $b_{00}(0) = 0$. Hence, we just proved that $a_{00}(0) = b_{00}(0) = 0$.

Now, to prove that $a_{10}(0) = a_{01}(0) = b_{10}(0) = b_{01}(0) = 0$, observe that (0.50) can be rewritten in the following way

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = \begin{bmatrix} a_{10}(\alpha) & a_{01}(\alpha) \\ b_{10}(\alpha) & b_{01}(\alpha) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} a_{00}(\alpha) + \frac{1}{2}a_{20}(\alpha)y_1^2 + a_{11}(\alpha)y_1 y_2 + \frac{1}{2}a_{02}(\alpha)y_2^2 + P_1(y,\alpha) \\ b_{00}(\alpha) + \frac{1}{2}b_{20}(\alpha)y_1^2 + b_{11}(\alpha)y_1 y_2 + \frac{1}{2}b_{02}(\alpha)y_2^2 + P_2(y,\alpha) \end{bmatrix}.$$

Let us focus on the matrix $\begin{bmatrix} a_{10}(\alpha) & a_{01}(\alpha) \\ b_{10}(\alpha) & b_{01}(\alpha) \end{bmatrix}$. At $\alpha = 0$, the matrix gives us $\begin{bmatrix} a_{10}(0) & a_{01}(0) \\ b_{10}(0) & b_{01}(0) \end{bmatrix}$. Since we get the system (0.50) at $\alpha = 0$, the matrix gives us $\begin{bmatrix} a_{10}(0) & a_{01}(0) \\ b_{10}(0) & b_{01}(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This gives us that $a_{10}(0) = b_{10}(0) = b_{01}(0) = 0$. However, since y_2 is accounted for in (0.50), then in order for the first equation of (0.50) to hold, then $a_{01}(0) = 0$. Hence, we have proven that (0.51) holds.

This completes ***step 0***.

(***<u>Step 1</u>***)

We will now define new coordinates (u_1, u_2), where u_2 is just $\dot{y_1}$ from (0.50) and u_1 is just y_1, i.e.,

$$\begin{cases} u_1 = y_1 \\ u_2 = y_2 + a_{00}(\alpha) + a_{10}(\alpha)y_1 + a_{01}(\alpha)y_2 + \frac{1}{2}a_{20}(\alpha)y_1^2 + a_{11}(\alpha)y_1y_2 + \frac{1}{2}a_{02}(\alpha)y_2^2 + P_1(y,\alpha) \end{cases} \tag{0.52}$$

Now, we will take the derivative to both sides of each equation in (0.52) with respect to time t. Now, to make the calculations easier, we will remove α from the expression $a_{kl}(\alpha)$ (i.e., $a_{kl} = a_{kl}(\alpha)$). For the first equation, we get

$$\dot{u_1} = \dot{y_1}$$

$$\dot{u_1} = y_2 + a_{00} + a_{10}y_1 + a_{01}y_2 + \frac{1}{2}a_{20}y_1^2 + a_{11}y_1y_2 + \frac{1}{2}a_{02}y_2^2 + P_1(y,.)$$

Hence, we have that $\dot{u_1} = u_2$. For the second equation, we get

$$\dot{u_2} = \dot{y_2} + a_{10}\dot{y_1} + a_{01}\dot{y_2} + a_{20}y_1\dot{y_1} + a_{11}(y_1\dot{y_2} + \dot{y_1}y_2) + a_{02}y_2\dot{y_2} + \dot{P_1}(y,.)$$

$$\dot{u_2} = (b_{00} + b_{10}y_1 + b_{01}y_2 + \tfrac{1}{2}b_{20}y_1^2 + b_{11}y_1y_2 + \tfrac{1}{2}b_{02}y_2^2 + P_2(y,.)) + a_{10}\dot{y_1} + a_{01}(b_{00} + b_{10}y_1 + b_{01}y_2 + \tfrac{1}{2}b_{20}y_1^2 + b_{11}y_1y_2 + \tfrac{1}{2}b_{02}y_2^2 + P_2(y,.))$$
$$+a_{20}y_1\dot{y_1} + a_{11}y_1(b_{00} + b_{10}y_1 + b_{01}y_2 + \tfrac{1}{2}b_{20}y_1^2 + b_{11}y_1y_2 + \tfrac{1}{2}b_{02}y_2^2 + P_2(y,.)) + a_{11}\dot{y_1}y_2$$
$$+a_{02}y_2(b_{00} + b_{10}y_1 + b_{01}y_2 + \tfrac{1}{2}b_{20}y_1^2 + b_{11}y_1y_2 + \tfrac{1}{2}b_{02}y_2^2 + P_2(y,.)) + P_1(\dot{y},.)$$

$$\dot{u_2} = b_{00} + b_{10}y_1 + b_{01}y_2 + \tfrac{1}{2}b_{20}y_1^2 + b_{11}y_1y_2 + \tfrac{1}{2}b_{02}y_2^2 + P_2(y,.) + a_{10}\dot{y_1} + a_{01}b_{00} + a_{01}b_{10}y_1 + a_{01}b_{01}y_2 + \tfrac{1}{2}a_{01}b_{20}y_1^2 + a_{01}b_{11}y_1y_2 + \tfrac{1}{2}a_{01}b_{02}y_2^2 + a_{01}P_2(y,.)$$
$$+a_{20}y_1\dot{y_1} + b_{00}a_{11}y_1 + b_{10}a_{11}y_1^2 + b_{01}a_{11}y_1y_2 + \tfrac{1}{2}b_{20}a_{11}y_1^3 + b_{11}a_{11}y_1^2y_2 + \tfrac{1}{2}b_{02}a_{11}y_1y_2^2$$
$$+a_{11}y_1P_2(y,.) + a_{11}\dot{y_1}y_2 + b_{00}a_{02}y_2 + b_{10}a_{02}y_1y_2 + b_{01}a_{02}y_2^2 + \tfrac{1}{2}b_{20}a_{02}y_1^2y_2 + b_{11}a_{02}y_1y_2^2 + \tfrac{1}{2}b_{02}a_{02}y_2^3 + a_{02}y_2P_2(y,.) + \dot{P_1}(y,.)$$

Now, using the relations $u_1 = y_1$, $u_2 = \dot{y_1}$, letting $y_2 = u_2 - a_{00} - a_{10}u_1 - \ldots$, and substituting these into the most recent equation given above gives us:

$$\dot{u_2} = b_{00} + b_{10}u_1 + b_{01}(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}b_{20}u_1^2 + b_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}b_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)^2 + \tilde{P_2}(u,.) + a_{10}u_2 + a_{01}b_{00} + a_{01}b_{10}u_1 + a_{01}b_{01}(u_2 - a_{00} - a_{10}u_1 - \ldots)$$
$$+\tfrac{1}{2}a_{01}b_{20}u_1^2 + a_{01}b_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}a_{01}b_{02}(u_2 - a_{00} - a_{10}u_1)^2 + a_{01}\tilde{P_2}(u,.) + \tfrac{1}{2}a_{01}b_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)^2 + a_{01}\tilde{P_2}(u,.) + a_{20}u_1u_2 + b_{00}a_{11}u_1 + b_{10}a_{11}u_1^2$$
$$+b_{01}a_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}b_{20}a_{11}u_1^3 + b_{11}a_{11}u_1^2(u_2 - a_{00} - a_{10}u_1 - \ldots)$$
$$+\tfrac{1}{2}b_{02}a_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots)^2 + a_{11}u_1\tilde{P_2}(u,.) + a_{11}u_2(u_2 - a_{00} - a_{10}u_1 - \ldots)$$
$$+b_{00}a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{10}a_{02}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{01}a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)^2 + \tfrac{1}{2}b_{20}a_{02}u_1^2(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{11}a_{02}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots)^2$$
$$+\tfrac{1}{2}b_{02}a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)^3 + a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)\tilde{P_2}(u,.) + \tilde{P_1}(u,.)$$

$$\dot{u_2} = b_{00} + b_{10}u_1 + b_{01}(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}b_{20}u_1^2 + b_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}b_{02}(a_{00}^2 + 2a_{00}a_{10}u_1 - 2a_{00}u_2 + a_{10}^2u_1^2 - 2a_{10}u_1u_2 + u_2^2 - \ldots) + \tilde{P_2}(u,.) + a_{10}u_2 + a_{01}b_{00} + a_{01}b_{10}u_1$$
$$+a_{01}b_{01}(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}a_{01}b_{20}u_1^2 + a_{01}b_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \tfrac{1}{2}a_{01}b_{02}(a_{00}^2$$
$$+2a_{00}a_{10}u_1 - 2a_{00}u_2 + a_{10}^2u_1^2 - 2a_{10}u_1u_2 + u_2^2 - \ldots) + a_{01}P_2(y,.) + a_{20}u_1u_2 + b_{00}a_{11}u_1$$

$+b_{10}a_{11}u_1^2 + b_{01}a_{11}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + \frac{1}{2}b_{20}a_{11}u_1^3 + b_{11}a_{11}u_1^2(u_2 - a_{00} - a_{10}u_1 - \ldots) + \frac{1}{2}b_{02}a_{11}u_1(a_{00}^2 + 2a_{00}a_{10}u_1 - 2a_{00}u_2 + a_{10}^2u_1^2 - 2a_{10}u_1u_2 + u_2^2 - \ldots) + a_{11}u_1\tilde{P}_2(u,.)$

$+a_{11}u_2(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{00}a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{10}a_{02}u_1(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{01}a_{02}(a_{00}^2 + 2a_{00}a_{10}u_1 - 2a_{00}u_2 + a_{10}^2u_1^2 - 2a_{10}u_1u_2 + u_2^2 - \ldots)$

$+\frac{1}{2}b_{20}a_{02}u_1^2(u_2 - a_{00} - a_{10}u_1 - \ldots) + b_{11}a_{02}u_1(a_{00}^2 + 2a_{00}a_{10}u_1 - 2a_{00}u_2 + a_{10}^2u_1^2 - 2a_{10}u_1u_2 + u_2^2 - \ldots) + \frac{1}{2}b_{02}a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)^3 + a_{02}(u_2 - a_{00} - a_{10}u_1 - \ldots)\tilde{P}_2(u,.) + \tilde{P}_1(u,.)$

$\dot{u_2} = b_{00} + b_{10}u_1 + b_{01}u_2 - a_{00}b_{01} - a_{10}b_{01}u_1 - b_{01}(\ldots) + \frac{1}{2}b_{20}u_1^2 + b_{11}u_1u_2 - a_{00}b_{11}u_1 - a_{10}b_{11}u_1^2 - b_{01}(\ldots) + \frac{1}{2}b_{02}a_{00}^2 + a_{00}a_{10}b_{02}u_1 - a_{00}b_{02}u_2 + \frac{1}{2}a_{10}^2b_{02}u_1^2 - a_{10}b_{02}u_1u_2 + \frac{1}{2}b_{02}u_2^2 - \frac{1}{2}b_{02}(\ldots)$

$+\tilde{P}_2(u,.) + a_{10}u_2 + a_{01}b_{00} + a_{01}b_{10}u_1 + a_{01}b_{01}u_2 - a_{00}a_{01}b_{01} - a_{10}a_{01}b_{01}u_1 - a_{01}b_{01}(\ldots) + \frac{1}{2}a_{01}b_{20}u_1^2 + a_{01}b_{11}u_1u_2 - a_{00}a_{01}b_{11}u_1 - a_{10}a_{01}b_{11}u_1^2 - a_{01}b_{01}u_1(\ldots)$

$+\frac{1}{2}a_{01}b_{02}a_{00}^2 + a_{00}a_{10}a_{01}b_{02}u_1 - a_{00}a_{01}b_{02}u_2 + \frac{1}{2}a_{10}^2a_{01}b_{02}u_1^2 - a_{10}a_{01}b_{02}u_1u_2 + \frac{1}{2}a_{01}b_{02}u_2^2 - \frac{1}{2}a_{01}b_{02}(\ldots) + a_{01}\tilde{P}_2(u,.) + a_{20}u_1u_2 + b_{00}a_{11}u_1 + b_{10}a_{11}u_1^2 + b_{01}a_{11}u_1u_2$

$-a_{00}b_{01}a_{11}u_1 - a_{10}b_{01}a_{11}u_1^2 - b_{01}a_{11}u_1(\ldots) + \frac{1}{2}b_{20}a_{11}u_1^3 + b_{11}a_{11}u_1^2u_2 - a_{00}b_{11}a_{11}u_1^2$

$-a_{10}b_{11}a_{11}u_1^3 - b_{11}a_{11}u_1^2(\ldots) + \frac{1}{2}a_{00}^2b_{02}a_{11}u_1 + a_{00}a_{10}b_{02}a_{11}u_1^2 - a_{00}b_{02}a_{11}u_1u_2$

$+\frac{1}{2}a_{10}^2b_{02}a_{11}u_1^3 - a_{10}b_{02}a_{11}u_1^2u_2 + \frac{1}{2}b_{02}a_{11}u_1u_2^2 - \frac{1}{2}b_{02}a_{11}u_1(\ldots) + a_{11}u_1\tilde{P}_2(u,.) + a_{11}u_2^2 - a_{00}a_{11}u_2 - a_{10}a_{11}u_1u_2 - a_{11}u_2(\ldots) + b_{00}a_{02}u_2 - a_{00}b_{00}a_{02} - a_{10}b_{00}a_{02}u_1 - b_{00}a_{02}(\ldots)$

$+b_{10}a_{02}u_1u_2 - a_{00}b_{10}a_{02}u_1 - a_{10}b_{10}a_{02}u_1^2 - b_{10}a_{02}u_1(\ldots) + b_{01}a_{02}a_{00}^2 + 2a_{00}a_{10}b_{01}a_{02}u_1 - 2a_{00}b_{01}a_{02}u_2 + a_{10}^2b_{01}a_{02}u_1^2 - 2a_{10}b_{01}a_{02}u_1u_2$

$+b_{01}a_{02}u_2^2 - b_{01}a_{02}(\ldots) + \frac{1}{2}b_{20}a_{02}u_1^2u_2 - \frac{1}{2}a_{00}b_{20}a_{02}u_1^2 - \frac{1}{2}a_{10}b_{20}a_{02}u_1^3 - \frac{1}{2}b_{20}a_{02}u_1^2(\ldots) + b_{11}a_{02}a_{00}^2u_1 + 2a_{00}a_{10}b_{11}a_{02}u_1^2 - 2a_{00}b_{11}a_{02}u_1u_2$

$+a_{10}^2b_{11}a_{02}u_1^3 - 2a_{10}b_{11}a_{02}u_1^2u_2 + b_{11}a_{02}u_1u_2^2 - b_{11}a_{02}u_1(\ldots) + \frac{1}{2}b_{02}a_{02}(u_2 - a_{00} - a_{10}u_1)^3 + (a_{02}u_2 - a_{00}a_{02} - a_{10}a_{02}u_1 - a_{02}(\ldots))\tilde{P}_2(u,.) + \tilde{P}_1(u,.)$

$\dot{u_2} = (b_{00} - a_{00}b_{01} + \frac{1}{2}b_{02}a_{00}^2 + a_{01}b_{00} - a_{00}a_{01}b_{01} + \frac{1}{2}a_{01}b_{02}a_{00}^2 - a_{00}b_{00}a_{02} + b_{01}a_{02}a_{00}^2)$

$+(b_{10} - a_{10}b_{01} - a_{00}b_{11} + a_{00}a_{10}b_{02} + a_{01}b_{10} - a_{10}a_{01}b_{01} - a_{00}a_{01}b_{11} + a_{00}a_{10}a_{01}b_{02}$

$+b_{00}a_{11} - a_{00}b_{01}a_{11} + \frac{1}{2}a_{00}^2b_{02}a_{11} - a_{10}b_{00}a_{02} - a_{00}b_{10}a_{02} + 2a_{00}a_{10}b_{01}a_{02} + b_{11}a_{02}a_{00}^2)u_1$

$+(b_{01} - a_{00}b_{02} + a_{10} + a_{01}b_{01} - a_{00}a_{01}b_{02} - a_{00}a_{11} + b_{00}a_{02} - 2a_{00}b_{01}a_{02})u_2$

$+\frac{1}{2}(b_{20} - 2a_{10}b_{11} + a_{10}^2b_{02} + a_{01}b_{20} - 2a_{10}a_{01}b_{11} - 2a_{00}^2a_{11}b_{02} - 2a_{00}b_{11}a_{11}$

$+2a_{00}a_{10}b_{02}a_{11} - 2a_{10}b_{10}a_{02} + 2a_{10}^2b_{01}a_{02} - \frac{1}{2}a_{00}b_{20}a_{02} + 4a_{00}a_{10}b_{11}a_{02})u_1^2$

$+(b_{11}-a_{10}b_{02}+a_{01}b_{11}-a_{10}a_{01}b_{02}+a_{20}+b_{01}a_{11}-a_{00}b_{02}a_{11}-a_{10}a_{11}+b_{10}a_{02}$

$-2a_{10}b_{01}a_{02}-2a_{00}b_{11}a_{02})u_1u_2$

$+\frac{1}{2}(b_{02}+a_{01}b_{02}+2a_{11}+2b_{01}a_{02})u_2^2+Q(u,.).$

Let $g_{00}=b_{00}-a_{00}b_{01}+\frac{1}{2}b_{02}a_{00}^2+a_{01}b_{00}-a_{00}a_{01}b_{01}+\frac{1}{2}a_{01}b_{02}a_{00}^2-a_{00}b_{00}a_{02}+b_{01}a_{02}a_{00}^2$,

$g_{10}=b_{10}-a_{10}b_{01}-a_{00}b_{11}+a_{00}a_{10}b_{02}+a_{01}b_{10}-a_{10}a_{01}b_{01}-a_{00}a_{01}b_{11}+a_{00}a_{10}a_{01}b_{02}+$ $b_{00}a_{11}-a_{00}b_{01}a_{11}+\frac{1}{2}a_{00}^2b_{02}a_{11}-a_{10}b_{00}a_{02}-a_{00}b_{10}a_{02}+2a_{00}a_{10}b_{01}a_{02}+b_{11}a_{02}a_{00}^2$,

$g_{01}=b_{01}-a_{00}b_{02}+a_{10}+a_{01}b_{01}-a_{00}a_{01}b_{02}-a_{00}a_{11}+b_{00}a_{02}-2a_{00}b_{01}a_{02}$,

$g_{20}=b_{20}-2a_{10}b_{11}+a_{10}^2b_{02}+a_{01}b_{20}-2a_{10}a_{01}b_{11}-2a_{00}b_{11}a_{11}+2a_{00}a_{10}b_{02}a_{11}$

$-2a_{10}b_{10}a_{02}+2a_{10}^2b_{01}a_{02}-\frac{1}{2}a_{00}b_{20}a_{02}+4a_{00}a_{10}b_{11}a_{02}$,

$g_{11}=b_{11}-a_{10}b_{02}+a_{01}b_{11}-a_{10}a_{01}b_{02}+a_{20}+b_{01}a_{11}-a_{00}b_{02}a_{11}-a_{10}a_{11}+b_{10}a_{02}$

$-2a_{10}b_{01}a_{02}-2a_{00}b_{11}a_{02}$, and

$g_{02}=b_{02}+a_{01}b_{02}+2a_{11}+2b_{01}a_{02}$.

Then we have that

$$\dot{u_2}=g_{00}+g_{10}u_1+g_{01}u_2+\frac{1}{2}g_{20}u_1^2+g_{11}u_1u_2+\frac{1}{2}g_{02}u_2^2+Q(u,.) \tag{0.53}$$

Putting $\dot{u_1}=u_2$ and (0.53) together gives us a new system (after putting α back in the expression g_{kl})

$$\begin{cases}\dot{u_1}=u_2\\ \dot{u_2}=g_{00}(\alpha)+g_{10}(\alpha)u_1+g_{01}(\alpha)u_2+\frac{1}{2}g_{20}(\alpha)u_1^2+g_{11}(\alpha)u_1u_2+\frac{1}{2}g_{02}(\alpha)u_2^2+Q(u,\alpha)\end{cases} \tag{0.54}$$

where $g_{kl}(\alpha)$ and $Q(u,\alpha)$ are smooth functions of their arguments. We also have that $Q(u,\alpha)=O(||u||^3)$. Note that

$g_{00}(0)=b_{00}(0)-a_{00}(0)b_{01}(0)+\frac{1}{2}b_{02}(0)a_{00}^2(0)+a_{01}(0)b_{00}(0)-a_{00}(0)a_{01}(0)b_{01}(0)$

$+\frac{1}{2}a_{01}(0)b_{02}(0)a_{00}^2(0) - a_{00}(0)b_{00}(0)a_{02}(0) + b_{01}(0)a_{02}(0)a_{00}^2(0)$. From (0.51), we have

$g_{00}(0) = 0-(0)(0)+\frac{1}{2}b_{02}(0)(0)+(0)(0)-(0)(0)(0)+\frac{1}{2}(0)b_{02}(0)(0)^2-(0)(0)a_{02}(0)+ (0)a_{02}(0)*(0)^2 = 0$.

Similar calculations show that $g_{10}(0) = g_{01}(0) = 0$. Hence, we have that

$$g_{00}(0) = g_{10}(0) = g_{01}(0) = 0.$$

Also note that

$g_{20}(0) = b_{20}(0) - 2a_{10}(0)b_{11}(0) + a_{10}^2(0)b_{02}(0) + a_{01}(0)b_{20}(0) - 2a_{10}(0)a_{01}(0)b_{11}(0)$
$-2a_{00}(0)b_{11}(0)a_{11}(0) + 2a_{00}(0)a_{10}(0)b_{02}(0)a_{11}(0) - 2a_{10}(0)b_{10}(0)a_{02}(0)$
$+2a_{10}^2(0)b_{01}(0)a_{02}(0) - \frac{1}{2}a_{00}(0)b_{20}(0)a_{02}(0) + 4a_{00}(0)a_{10}(0)b_{11}(0)a_{02}(0)$.

Again, from (0.51), we have

$g_{20}(0) = b_{20}(0)-2(0)b_{11}(0)+(0)^2b_{02}(0)+(0)b_{20}(0)-2(0)(0)b_{11}(0)-2(0)b_{11}(0)a_{11}(0)+ 2(0)(0)b_{02}(0)a_{11}(0)-2(0)(0)a_{02}(0)+2(0)(0)a_{02}(0)-\frac{1}{2}(0)b_{20}(0)a_{02}(0)+4(0)(0)b_{11}(0)a_{02}(0) = b_{20}(0)$.

Similar calculations show that

$g_{11}(0) = b_{11}(0) + a_{20}(0)$, and $g_{02}(0) = b_{02}(0) + 2a_{11}(0)$.

So, we get that

$$\begin{cases} g_{20}(0) = b_{20}(0) \\ g_{11}(0) = b_{11}(0) + a_{20}(0) \\ g_{02}(0) = b_{02}(0) + 2a_{11}(0) \end{cases}.$$

Assume that (0.36) holds. This completes ***step 1***.

(***Step 2***)

We now go through another change of variables. Here, we go through a parameter - dependent shift. Define

$$\begin{cases} u_1 = v_1 + \delta(\alpha) \\ u_2 = v_2 \end{cases} . \tag{0.55}$$

We now take the derivative of both sides of each equation of (0.55) with respect to t. For the first equation of (0.55), we have

$\dot{v_1} = \dot{u_1}$

$\dot{v_1} = u_2$

$$\dot{v_1} = v_2 \tag{0.56}$$

For the second equation of (0.55), (again we remove α temporarily so we have $g_{kl} = g_{kl}(\alpha)$), we have

$\dot{v_2} = \dot{u_2}$

$\dot{v_2} = g_{00} + g_{10}u_1 + g_{01}u_2 + \frac{1}{2}g_{20}u_1^2 + g_{11}u_1u_2 + \frac{1}{2}g_{02}u_2^2 + Q(u,.)$.

Using the relations from (0.55) gives us

$\dot{v_2} = g_{00} + g_{10}(v_1+\delta) + g_{01}v_2 + \frac{1}{2}g_{20}(v_1+\delta)^2 + g_{11}(v_1+\delta)v_2 + \frac{1}{2}g_{02}v_2^2 + Q(u,.)$

$\dot{v_2} = g_{00} + g_{10}v_1 + g_{10}\delta + g_{01}v_2 + \frac{1}{2}g_{20}(v_1^2 + 2\delta v_1 + \delta^2) + g_{11}v_1v_2 + g_{11}\delta v_2 + \frac{1}{2}g_{02}v_2^2 + Q(u,.)$

$\dot{v_2} = g_{00} + g_{10}v_1 + g_{10}\delta + g_{01}v_2 + \frac{1}{2}g_{20}v_1^2 + g_{20}\delta v_1 + \frac{1}{2}g_{20}\delta^2 + g_{11}v_1v_2 + g_{11}\delta v_2 + \frac{1}{2}g_{02}v_2^2 + Q(u,.)$

$$\dot{v_2} = (g_{00} + g_{10}\delta + \tfrac{1}{2}g_{20}\delta^2) + (g_{10} + g_{20}\delta)v_1 + (g_{01} + g_{11}\delta)v_2 + \tfrac{1}{2}g_{20}v_1^2 + g_{11}v_1v_2 + \tfrac{1}{2}g_{02}v_2^2 + O(||v||^3)$$

$$\dot{v_2} = (g_{00} + g_{10}\delta + O(\delta^2)) + (g_{10} + g_{20}\delta + O(\delta^2))v_1 + (g_{01} + g_{11}\delta + O(\delta^2))v_2$$

$$+\frac{1}{2}(g_{20} + O(\delta))v_1^2 + (g_{11} + O(\delta))v_1v_2 + \frac{1}{2}(g_{02} + O(\delta))v_2^2 + O(||v||^3). \tag{0.57}$$

Putting (0.56) and (0.57) together gives us the system

$$\begin{cases} \dot{v_1} = v_2 \\ \dot{v_2} = (g_{00} + g_{10}\delta + O(\delta^2)) + (g_{10} + g_{20}\delta + O(\delta^2))v_1 + (g_{01} + g_{11}\delta + O(\delta^2))v_2 \\ +\tfrac{1}{2}(g_{20} + O(\delta))v_1^2 + (g_{11} + O(\delta))v_1v_2 + \tfrac{1}{2}(g_{02} + O(\delta))v_2^2 + O(||v||^3) \end{cases} . \tag{0.58}$$

Define the function $G(\alpha, \delta) = g_{01}(\alpha) + g_{11}(\alpha)\delta + O(\delta^2)$. Then we have that G is continuously differentiable, and $G(0,0) = g_{01}(0) + g_{11}(0) * 0 + O(0) = 0 + 0 + 0 = 0$. Taking the partial derivative of G with respect to δ gives us $G_\delta(\alpha, \delta) = g_{11}(\alpha) + O(\delta)$. Since we have that $G_\delta(\alpha, \delta) = g_{11}(0) + O(0) = g_{11}(0) + 0 = g_{11}(0) \neq 0$, then by the Implicit Function Theorem, there exists a smooth function $\delta = \delta(\alpha) \approx -\frac{g_{01}(\alpha)}{g_{11}(0)}$ near $(0,0)$.

From the results given above about the Implicit Function Theorem, then substituting $\delta(\alpha) = -\frac{g_{01}(\alpha)}{g_{11}(0)}$ in for δ into (0.58) gives us

$$\begin{cases} \dot{v_1} = v_2 \\ \dot{v_2} = (g_{00}(\alpha) + g_{10}(\alpha)(-\frac{g_{01}(\alpha)}{g_{11}(0)}) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})^2) + (g_{10}(\alpha) + g_{20}(\alpha)(-\frac{g_{01}(\alpha)}{g_{11}(0)}) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})^2))v_1 \\ +(g_{01}(\alpha) + g_{11}(\alpha)(-\frac{g_{01}(\alpha)}{g_{11}(0)}) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})^2))v_2 + \frac{1}{2}(g_{20}(\alpha) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})))v_1^2 \\ +(g_{11}(\alpha) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})))v_1 v_2 + \frac{1}{2}(g_{02}(\alpha) + O((-\frac{g_{01}(\alpha)}{g_{11}(0)})))v_2^2 + R(v,) \end{cases},$$

$$\begin{cases} \dot{v_1} = v_2 \\ \dot{v_2} = h_{00}(\alpha) + h_{10}(\alpha)v_1 + \frac{1}{2}h_{20}(\alpha)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 \end{cases} \tag{0.59}$$

where we have

$$h_{00}(\alpha) = g_{00}(\alpha) - \frac{g_{01}(\alpha)}{g_{11}(0)} * g_{10}(\alpha) + O(\delta^2)$$
$$h_{10}(\alpha) = g_{10}(\alpha) - \frac{g_{01}(\alpha)}{g_{11}(0)} * g_{20}(\alpha)\delta + O(\delta^2)$$
$$h_{20}(\alpha) = g_{20}(\alpha) + O(\delta)$$
$$h_{11}(\alpha) = g_{11}(\alpha) + O(\delta)$$
$$h_{02}(\alpha) = g_{02}(\alpha) + O(\delta).$$

Observe that since we have that $g_{00}(0) = g_{10}(0) = g_{01}(0) = 0$, we have that $h_{00}(0) = g_{00}(0) - \frac{g_{01}(0)}{g_{11}(0)} g_{10}(0) + O(\frac{(g_{01}(0))^2}{(g_{11}(0))^2}) = 0 - (\frac{0}{g_{11}(0)})(0) = 0 - 0 + 0 = 0$. Similarly, we have that $h_{10}(0) = g_{10}(0) - \frac{g_{01}(0)}{g_{11}(0)} g_{20}(0) + O(\frac{(g_{01}(0))^2}{(g_{11}(0))^2}) = 0 - (\frac{0}{g_{11}(0)})(0) + O(0) = 0 - 0 + 0 = 0$. Hence, we have that $h_{00}(0) = h_{10}(0) = 0$. We also have that $h_{20}(0) = g_{20}(0) + O(-\frac{g_{01}(0)}{g_{11}(0)}) = g_{20}(0) + O(-\frac{0}{g_{11}(0)}) = g_{20}(0) + O(0) = g_{20}(0) + 0 = g_{20}(0)$. Similar calculations shows that $h_{11}(0) = g_{11}(0)$, and $h_{02}(0) = g_{02}(0)$. So, we get that

$$\begin{cases} h_{20}(0) = g_{20}(0) \\ h_{11}(0) = g_{11}(0) \\ h_{02}(0) = g_{02}(0) \end{cases}.$$

This completes ***step 2***.

(***Step 3***)

Now, we consider changing the time parameter from t to τ by the equation $dt = (1+\theta v_1)d\tau$, where $\theta = \theta(\alpha)$ is a smooth function (which will be defined later). We claim that the direction of time is preserved near the origin for small $||\alpha||$. This will be proven later.

So, the first equation of (0.59) becomes (using the chain rule for differentiation)

$$\dot{v_1} = \frac{dv_1}{dt} \times \frac{dt}{d\tau} = v_2(1+\theta v_1) = v_2 + \theta v_1 v_2 \tag{0.60}$$

.

The second equation of (0.59) becomes

$\dot{v_2} = \frac{dv_2}{dt} \times \frac{dt}{d\tau} = (h_{00}(\alpha) + h_{10}(\alpha)v_1 + + \frac{1}{2}h_{20}(\alpha)v_1^2 + h_{11}(\alpha)v_1v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + R(v,))(1+\theta v_1)$

$\dot{v_2} = h_{00}(\alpha) + h_{10}(\alpha)v_1 + \frac{1}{2}h_{20}(\alpha)v_1^2 + h_{11}(\alpha)v_1v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + R(v,) + h_{00}(\alpha)\theta v_1$
$+h_{10}(\alpha)\theta v_1^2 + \frac{1}{2}h_{20}(\alpha)\theta v_1^3 + h_{11}(\alpha)\theta v_1^2 v_2 + \frac{1}{2}h_{02}(\alpha)\theta v_1 v_2^2 + R(v,)$

$$\dot{v_2} = h_{00}(\alpha) + (h_{10}(\alpha) + h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)v_1^2 + h_{11}(\alpha)v_1v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3). \tag{0.61}$$

Putting (0.60) and (0.61) together gives us the system

$$\begin{cases} \dot{v_1} = v_2 + \theta v_1 v_2 \\ \dot{v_2} = h_{00}(\alpha) + (h_{10}(\alpha) + h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3). \end{cases} \tag{0.62}$$

Here we (once again) change coordinates in the following way

$$\begin{cases} \xi_1 = v_1 \\ \xi_2 = v_2 + \theta v_1 v_2 \end{cases} \tag{0.63}$$

Now, we differentiate both sides of both equations of (0.63) with respect to τ. For the first equation, we have

$\dot{\xi_1} = \dot{v_1} = v_2 + \theta v_1 v_2 = \xi_2$. Hence, we have

$$\dot{\xi_1} = \xi_2 \tag{0.64}$$

Now, taking the derivative of the second equation of (0.63) gives us

$\dot{\xi_2} = \dot{v_2} + \theta v_1 \dot{v_2} + \theta \dot{v_1} v_2$

$\dot{\xi_2} = (h_{00}(\alpha) + (h_{10}(\alpha) + h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3)) + \theta v_1(h_{00}(\alpha) + (h_{10}(\alpha) + h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3)) + \theta \dot{v_1} v_2$

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3) + h_{00}(\alpha)\theta v_1 + (h_{10}(\alpha) + h_{00}(\alpha)\theta)\theta v_1^2 + \frac{1}{2}(h_{20}(\alpha) + 2h_{10}(\alpha)\theta)\theta v_1^3 + h_{11}(\alpha)\theta v_1^2 v_2 + \frac{1}{2}h_{02}(\alpha)\theta v_1 v_2^2 + \theta v_1 O(||v||^3) + \theta \dot{v_1} v_2$

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta)v_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta + 2h_{00}(\alpha)\theta^2)v_1^2 + h_{11}(\alpha)v_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3) + \theta \dot{v_1} v_2$

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta)\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta + 2h_{00}(\alpha)\theta^2)\xi_1^2 + h_{11}(\alpha)\xi_1 v_2 + \frac{1}{2}h_{02}(\alpha)v_2^2 + O(||v||^3) + \theta\xi_2 v_2.$

Letting $v_2 = \dot{v_1} - \theta v_1 v_2 = \xi_2 - \theta\xi_1 v_2$ gives us the following

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta)\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta + 2h_{00}(\alpha)\theta^2)\xi_1^2 + h_{11}(\alpha)\xi_1(\xi_2 - \theta\xi_1 v_2) + \frac{1}{2}h_{02}(\alpha)(\xi_2 - \theta\xi_1 v_2)^2 + O(||v||^3) + \theta\xi_2(\xi_2 - \theta\xi_1 v_2)$

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta)\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta + 2h_{00}(\alpha)\theta^2)\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 - h_{11}\theta\xi_1^2 v_2 + \frac{1}{2}h_{02}(\alpha)\xi_2^2 - h_{02}\theta\xi_1\xi_2 v_2 + \frac{1}{2}h_{02}\theta^2\xi_1^2 v_2^2 + O(||v||^3) + \theta\xi_2^2$
$-\theta^2\xi_1\xi_2 v_2.$

Observe that everything after ξ_2 from the equation $v_2 = \xi_2 - \theta\xi_1 v_2$ in the previous equation would be of degree 3, so we overlook those terms to give us

$\dot{\xi_2} = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta)\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta + 2h_{00}(\alpha)\theta^2)\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}(h_{02}(\alpha) + 2\theta)\xi_2^2 + O(||\xi||^3).$

Letting

$f_{00}(\alpha) = h_{00}(\alpha)$
$f_{10}(\alpha) = h_{10}(\alpha) + 2h_{00}(\alpha)\theta(\alpha)$
$f_{20}(\alpha) = h_{20}(\alpha) + 4h_{10}(\alpha)\theta(\alpha) + O(\theta^2)$
$f_{11}(\alpha) = h_{11}(\alpha)$
$f_{02}(\alpha) = h_{02}(\alpha) + 2\theta(\alpha),$

gives us the equation

$$\dot{\xi_2} = f_{00}(\alpha) + f_{10}(\alpha)\xi_1 + \frac{1}{2}f_{20}(\alpha)\xi_1^2 + f_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}f_{02}(\alpha)\xi_2^2 + O(||\xi||^3). \quad (0.65)$$

Putting (0.64) and (0.65) together gives us the system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = f_{00}(\alpha) + f_{10}(\alpha)\xi_1 + \frac{1}{2}f_{20}(\alpha)\xi_1^2 + f_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}f_{02}(\alpha)\xi_2^2 + O(||\xi||^3). \end{cases} \tag{0.66}$$

Now, to eliminate the ξ_2^2 term in (0.66), we let $\theta(\alpha) = -\frac{h_{02}(\alpha)}{2}$ (and also specify the time reparametrization). Substituting in this choice of $\theta(\alpha)$ gives us

$$\dot{\xi}_2 = f_{00}(\alpha) + f_{10}(\alpha)\xi_1 + \frac{1}{2}f_{20}(\alpha)\xi_1^2 + f_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}f_{02}(\alpha)\xi_2^2 + O(||\xi||^3)$$

$$\dot{\xi}_2 = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)\theta(\alpha))\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)\theta(\alpha) + O(\theta^2))\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}(h_{02}(\alpha) + 2\theta(\alpha))\xi_2^2 + O(||\xi||^3)$$

$$\dot{\xi}_2 = h_{00}(\alpha) + (h_{10}(\alpha) + 2h_{00}(\alpha)(-\frac{h_{02}(\alpha)}{2}))\xi_1 + \frac{1}{2}(h_{20}(\alpha) + 4h_{10}(\alpha)(-\frac{h_{02}(\alpha)}{2}) + O((-\frac{h_{02}(\alpha)}{2})^2))\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}(h_{02}(\alpha) + 2(-\frac{h_{02}(\alpha)}{2}))\xi_2^2 + O(||\xi||^3)$$

$$\dot{\xi}_2 = h_{00}(\alpha) + (h_{10}(\alpha) - h_{00}(\alpha)h_{02}(\alpha))\xi_1 + \frac{1}{2}(h_{20}(\alpha) - 2h_{10}(\alpha)h_{02}(\alpha) + O(\frac{h_{02}^2(\alpha)}{4}))\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 + \frac{1}{2}(h_{02}(\alpha) - h_{02}(\alpha))\xi_2^2 + O(||\xi||^3)$$

$$\dot{\xi}_2 = h_{00}(\alpha) + (h_{10}(\alpha) - h_{00}(\alpha)h_{02}(\alpha))\xi_1 + \frac{1}{2}(h_{20}(\alpha) - 2h_{10}(\alpha)h_{02}(\alpha) + O(\frac{h_{02}^2(\alpha)}{4}))\xi_1^2 + h_{11}(\alpha)\xi_1\xi_2 + O(||\xi||^3).$$

Now, let

$\mu_1(\alpha) = h_{00}(\alpha)$

$\mu_2(\alpha) = h_{10}(\alpha) - h_{00}(\alpha)h_{02}(\alpha)$

$A(\alpha) = \frac{1}{2}(h_{20}(\alpha) - 2h_{10}(\alpha)h_{02}(\alpha) + O(\frac{h_{02}^2(\alpha)}{4}))$

$B(\alpha) = h_{11}(\alpha)$.

Then we have that

$$\dot{\xi}_2 = \mu_1(\alpha) + \mu_2(\alpha)\xi_1 + A(\alpha)\xi_1^2 + B(\alpha)\xi_1\xi_2 + O(||\xi||^3).$$

Then the system (0.66) becomes the system

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \mu_1(\alpha) + \mu_2(\alpha)\xi_1 + A(\alpha)\xi_1^2 + B(\alpha)\xi_1\xi_2 + O(||\xi||^3). \end{cases} \quad . \tag{0.67}$$

We claimed earlier that the direction of time is preserved for small $||\alpha||$. We now prove that this is true. Observe that we have

$\theta(\alpha) = -\frac{h_{02}(\alpha)}{2} = -\frac{1}{2}(g_{02}(\alpha) + O(-\frac{g_{01}(\alpha)}{g_{11}(0)})) = -\frac{1}{2}(b_{02}(\alpha) + a_{01}(\alpha)b_{02}(\alpha)$
$+2a_{11}(\alpha) + 2b_{01}(\alpha)a_{02}(\alpha) + O(-\frac{g_{01}(\alpha)}{g_{11}(0)})).$

So, we have that $\theta(\alpha)$ can be related to the system (0.50). Since $y = 0$ is the equilibrium point of the system (0.50), then we can choose α small enough so that $\theta(\alpha)v_1 < 1$. So, from this and from $dt = (1 + \theta v_1)d\tau$, we have that the direction of time is preserved (for small $||\alpha||$), which proves the claim.

This completes ***step 3***.

(***Step 4***) Introduce a new time (denoted by t again) by

$$t = |\frac{B(\alpha)}{A(\alpha)}|\tau.$$

Observe that we have $B(0) = h_{11}(0) = g_{11}(0) = a_{20}(0) + b_{11}(0) \neq 0$ (by (0.36)), and we have $2A(0) = h_{20}(0) = g_{20}(0) = b_{20}(0) \neq 0$ (by (0.37)). Now, we do another change of coordinates by

$$\begin{cases} \eta_1 = \frac{A(\alpha)}{B^2(\alpha)}\xi_1 \\ \eta_2 = sign(\frac{B(\alpha)}{A(\alpha)})\frac{A^2(\alpha)}{B^3(\alpha)}\xi_2 \end{cases}$$

which scales the coordinates of the previous system. Taking the derivative to both sides of the first equation from above with respect to t gives us (again removing α from $A(\alpha)$ and $B(\alpha)$)

$\dot{\eta}_1 = \frac{d\eta_1}{dt} = \frac{d\eta_1}{d\tau} \times \frac{d\tau}{dt}$

$\dot{\eta}_1 = \frac{A(\alpha)}{B^2(\alpha)}\dot{\xi}_1 \times |\frac{A}{B}|$

$\dot{\eta}_1 = \frac{A(\alpha)}{B^2(\alpha)}\dot{\xi}_1 \times sign(\frac{A}{B}) \times \frac{A}{B}$

$\dot{\eta}_1 = sign(\frac{B}{A}) \times \frac{A^2}{B^3} \times \xi_2$

$$\dot{\eta}_1 = \eta_2. \tag{0.68}$$

Now, taking the derivative to both sides of the second equation from above with respect to t gives us

$\dot{\eta}_2 = \frac{d\eta_2}{dt} = \frac{d\eta_2}{d\tau} \times \frac{d\tau}{dt}$

$\dot{\eta}_2 = sign(\frac{B}{A}) \times \frac{A^2}{B^3} \times \dot{\xi}_2 \times |\frac{A}{B}|$

$\dot{\eta}_2 = sign(\frac{B}{A}) \times \frac{A^2}{B^3} \times \dot{\xi}_2 \times sign(\frac{A}{B}) \times \frac{A}{B}$

$\dot{\eta}_2 = sign(\frac{B}{A}) \times sign(\frac{B}{A}) \times \frac{A^3}{B^4} \times \dot{\xi}_2$

$\dot{\eta}_2 = \frac{A^3}{B^4} \times \dot{\xi}_2$

$\dot{\eta}_2 = \frac{A^3}{B^4} \times \dot{\xi}_2$

$\dot{\eta}_2 = \frac{A^3}{B^4}[\mu_1 + \mu_2\xi_1 + A\xi_1^2 + B\xi_1\xi_2 + O(||\xi||^3)]$

$\dot{\eta}_2 = \frac{A^3}{B^4}\mu_1 + \frac{A^3}{B^4}\mu_2\xi_1 + \frac{A^4}{B^4}\xi_1^2 + \frac{A^3}{B^3}\xi_1\xi_2 + O(||\xi||^3)$

$\dot{\eta}_2 = \frac{A^3}{B^4}\mu_1 + \frac{A^2}{B^2}\mu_2 \times \frac{A}{B^2}\xi_1 + A^2 \times \frac{A^2}{B^4}\xi_1^2 + \frac{A}{B^2}\xi_1 \times \frac{A^2}{B}\xi_2 + O(||\xi||^3)$

$\dot{\eta}_2 = \frac{A^3}{B^4}\mu_1 + \frac{A^2}{B^2}\mu_2\eta_1 + A^2\eta_1^2 + \eta_1 \times sign(\frac{B}{A}) \times B^2 \times \eta_2 + O(||\xi||^3)$

$\dot{\eta}_2 = \frac{A^3}{B^4}\mu_1 + \frac{A^2}{B^2}\mu_2\eta_1 + A^2\eta_1^2 + sign(\frac{B}{A})B^2\eta_1\eta_2 + O(||\xi||^3).$

$\dot{\eta}_2 = \frac{A^3}{B^4}\mu_1 + \frac{A^2}{B^2}\mu_2\eta_1 + A^2\eta_1^2 \pm B^2\eta_1\eta_2 + O(||\xi||^3).$

Letting $\beta_1(\alpha) = \frac{A^3(\alpha)}{B^4(\alpha)}\mu_1(\alpha)$ and $\beta_2(\alpha) = \frac{A^2(\alpha)}{B^2(\alpha)}\mu_2(\alpha)$ gives us

$$\dot{\eta}_2 = \beta_1(\alpha) + \beta_2(\alpha)\eta_1 + A^2\eta_1^2 \pm B^2\eta_1\eta_2 + O(||\xi||^3). \tag{0.69}$$

Putting (0.68) and (0.69) together gives us (the final) system

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + A^2\eta_1^2 \pm B^2\eta_1\eta_2 + O(||\xi||^3). \end{cases} \tag{0.70}$$

We claim that $\beta_1(0) = \beta_2(0) = 0$. To show this, observe that

$\beta_1(0) = \frac{A^3(0)}{B^4(0)}\mu_1(0) = \frac{A^3(0)}{B^4(0)}[h_{00}(0)] = \frac{A^3(0)}{B^4(0)} \times 0 = 0$ (since we have that $h_{00}(0) = 0$). Similarly, we have

$\beta_2(0) = \frac{A^2(0)}{B^2(0)}\mu_2(0) = \frac{A^2(0)}{B^2(0)}[h_{10}(0) - h_{00}(0)h_{02}(0)] = \frac{A^2(0)}{B^2(0)}[0 - 0 \times h_{02}(0)] = \frac{A^2(0)}{B^2(0)} \times 0 = 0$, since $h_{00}(0) = h_{10}(0) = 0$. This proves the claim. Since (0.38) holds, then *step 4* and the proof of Theorem 41 is done. ■

REMARK 42. The system given in Theorem 41 can be rescaled further to the following system

$$\begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = \omega_1(\alpha) + \omega_2(\alpha)\eta_1 + \eta_1^2 \pm \eta_1\eta_2 + O(||\xi||^3). \end{cases} \tag{0.71}$$

since (0.70) and (0.71) are topologically equivalent (meaning that the two systems are "similar" from a topological point of view).[6] △

6.3 Example

Here, we provide an example of a system that will experience a Bogdanov-Takens bifurcation.

EXAMPLE 43. Consider the system given as follows:

$$\begin{cases} \dot{x} = x - \frac{xy}{1+ax} \\ \dot{y} = \frac{xy}{1+ax} - 2y - by^2 \end{cases}$$

[6] This result can be found in [3].

where $x(t)$ is the population of a prey, $y(t)$ is the population of a predator, and a and b are positive parameters.

Here is the bifurcation diagram that we see with this example (where $a = 0.1$ and $b = 0.22$). (See F27.)

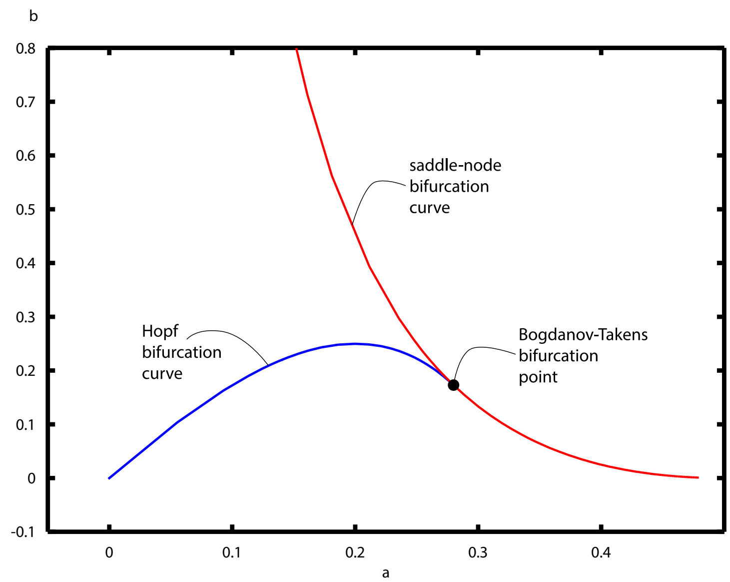
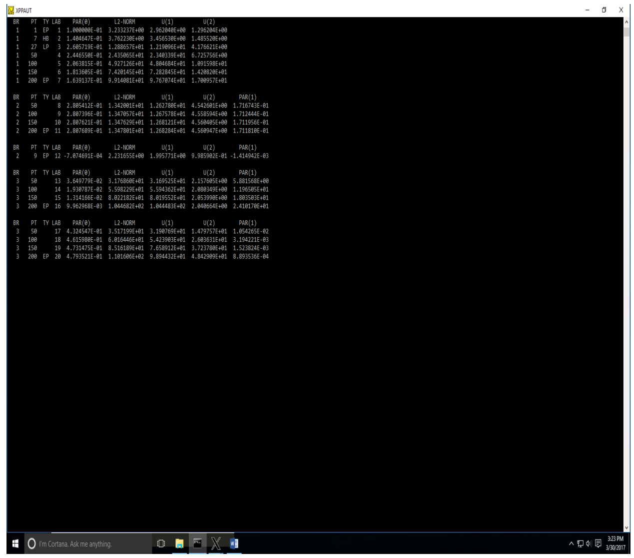


Figure 27. Bifurcation diagram of Bogdanov-Takens bifurcation for Example 43.



XPPAUT

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)
1	1	EP	1	1.000000E-01	3.233237E+00	2.962040E+00	1.296204E+00
1	7	HB	2	1.404647E-01	3.762230E+00	3.456530E+00	1.485520E+00
1	27	LP	3	2.605719E-01	1.288657E+01	1.219096E+01	4.176621E+00
1	50		4	2.446550E-01	2.435065E+01	2.340339E+01	6.725756E+00
1	100		5	2.063815E-01	4.927126E+01	4.804684E+01	1.091598E+01
1	150		6	1.813605E-01	7.420145E+01	7.282845E+01	1.420820E+01
1	200	EP	7	1.639137E-01	9.914081E+01	9.767074E+01	1.700957E+01

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)	PAR(1)
2	50		8	2.805412E-01	1.342001E+01	1.262780E+01	4.542601E+00	1.716743E-01
2	100		9	2.807396E-01	1.347057E+01	1.267578E+01	4.558594E+00	1.712444E-01
2	150		10	2.807621E-01	1.347629E+01	1.268121E+01	4.560405E+00	1.711956E-01
2	200	EP	11	2.807689E-01	1.347801E+01	1.268284E+01	4.560947E+00	1.711810E-01

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)	PAR(1)
2	9	EP	12	-7.074691E-04	2.231655E+00	1.995771E+00	9.985902E-01	-1.414942E-03

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)	PAR(1)
3	50		13	3.649779E-02	3.176860E+01	3.169525E+01	2.157605E+00	5.881568E+00
3	100		14	1.930787E-02	5.598229E+01	5.594362E+01	2.080349E+00	1.196505E+01
3	150		15	1.314166E-02	8.022182E+01	8.019552E+01	2.053990E+00	1.803503E+01
3	200	EP	16	9.962968E-03	1.044682E+02	1.044483E+02	2.040664E+00	2.410170E+01

BR	PT	TY	LAB	PAR(0)	L2-NORM	U(1)	U(2)	PAR(1)
3	50		17	4.324547E-01	3.517199E+01	3.190769E+01	1.479757E+01	1.054265E-02
3	100		18	4.615980E-01	6.016446E+01	5.423903E+01	2.603631E+01	3.194221E-03
3	150		19	4.731475E-01	8.516189E+01	7.658912E+01	3.723780E+01	1.523824E-03
3	200	EP	20	4.793521E-01	1.101606E+02	9.894432E+01	4.842909E+01	8.893536E-04

Figure 28. Screenshot of data from XPPAUT which describes a Bogdanov-Takens bifurcation.

In F27, we have that a and b are both free parameters. So, if you fix $b = 0.2$, then there are two Hopf bifurcations and a saddle-node bifurcation. The point where the two curves intersect is the Bogdanov-Takens bifurcation.

To better understand what is going on with the Bogdanov-Takens bifurcation, we provide another example of a system that will undergo this bifurcaton.

Consider the system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = a + bx + x^2 - xy \end{cases} \tag{0.72}$$

where $\alpha = \begin{bmatrix} a & b \end{bmatrix}^T \in \mathbb{R}^2$. The bifurcation diagram of (0.72) is given as follows (in F29):

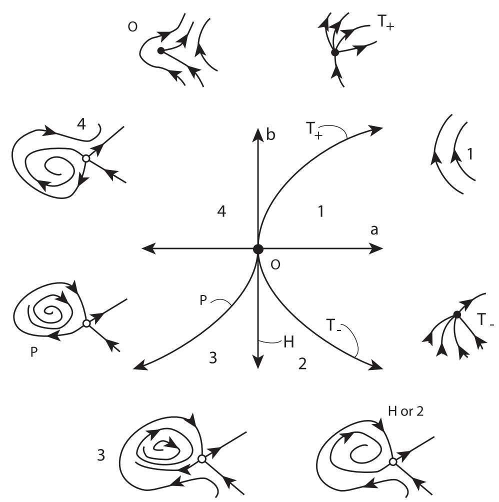


Figure 29. Bifurcation diagram of Bogdanov-Takens bifurcation for Example 44.

It can be shown that if $a = b = 0$, then the origin is a cusp.[7] The phase portrait for this is as follows:

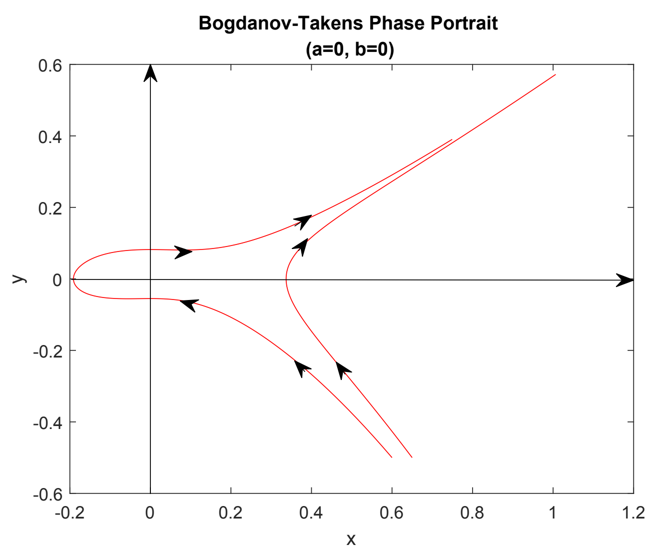


Figure 30. Phase Portrait of (0.72) at $(a,b) = (0,0)$.

[7]This can be shown by applying Theorem 17 to the system (0.72).

Now, the equilibrium points of system (0.72) are along the x-axis (i.e., when $y = 0$), and satisfies the equation $a + bx + x^2 = 0$. So, the solutions of the equation are of the form $x = \frac{-b \pm \sqrt{b^2 - 4a}}{2}$. Hence, the number of real solutions are based on the sign of the discriminant $b^2 - 4a$. The phase portrait for when $b^2 - 4a < 0$ is as follows:

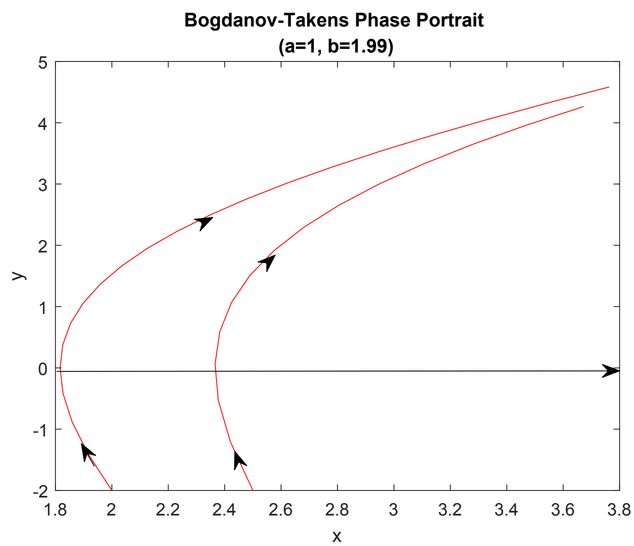


Figure 31. Phase Portrait of (0.72) in region **1**.

We claim that the (discriminant) parabola $T = \{(a,b) \mid b^2 - 4a = 0\}$ gives us a saddle-node bifurcation, which we shall prove in a moment. Now, along the curve T, we claim that the system (0.72) has an equilibrium point with a zero eigenvalue. To show this, consider the equilibrium point $(-\frac{b}{2}, 0) \in T$. Observe that the Jacobian of the system (0.72) is

$$Df(x,y) = \begin{bmatrix} 0 & 1 \\ b + 2x - y & -x \end{bmatrix}. \tag{0.73}$$

The Jacobian evaluated at $(-\frac{b}{2}, 0)$ is

$$Df(-\frac{b}{2}, 0) = \begin{bmatrix} 0 & 1 \\ b + 2(-\frac{b}{2}) - (0) & -(-\frac{b}{2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{b}{2} \end{bmatrix}.$$

It can be shown that the eigenvalues of $Df(-\frac{b}{2},0)$ are $\lambda_1 = 0$, and $\lambda_2 = \frac{b}{2}$, which proves the claim. Furthermore, we claim that the equilibrium point $(-\frac{b}{2},0)$ will be a saddle-node.[8] The phase portrait for this case is given as follows:

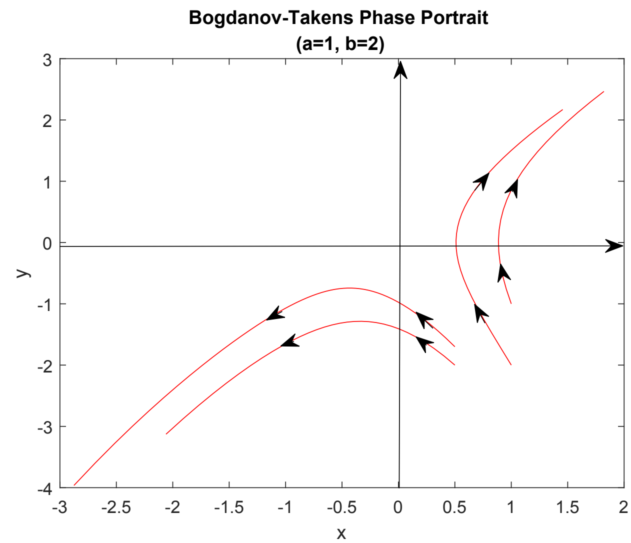


Figure 32. Phase Portrait of (0.72) on T_+.

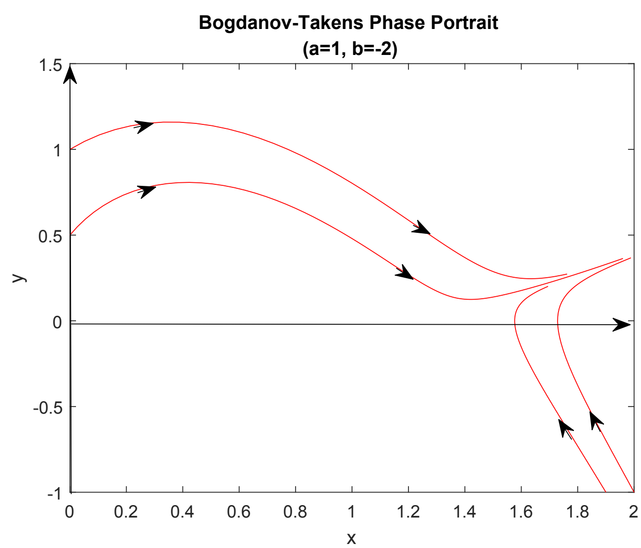


Figure 33. Phase Portrait of (0.72) on T_-.

We will now prove that points on T gives us a saddle-node bifurcation. Now, the point $\alpha = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ separates the curve T into two parts: T_- and T_+, where $b < 0$ and $b > 0$

[8]This can be proven by fixing a point on T, shift it to the origin, and apply Theorem 15 to the new system.

respectively. We claim that E_2 is a saddle equilibrium point, whether the point $\alpha = \begin{bmatrix} a & b \end{bmatrix}^T$ passes through T_- or T_+, and for all parameters to the left of the curve T. To prove this, assume that $b^2 - 4a > 0$ so that E_2 exists. Plugging in E_2 into (0.73) gives us:

$$Df(E_2) = \begin{bmatrix} 0 & 1 \\ b + 2(\frac{-b+\sqrt{b^2-4a}}{2}) - (0) & -(\frac{-b+\sqrt{b^2-4a}}{2}) \end{bmatrix}$$

$$Df(E_2) = \begin{bmatrix} 0 & 1 \\ \sqrt{b^2 - 4a} & \frac{b-\sqrt{b^2-4a}}{2} \end{bmatrix}. \tag{0.74}$$

Observe that the determinant of the matrix (0.74) is $D = det(Df(E_2)) = -\sqrt{b^2 - 4a} < 0$. So, by the Trace-Determinant Analysis, E_2 is a saddle equilibrium point. We also claim that E_1 is a stable node if $b < 0$, and is an unstable node if $b > 0$. Plugging in E_1 into (0.73) gives us:

$$Df(E_1) = \begin{bmatrix} 0 & 1 \\ b + 2(\frac{-b-\sqrt{b^2-4a}}{2}) - (0) & -(\frac{-b-\sqrt{b^2-4a}}{2}) \end{bmatrix}$$

$$Df(E_1) = \begin{bmatrix} 0 & 1 \\ -\sqrt{b^2 - 4a} & \frac{b+\sqrt{b^2-4a}}{2} \end{bmatrix} \tag{0.75}$$

The eigenvalues of (0.75) are $\lambda_{1,2} = \frac{1}{2}(\frac{b+\sqrt{b^2-4a}}{2} \pm \sqrt{(\frac{b+\sqrt{b^2-4a}}{2})^2 - 4\sqrt{b^2 - 4a}})$. We claim that E_1 is a node. To prove this, consider the following function,

$$f(a) = \lambda_1 + \lambda_2$$
$$f(a) = [\frac{1}{2}(\frac{b+\sqrt{b^2-4a}}{2} + \sqrt{(\frac{b+\sqrt{b^2-4a}}{2})^2 - 4\sqrt{b^2 - 4a}})]$$

$+[\frac{1}{2}(\frac{b+\sqrt{b^2-4a}}{2}-\sqrt{(\frac{b+\sqrt{b^2-4a}}{2})^2-4\sqrt{b^2-4a}})]$
$f(a)=\frac{1}{2}\times\frac{b+\sqrt{b^2-4a}}{2}+\frac{1}{2}\times\frac{b+\sqrt{b^2-4a}}{2}$
$f(a)=\frac{b+\sqrt{b^2-4a}}{2}$.

Observe that f is continuous for $b^2-4a\geq 0$, and that $f(0)=0$ (as we will discuss this below). Since this is the case, this indicates that λ_1 and λ_2 are either real or pure imaginary. This is because if λ_1 and λ_2 can be written in the form: $\lambda_1=A+iB$ and $\lambda_2=A-iB$. Then we have that $\lambda_1+\lambda_2=[A+iB]+[A-iB]=2A$, which does not follow the assumption about the form of λ_1 and λ_2. Now, if we assume that $\lambda_{1,2}=\pm iB$. Then we have that the real part of $\lambda_1=0$. However, the real part of λ_1 is $\frac{b+\sqrt{b^2-4a}}{2}$, which is a contradiction since we have that $\sqrt{b^2-4a}>0$. This shows that $\lambda_{1,2}\in\mathbb{R}$. Now, observe that we have that $\sqrt{(\frac{b+\sqrt{b^2-4a}}{2})^2-4\sqrt{b^2-4a}}<\sqrt{(\frac{b+\sqrt{b^2-4a}}{2})^2}=|\frac{b+\sqrt{b^2-4a}}{2}|$. So, this shows that the sign of the eigenvalues $\lambda_{1,2}$ does not change, which implies that the equilibrium point E_1 is a node.

Now, we observe that if $b>0$, then $\frac{b+\sqrt{b^2-4a}}{2}>0$ since $\sqrt{b^2-4a}>0$. Hence, if $b>0$, then E_1 is an unstable node. Similarly, if $b<0$, then $\frac{b+\sqrt{b^2-4a}}{2}<\frac{b+|b|}{2}=\frac{b-b}{2}=0$, since we have that $a>0$. Hence, if $b<0$, then E_1 is a stable node.

It turns out that there is a nonbifurcation curve (which is not shown in Figure 29) at $a>0$, where passing through the origin causes the equilibrium point E_1 to change from a node to a focus.

Now assume that $a=0$. Then (0.75) becomes

$$Df(E_1)=\begin{bmatrix}0 & 1\\ -\sqrt{b^2} & \frac{b+\sqrt{b^2}}{2}\end{bmatrix}$$

$$Df(E_1)=\begin{bmatrix}0 & 1\\ -|b| & \frac{b+|b|}{2}\end{bmatrix}. \tag{0.76}$$

So the eigenvalues of (0.76) are $\lambda_{1,2} = \frac{1}{2}(\frac{b+|b|}{2} \pm \sqrt{(\frac{b+|b|}{2})^2 - 4|b|})$. For $b < 0$, the curve $H = \{(a,b) \mid a = 0,\ b < 0\}$ corresponds to a Hopf bifurcation.[9] We claim that the Hopf bifurcation H will form a stable limit cycle. Indeed for if we let $(a,b) \in H$, then the system (0.72) will simplify to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = bx + x^2 - xy \end{cases} \tag{0.77}$$

Now, we compute the Lyapunov coefficient for the system (0.77). Recall that the Lyapunov coefficient is defined as follows:

$$\sigma = \frac{-3\pi}{2b'D^{3/2}}\{[a'c'(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a'b'(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) + c'^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a'c'(b_{02}^2 - a_{20}a_{02})$$

$$-2a'b'(a_{20}^2 - b_{20}b_{02}) - b'^2(2a_{20}b_{20} + b_{11}b_{20}) + (b'c' - 2a'^2)(b_{11}b_{02} - a_{11}a_{20}) - (a'^2 + b'c')[3(c'b_{03} - b'a_{30})$$

$$+2a'(a_{21} + b_{12}) + (c'a_{12} - b'b_{21})]\}$$

where a', b', c', d' represent the entries of the matrix $A = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, $D = det(A) = a'd' - b'c' > 0$, a_{ij} are the coefficients of an analytic function $p(\eta_1, \eta_2)$, and b_{ij} are the coefficients of an analytic function $q(\eta_1, \eta_2)$. For the system (0.77), we have that $a' = 0$, $b' = 1$, $c' = b$, $d' = 0$, $a_{20} = 0$, $a_{11} = 0$, $a_{02} = 0$, $a_{30} = 0$, $a_{21} = 0$, $a_{12} = 0$, $a_{03} = 0$, $b_{20} = 1$, $b_{11} = -1$, $b_{02} = 0$, $b_{30} = 0$, $b_{21} = 0$, $b_{12} = 0$, $b_{03} = 0$, and $D = -b > 0$ (since $b < 0$). Plugging in all of these numbers into σ as defined above gives us:

$$\sigma = \frac{-3\pi}{2(1)(-b)^{3/2}}\{[(0)(b)(0^2 + (0)(0) + (0)(-1)) + (0)(1)((-1)^2 + (0)(-1) + (0)(0)) + b^2((0)(0) + 2(0)(0)) - 2(0)(b)(0^2 - (0)(0))$$

$$-2(0)(1)(0^2 - (1)(0)) - (1)^2(2(0)(1) + (-1)(1)) + ((1)(b) - 2(0)^2)((-1)(0) - (0)(0)) - (0^2 + (1)(b))[3((b)(0) - (1)(0)$$

[9]This can be proven by using Theorem 31 (after fixing a point in H, and shifting the coordinates).

$+2(0)(0+0)+((b)(0)-(1)(0))]\}$

$\sigma = \frac{-3\pi}{2(-b)^{3/2}}\{1\}$

$\sigma = \frac{-3\pi}{2(-b)^{3/2}} < 0.$

Since $\sigma < 0$, then by Theorem 31, this proves the claim that the Hopf bifurcation H forms a stable limit cycle. The phase portrait for this case is as follows:

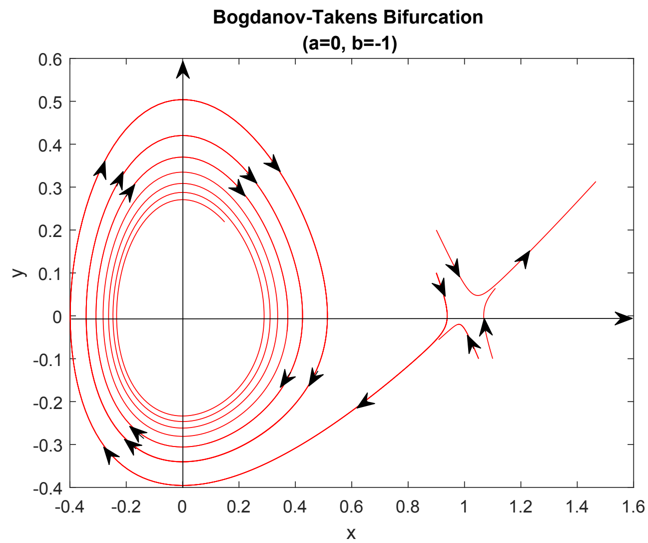


Figure 34. Phase Portrait of (0.72) on H.

Since we have a Hopf bifurcation on the curve H, then the limit cycle will still exist for parameters near the curve H (in particular, for $a < 0$ near H). The phase portrait for this case is as follows:

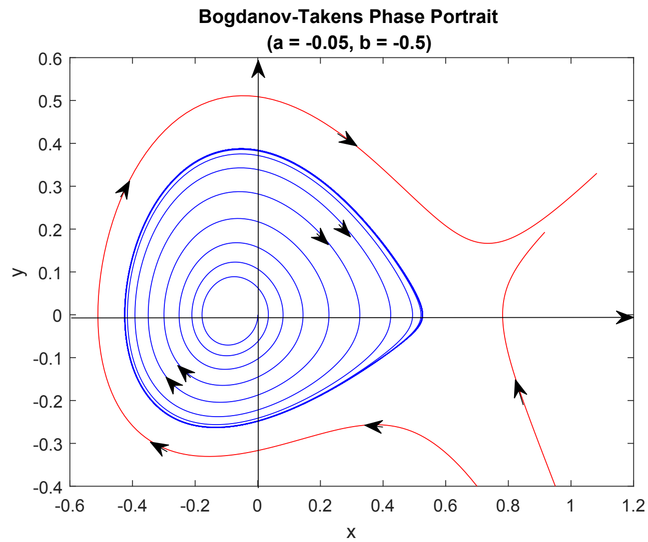


Figure 35. Phase Portrait of (0.72) in region **3**.

As the bifurcation parameters move away from the curve H, the "size" of the limit cycle gets larger, and eventually will give us a global bifurcation P. Once we pass through P, the limit cycle disappears (which is region **4** in Figure 29.). The phase portrait for this case is as follows:

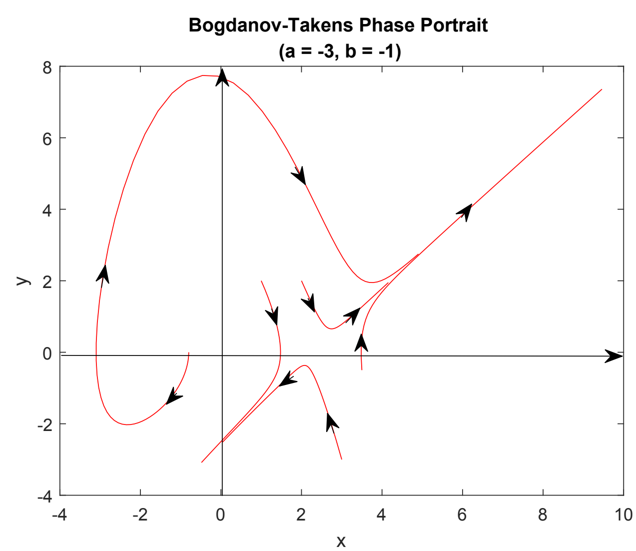


Figure 36. Phase Portrait of (0.72) in region **4**.

To summarize what we discussed in this example, we will make a round trip near the origin. (We refer back to Figure 29.) We will begin in region **1**, and work clockwise. In region **1**, we see that there are no equilibrium points (and no limit cycles). Going from region **1** to region **2** through T_-, we get two equilibrium points: a stable node E_1, and a saddle point E_2 (hence a saddle-node bifurcation on T_-). Then the node turns into a (stable) focus, and looses its stability as it passes through the Hopf bifurcation curve H. We have a set stable limit cycles for parameter values to the left of the curve H. Continuing from H (in region **3**) heading back to region **1**, the "size" of the limit cycle increases, and then eventually the limit cycle disappears. Hence, there must be a global bifurcation[10] which "destroys" the cycle somewhere between H and T_+ (in region **4**). It turns out that there are only two types of codimension-1 global bifurcations we could have: saddle homoclinic bifurcation[11] and saddle-node homoclinic bifurcation[12]. Since a saddle-node equilibrium at the saddle-node bifurcation cannot have a homoclinic orbit, then the only possible global bifurcation is the appearance of an orbit homoclinic to the saddle E_2. Hence, there is at least one bifurcation curve originating at $\alpha = (0,0)$ along which (0.72) undergoes a saddle homoclinic bifurcation. As we follow the homoclinic orbit along the curve P toward the Bogdanov-Takens point, the looplike orbit shrinks and will eventually disappear. In region **4** heading back to region **1**, we go from having E_2 be a saddle and E_1 be an unstable node to no equilibrium points as we pass through T_+ (that is, we have a saddle-node bifurcation on T_+). $\triangledown$

[10] Global bifurcations are beyond the scope of this thesis, but for more information, see [1].

[11] A saddle homoclinic bifurcation is a codimension-1 bifurcation that has a homoclinic orbit to a saddle point.

[12] A saddle-node homoclinic bifurcation is a codimension-1 bifurcation that has a homoclinic orbit to a saddle-node point.

CHAPTER 7: CONCLUSION

We have studied a codimension-2 bifurcation, namely the Bogdanov-Takens bifurcation, which happens when two codimension-1 bifurcations (namely the Hopf and saddle-node bifurcations) collide to get a codimension-2 bifurcation.

The main contribution of this thesis is providing full details of the proof for Theorem 41. While the proof of this theorem can be found in [2], it turns out that most of the details of this proof were left out, and to the best of the author's knowledge, a full and complete proof of this theorem (in all details) was lacking in the literature. The dynamics of Bogdanov-Takens bifurcations are very rich, which implies the existence of many things such as global bifurcation, saddle-node bifurcation, Hopf bifurcation (and limit cycle), and cusp equilibrium point as the bifurcation parameters vary. (See Figure 29.) For more information on global bifurcations, see [2] and [5].

There are several applications of Bogdanov-Takens bifurcations, particularly in Biology and Population Dynamics (see [6]).

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