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Energy Calculations and Wave Equations

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ENERGY CALCULATIONS AND WAVE EQUATIONS

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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Mathematics

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ABSTRACT

The focus of this thesis is to show how methods of Fourier analysis, in particular Parseval's equality, can be used to provide explicit energy calculations for solutions of wave equations in one dimension. These calculations are discussed for simple examples and then extended to fit the general wave equation with Robin boundary conditions. Ideas from Sobolev space theory are used to provide justification of the method.

KEYWORDS: wave equation, energy, Fourier series, Fourier coefficients, partial differential equations

This abstract is approved as to form and content

Dr. William O. Bray
Chairperson, Advisory Committee
Missouri State University

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1. INTRODUCTION

1.1 The Wave Equation

Consider the motion of a thin, flexible string, stretched taught horizontally. Suppose that string is displaced or plucked from its resting position and then allowed to vibrate. Classically, this string can be represented by the following equation with variables in time, t, and position, x:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < L, \quad t > 0. \tag{1.1}$$

In this equation, $\rho(x)$ and $k(x)$ are weight functions, $\rho(x)$ represents the mass density over the length of the string, $k(x)$ represents tension in the string, and L is the length of the string. Given appropriate information about the behavior of the string at the boundries (i.e. at $x = 0$ and $x = L$) and the initial state, this equation can be solved for the solution $u(x,t)$, describing the position of the string at point x and time t.

Once a solution is found it is also relevant to calculate other physical values to describe the motion of the string in space-time, namely potential energy and kinetic energy. These quantities may be expressed using the following integrals:

$$\begin{aligned} \text{PE:} &= \frac{1}{2}\int_0^L u_x^2(x,t)k(x)dx \\ \text{KE} &= \frac{1}{2}\int_0^L u_t^2(x,t)\rho(x)dx. \end{aligned} \tag{1.2}$$

Motivation for these quantities will be given in Section 4. A focus of this thesis is their computation through Fourier analysis.

1.2 The Clamped String

As an example, let us first consider the special case of a string that is clamped on either end (Dirichlet conditions). Under ideal conditions, tension and density are presumed to be constant and equal to one. This reduces (1.1) to

$$u_{tt} = u_{xx}. \tag{1.3}$$

For simplicity's sake, suppose the length of the string is π. The boundary conditions and initial conditions are as follows:

$$\text{BC:} \begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases} \qquad \text{IC:} \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases}$$

One form of the solution for such a problem can be found by the method of separation of variables. For this problem, the solutions are assumed to have the form $u(x,t) = X(x)T(t)$. From here, by substituting into the differential equation, the problem can be reduced to the following two problems with one variable with corresponding boundary conditions:

$$X'' + \mu X = 0, \quad 0 < x < \pi$$
$$\text{BC:} \begin{cases} X(0) = 0 \\ X(\pi) = 0 \end{cases}$$

and

$$T'' + \mu T = 0, \quad t > 0$$
$$\text{BC:} \left\{ T'(0) = 0. \right.$$

The solutions of these two equation will then depend on the value of μ, specifically whether μ is positive, negative, or zero. In this problem, $\mu = n^2$, for $n = 1, 2, 3, \ldots$ (Bray, 2012, p. 130). For the problem in x, the solutions are $X_n(x) = \sin nx$ and for the equation in t, the solutions are $T_n(t) = \cos nt$. So the separated solutions for the total wave equation are

$$u_n(x,t) = \cos nt \sin nx, \quad \text{for } n = 1, 2, 3, \ldots \tag{1.4}$$

This suggests a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} \hat{f}_s(n) \cos nt \sin nx$$

where $\hat{f}_s(n)$ are the Fourier sine coefficients for the initial state $f(x)$, specifically

$$\hat{f}_s(n) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \tag{1.5}$$

Now that we have a form for the solutions of the wave equation, we can continue learn more about the motion of this vibrating string by calculating its potential and kinetic energy.

1.3 Energy Calculations for the Clamped String

Since the formulas for the kinetic and potential energies have the form of the square of the L^2-norm of a function, it is natural to employ Parseval's equality, described as follows. Let $L^2([a,b], \rho dx)$ be the vector space of real valued square integrable functions with respect to $\rho = \rho(x) > 0$, called the weight function. This space is an inner product space with inner product

$$(f,g)\rho = \int_a^b f(x)g(x)\rho(x)dx \tag{1.6}$$

and norm

$$||f||_{2,\rho}^2 = \int_0^\pi f^2(x)\rho(x)dx = (f,f)_\rho. \tag{1.7}$$

Let $\{X_n\}_1^\infty$ be an orthogonal system in $L^2([a,b],\rho dx)$. If $f \in L^2([a,b],\rho dx)$, the Fourier series of f has the form

$$f \sim \sum_n \hat{f}(n)X_n(x),$$

where $\hat{f}(n)$ are the Fourier coefficients of f,

$$\hat{f}(n) = \frac{1}{||X_n||_2^2}\int_a^b f(x)X_n(x)\rho(x)dx. \tag{1.8}$$

The orthogonal system $\{X_n\}$ is complete in $L^2([a,b],\rho dx)$ if $\int_a^b f(x)X_n(x)\rho(x)dx = 0$ for all n implies that $f = 0$. For a complete orthogonal system, Parseval's equality is then

$$||f||_2^2 = \sum_{n=1}^{\infty} |\hat{f}(n)|^2||X_n||_2^2 \tag{1.9}$$

(Bray, 2012, p. 142). In the setting of the clamped string $\{\sin nx\}_{n=1}^\infty$ and $\{\frac{1}{2}, \cos nx\}_{n=1}^\infty$ form complete orthogonal sets in $L^2[0,\pi]$.

Fourier Coefficients for $\{\sin nx\}_{n=1}^\infty$ and $\{\frac{1}{2}, \cos nx\}_{n=1}^\infty$. Specifically for $\{\sin nx\}_1^n$,

$$\int_0^\pi \sin nx \sin mx = \begin{cases} \frac{\pi}{2} & n = m \\ 0 & n \neq m. \end{cases}$$

The Fourier coefficients have form

$$\hat{f}_s(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \dots$$

and Parseval (1.9) takes the explicit form

$$||f||_2^2 = \int_0^{\pi} f^2(x)dx = \frac{\pi}{2}\sum_{n=1}^{\infty} \hat{f}_s(n)^2.$$

Likewise, for the system $\{\frac{1}{2}, \cos nx\}_1^n$,

$$\int_0^{\pi} \cos nx \cos mx = \begin{cases} \frac{\pi}{2} & n = m \\ 0 & n \neq m. \end{cases}$$

The Fourier coefficients have form

$$\hat{f}_c(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 0, 1, \dots$$

and Parseval (1.9) takes the form

$$||f||_2^2 = \frac{\pi}{2}\sum_{n=0}^{\infty} \hat{f}_c(n)^2.$$

Similar explicit formulas hold in the case of Neumann boundary conditions and mixed boundary conditions (a Dirichlet condition on one end and a Neumann condition on the other), i.e.,

$$\text{Neumann BC:} \begin{cases} u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \end{cases}$$

$$\text{Mixed BC:} \begin{cases} u_x(0,t) = 0 & \text{or} \qquad u(0,t) = 0 \\ u(\pi,t) = 0 & \qquad\quad u_x(\pi,t) = 0 \end{cases}$$

Kinetic and Potential Energy. Using Parseval's equality (1.9) we can perform the following *formal* calculations. Since,

$$u(x,t) = \sum_{n=1}^{\infty} \hat{f}_s(n) \cos nt \sin nx,$$

the derivative with respect to t is

$$u_t(x,t) = -\sum_{n=1}^{\infty} n\hat{f}_s(n) \sin nt \sin nx.$$

Hence,

$$KE(t) = \frac{1}{2}\int_0^{\pi} u_t^2(x,t)dx = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) \sin^2 nt. \tag{1.10}$$

For the potential energy, we take the derivative of $u(x,t)$ with respect to x and find

$$u_x(x,t) = \sum_{n=1}^{\infty} n\hat{f}_s(n) \cos nt \cos nx.$$

Thus,

$$PE(t) = \frac{1}{2}\int_0^{\pi} u_x^2(x,t)dx = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) \cos^2 nt. \tag{1.11}$$

Note that from these representations of kinetic and potential energy the conservation law comes immediately. The total energy, E, can written as the sum of $KE(t)$ and $PE(t)$. Thus,

$$KE(t) + PE(t) = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) \sin^2 nt + \frac{\pi}{2}\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) \cos^2 nt = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) = E.$$

So the total energy is constant and independent of time.

These calculations were easily achievable as both $\{X_n\}$ and $\{X_n'\}$ were complete orthogonal sets, which allowed the use of Parseval's equality to calculate the integrals. However, this is not always the case. Some boundary conditions will not provide orthogonality for $\{X_n'\}$, which creates difficulty in the calculation for the potential energy from the string. In these cases, other methods must be found to calculate the kinetic and potential energies for the solution. One such case will be illustrated over the next sections.

2. WAVE EQUATION WITH ROBIN BOUNDARY CONDITIONS

The wave equation with general Robin boundary conditions on both ends of the string can be stated as follows:

$$u_{tt} = u_{xx}$$

$$\text{BC:} \begin{cases} a_{11}u_x(0,t) + a_{12}u(0,t) = 0, \quad a_{11}a_{12} \leq 0 \\ a_{21}u_x(\pi,t) + a_{22}u(\pi,t) = 0, \quad a_{21}a_{22} \geq 0 \end{cases} \tag{2.12}$$

$$\text{IC:} \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases}$$

It is elementary to transform these boundary conditions into conditions in which the constants a_{12} and a_{22} are -1 and 1, respectively. This can be achieved by dividing the first equation through by a_{12} and the second by a_{22}, resulting in boundary conditions of the form below.

$$\text{BC:} \begin{cases} b_1 u_x(0,t) - u(0,t) = 0, \\ b_2 u_x(\pi,t) + u(\pi,t) = 0, \quad b_1, b_2 \geq 0 \end{cases} \tag{2.13}$$

Proceeding by the method of separation of variables used in Section 1.2, the wave equation can be reduced to a problem in one variable, x, with boundary conditions derived from the conditions for the original differential equation.

$$X'' + \mu X = 0, \qquad 0 < x < \pi, \tag{2.14}$$

$$\begin{cases} b_1 X'(0) - X(0) = 0, \\ b_2 X'(\pi) + X(\pi) = 0, \quad b_1, b_2 \geq 0. \end{cases} \tag{2.15}$$

The form of a function, $X(x)$, that satisfies equation 2.14 depends on the value of μ, specifically whether μ is positive, μ is negative, or μ is equal to zero. So, in seeking the functions $X(x)$ that satisfy 2.14 and the boundary conditions 2.15, we address three cases based on these values for μ.

1. If $\mu = -\lambda^2 < 0$,

$$\begin{cases} X(x) = c_1 \cosh \lambda x + c_2 \sinh \lambda x \\ X'(x) = \lambda c_1 \sinh \lambda x + \lambda c_2 \cosh \lambda x \end{cases}$$

   Thus, from the first boundary condition (BC_1),

$$0 = b_1 \lambda c_2 \cosh 0 - c_1 \cosh 0$$
$$c_1 = b_1 \lambda c_2$$

   From the second boundary condition (BC_2),

$$0 = b_2 \lambda c_1 \sinh \lambda\pi + b_2 \lambda c_2 \cosh \lambda\pi + c_1 \cosh \lambda\pi + c_2 \sinh \lambda\pi,$$
$$0 = (b_2 \lambda c_1 + c_2) \sinh \lambda\pi + (b_2 \lambda c_2 + c_1) \cosh \lambda\pi,$$
$$0 = c_2[(b_1 b_2 \lambda^2 + 1) \sinh \lambda\pi + (b_2 \lambda + b_1 \lambda) \cosh \lambda\pi].$$

   In general, the expression in brackets is positive, so $c_2 = 0$. This implies that $c_1 = 0$, so the first case is trivial.

2. If $\mu = 0$,

$$\begin{cases} X(x) = c_1 x + c_2 \\ X'(x) = c_1 \end{cases}$$

   So from BC_1, $0 = b_1 c_1 - c_2$, implying that $c_2 = b_1 c_1$.

   And from BC_2, $0 = b_2 c_1 + c_1 \pi + c_2 = (b_2 + \pi + b_1) c_1$.

Clearly, $(b_2 + \pi + b_1) > 0$, so $c_1 = 0$ and thus, $c_2 = 0$. So Case 2 is also trivial.

3. Finally, if $\mu = \lambda^2 > 0$,

$$\begin{cases} X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x \\ X'(x) = -\lambda c_1 \sin \lambda x + \lambda c_2 \cos \lambda x \end{cases}$$

Thus, from BC_1, $\quad 0 = b_1 \lambda c_2 - c_1 \rightarrow c_1 = b_1 \lambda c_2$.

From BC_2,

$$0 = -b_2 \lambda c_1 \sin \lambda\pi + b_2 \lambda c_2 \cos \lambda\pi + c_1 \cos \lambda\pi + c_2 \sin \lambda\pi$$
$$0 = (c_2 - b_2 \lambda c_1) \sin \lambda\pi + (c_1 + b_2 \lambda c_2) \cos \lambda\pi$$
$$0 = c_2[(1 - b_1 b_2 \lambda^2) \sin \lambda\pi + (b_1 \lambda + b_2 \lambda) \cos \lambda\pi]$$

Now, $c_2 \neq 0$, as this would make all solutions trivial. So, the bracketed expression must be equal to zero. Thus,

$$(b_1 b_2 \lambda^2 - 1) \sin \lambda\pi = (b_1 \lambda + b_2 \lambda) \cos \lambda\pi,$$

or

$$\tan \lambda\pi = \frac{(b_1 \lambda + b_2 \lambda)}{(b_1 b_2 \lambda^2 - 1)}.$$

This equation provides a scenario in which every λ_n that satisfies the equation corresponds to a solution, X_n. So solutions take the following form:

$$\begin{cases} X_n(x) = b_1 \lambda_n \cos \lambda_n x + \sin \lambda_n x \\ \tan \lambda\pi = \frac{(b_1 \lambda + b_2 \lambda)}{(b_1 b_2 \lambda^2 - 1)}. \end{cases} \tag{2.16}$$

2.1 The Tangent Condition

The equation determining the values of the λ_ns is the transcendental equation $\tan \lambda\pi = \frac{(b_1\lambda + b_2\lambda)}{(b_1 b_2 \lambda^2 - 1)}$. Graphically, we can look at the intersections of the graphs of $\tan \lambda\pi$ and $\frac{(b_1\lambda + b_2\lambda)}{(b_1 b_2 \lambda^2 - 1)}$, restricting the domain to $\lambda > 0$, to visualize what values these λ_ns take on and determine if there are patterns in their distribution. Consider the graphs (Figures 1 and 2) with two different sets of conditions on b_1 and b_2.

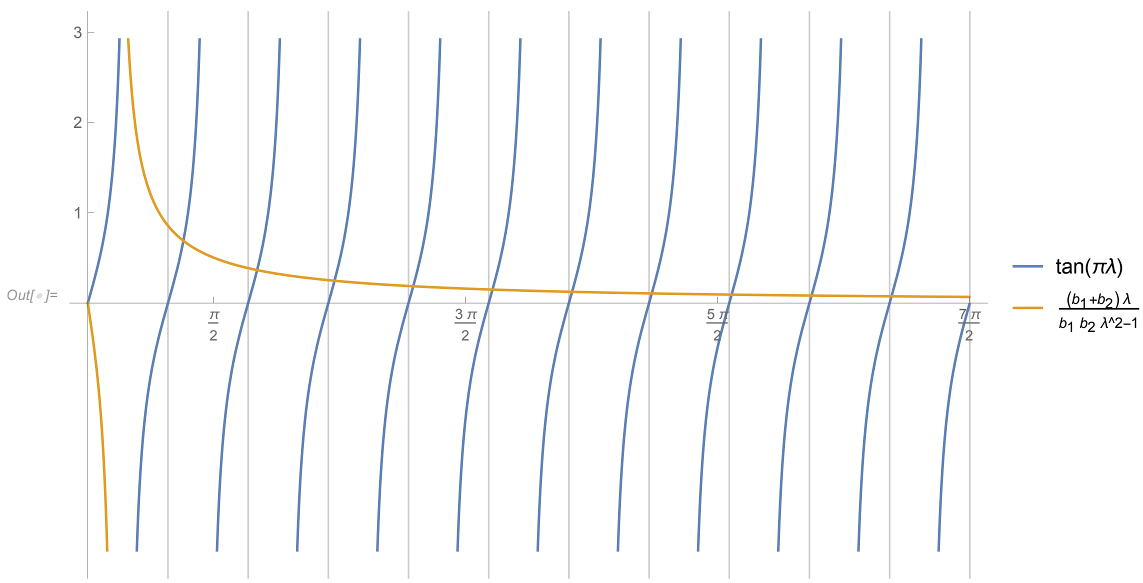

Figure 1: Tangent Condition with $b_1 = 4, b_2 = 2$

In both figures, the fractional expression consists of a negative and a positive portion and intersects with the tangent graph an infinite number of times. These particular examples demonstrate two possibilities: one in which there are no intersections below the x-axis (Figure 1) and one in which there are one or more intersections below the graph (Figure 2). While this is worth observing, the intersections of most interest are those that occur as λ approaches infinity. Considering the positive portion of the rational expression in both figures, we can begin to see some regularity in how often the intersections occur. As the graph approaches zero, intersections occur once per period relative to the tangent function, and these intersec-

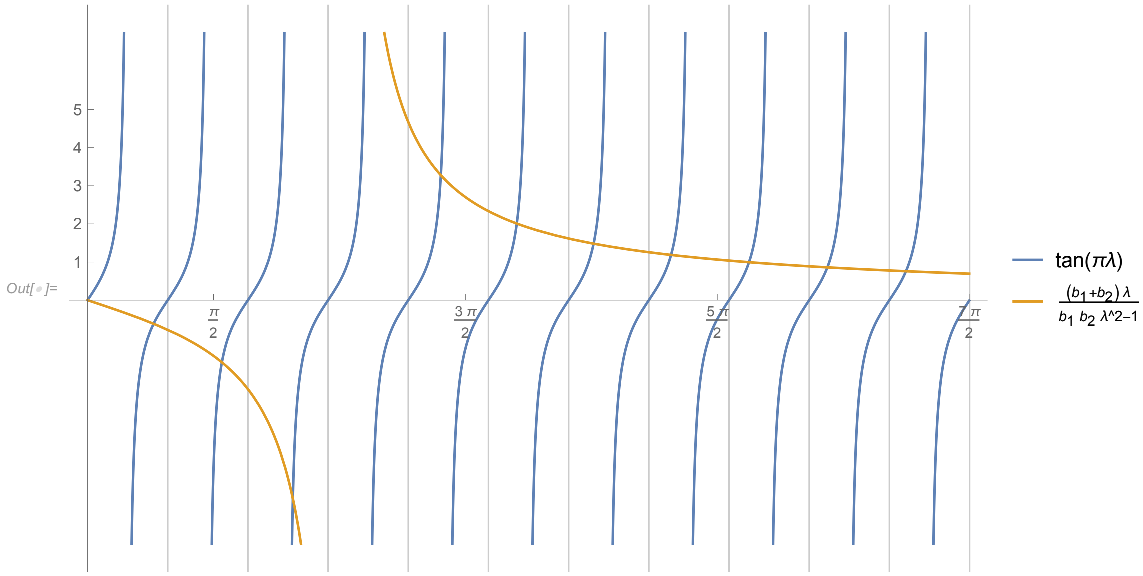

Figure 2: Tangent Condition with $b_1 = 0.5, b_2 = 0.2$

tions begin to occur closer and closer to the inflection points of the tangent graph. In other words, the values of λ_n approach the integers, which are represented by grey vertical lines in Figures 1 and 2.

2.2 The Value of λ_n

Now, through observation, we have seen that the values of the λ_n's appears to approach the integers as $n \to \infty$. This can be demonstrated analytically, as follows. First, recall:

$$X'' + \mu X = 0$$
$$\begin{cases} b_1 X'(0) - X(0) = 0 \\ b_2 X'(\pi) + X(\pi) = 0 \end{cases}$$

and

$$\begin{cases} \tan \lambda\pi = \frac{(b_1\lambda + b_2\lambda)}{(b_1 b_2 \lambda^2 - 1)} \\ X(x) = b_1\lambda \cos \lambda x + \sin \lambda x. \end{cases}$$

Note that $\tan \lambda\pi$ from the equation above, has zeros at the integers, $n = 1, 2, 3, \ldots$

Applying Taylor's theorem at $x = n$,

$$\begin{aligned} \tan \lambda\pi &= \tan(n) + \frac{d}{d\lambda}(\tan n\pi)(\lambda - n) + \mathcal{O}((\lambda - n)^2) \\ &= \pi \sec^2(n\pi)(\lambda - n) + \mathcal{O}((\lambda - n)^2) \\ &= \pi(\lambda - n) + \mathcal{O}((\lambda - n)^2) \end{aligned}$$

Set $\epsilon_n = \lambda_n - n$, then the expansion can be rewritten as follows:

$$\begin{aligned} \pi(\lambda - n) + \mathcal{O}((\lambda - n)^2) &= \frac{(b_1 + b_2)\lambda}{(b_1 b_2 \lambda^2 - 1)} \\ -\pi(\epsilon_n) + \mathcal{O}(\epsilon_n^2) &= \frac{(b_1 + b_2)(\epsilon_n + n)}{(b_1 b_2 (\epsilon_n + n)^2 - 1)} = \mathcal{O}(\frac{1}{n}) \end{aligned}$$

It follows that $\epsilon_n = \mathcal{O}(\frac{1}{n})$ and thus, $\lambda_n = n + \mathcal{O}(\frac{1}{n})$. Therefore, as $n \to \infty$, $\lambda_n \to n$.

2.3 Orthogonality of $\{X_n(x)\}$

Now, that we have a sense of the type of solutions that this Robin conditioned case produces, let us return to the subject of orthogonality. In order to calculate the potential and kinetic energies using Parseval's equality, the sets $\{X_n(x)\}$ and $\{X_n'(x)\}$ must both be orthogonal.

Theorem 2.1: Given the boundary conditions from (2.13), $\{X_n(x)\}$ is an orthogonal set.

Proof. By definition, the set $\{X_n(x)\}$ from (2.13) is orthogonal if and only if

$\int_0^\pi X_n(x)X_m(x)dx = 0$ for all $n \neq m$. Using the differential equation and then integrating by parts we get the following:

$$
\begin{aligned}
-\mu_n \int_0^\pi X_n(x)X_m(x)dx &= \int_0^\pi X_n'' X_m dx \qquad (2.17)\\
&= [X_n' X_m|_0^\pi - \int_0^\pi X_n' X_m' dx \\
&= [X_n' X_m|_0^\pi - [X_n X_m'|_0^\pi + \int_0^\pi X_n X_m'' dx \\
&= [X_n' X_m|_0^\pi - [X_n X_m'|_0^\pi - \mu_m \int_0^\pi X_n X_m dx,
\end{aligned}
$$

Then, by collecting the integral terms on the left-hand side and applying the boundary conditions,

$$
\begin{aligned}
(\mu_m - \mu_n) \int_0^\pi X_n(x)X_m(x)dx &= [X_n' X_m]_0^\pi - [X_n X_m']_0^\pi \\
&= X_n'(\pi)X_m(\pi) - X_n'(0)X_m(0) - X_n(\pi)X_m'(\pi) + X_n(0)X_m'(0) \\
&= -b_2 X_n'(\pi)X_m'(\pi) - b_1 X_n'(0)X_m'(0) + b_2 X_n'(\pi)X_m'(\pi) + b_1 X_n'(0)X_m'(0) = 0
\end{aligned}
$$

So $\int_0^\pi X_n(x)X_m(x)dx = 0$, and since n and m are arbitrary, $\{X_n(x)\}$ is an orthogonal set in $L^2[0,\pi]$. □

REMARK 2.2: Formula 2.17 may be used to demonstrate orthogonality for any wave equation with Dirichlet, Neumann, or mixed boundary conditions.

2.4 Non-Orthogonality of $\{X_n'(x)\}$

Here we prove the following result:

THEOREM 2.3: For (2.13), $\{X_n'(x)\}$ is not orthogonal for $b_1, b_2 > 0$. Furthermore, $\{X_n'(x)\}$ can only be an orthogonal set in the case where $b_1 = b_2 = 0$.

Proof. $\{X_n'(x)\}$ not orthogonal means that $\int_0^\pi X_n'(x)X_m'(x)dx \neq 0$, for some n and

m. By integrating by parts we see that,

$$\int_0^\pi X_n'(x)X_m'(x)dx = [X_n'X_m]_0^\pi - \int_0^\pi X_n''(x)X_m(x)dx$$
$$\int_0^\pi X_n'(x)X_m'(x)dx + \int_0^\pi X_m''(x)X_m(x)dx = X_n'(\pi)X_m(\pi) - X_n'(0)X_m(0)$$
$$\int_0^\pi X_n'(x)X_m'(x)dx - \mu_1\int_0^\pi X_m(x)X_m(x)dx = X_n'(\pi)X_m(\pi) - X_n'(0)X_m(0)$$

But $\{X_n(x)\}$ is an orthogonal set, so $\int_0^\pi X_n(x)X_m(x)dx = 0$. Thus,

$$\begin{aligned}\int_0^\pi X_n'(x)X_m'(x)dx &= X_n'(\pi)X_m(\pi) - X_n'(0)X_m(0)\\ &= (-b_1\lambda_n^2\sin\lambda_n\pi + \lambda_n\cos\lambda_n\pi)(b_1\lambda_m\cos\lambda_m\pi + \sin\lambda_m\pi) - b_1\lambda_n\lambda_m\\ &= (-b_1\lambda_n^2\tan\lambda_n\pi + \lambda_n)(b_1\lambda_m + \tan\lambda_m\pi)\cos\lambda_n\pi\cos\lambda_m\pi - b_1\lambda_n\lambda_m\end{aligned}$$

Recall that $\tan\lambda\pi = \frac{(b_1+b_2)\lambda}{(b_1b_2\lambda^2-1)}$. For clarity, let $\Delta_n = b_1b_2\lambda^2 - 1$. Then, the right hand side of the above becomes,

$$\begin{aligned}&\left(\frac{-b_1\lambda_n^2(b_1+b_2)\lambda_n}{\Delta_n} + \lambda_n\right)\left(b_1\lambda_m + \frac{(b_1+b_2)\lambda_m}{\Delta_m}\right)\cos\lambda_n\pi\cos\lambda_m\pi - b_1\lambda_n\lambda_m\\ &= \lambda_n\lambda_m\left(\frac{-b_1\lambda_n^2(b_1+b_2)}{\Delta_n} + 1\right)\left(b_1 + \frac{(b_1+b_2)}{\Delta_m}\right)\cos\lambda_n\pi\cos\lambda_m\pi - b_1\lambda_n\lambda_m\\ &= \lambda_n\lambda_m\left(\frac{-b_1^2\lambda_n^2 - b_1b_2\lambda_n^2 + b_1b_2\lambda_n^2 - 1}{\Delta_n}\right)\left(\frac{b_1^2b_2\lambda_m^2 - b_1 + b_1 + b_2}{\Delta_m}\right)\cos\lambda_n\pi\cos\lambda_m\pi\\ &\qquad - b_1\lambda_n\lambda_m\\ &= -\left[b_2\lambda_n\lambda_m\left(\frac{(b_1^2\lambda_n^2+1)(b_1^2\lambda_m^2+1)}{\Delta_n\Delta_m}\right)\cos\lambda_n\pi\cos\lambda_m\pi + b_1\lambda_n\lambda_m\right]\end{aligned}$$

Now, $\lambda_n\lambda_m$ is positive and $(b_1^2\lambda_n^2+1)(b_1^2\lambda_m^2+1)$ is positive. We can choose $\cos\lambda_n\pi$ and $\cos\lambda_m\pi$ to have the same sign. We know that such values for λ exist by considering Figure 3, which displays Figure 1 overlayed with $\cos\lambda_n\pi$.

Clearly, since the intersections (the desired values of λ) occur once per period in the tangent graph and since the cosine graph alternates between positive

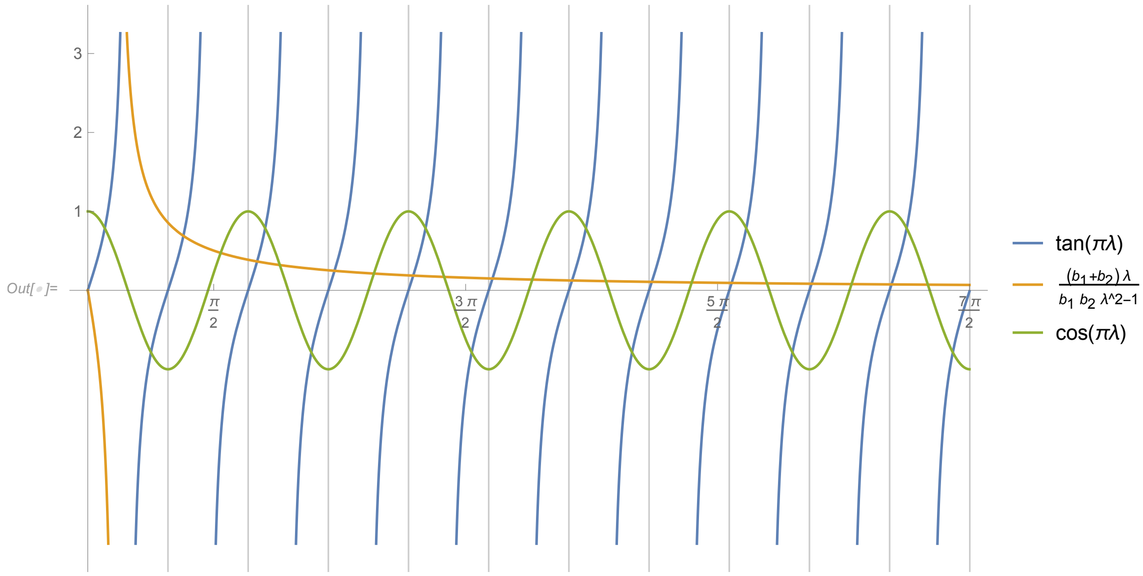

Figure 3: Tangent Condition with $b_1 = 4, b_2 = 2$ and $\cos \lambda_n \pi$

and negative values and has a period twice as long as the tangent graph, there will exist values for λ such that $\cos \lambda_n \pi$ is positive and values such that $\cos \lambda_n \pi$ is negative. So, if the values of $\cos \lambda_n \pi$ and $\cos \lambda_m \pi$ have the same sign, and $b_1, b_2 > 0$, this quantity is strictly non-zero. Thus, $\int_0^\pi X_n'(x) X_m'(x) dx \neq 0$ and $\{X_n'(x)\}$ is not an orthogonal set.

If b_1 is allowed to equal 0, the expression reduces to

$$\int_0^\pi X_n'(x) X_m'(x) dx = -b_2 \lambda_n \lambda_m \cos \lambda_n \pi \cos \lambda_m \pi,$$

which can only reduced to zero (making $\{X_n'(x)\}$ orthogonal) if $b_2 = 0$. Likewise, if b_2 is allowed to equal 0, the expression reduces to

$$\int_0^\pi X_n'(x) X_m'(x) dx = -b_1 \lambda_n \lambda_m,$$

again requiring $b_1 = 0$ to produce orthogonality in $\{X_n'(x)\}$. $\square$

Because of the above non-orthogonality, direct use of Parseval's equality in the case of Robin boundary conditions is impossible.

3. SOBOLEV SPACE

3.1 General Ideas

In this section, we describe essential ideas behind Sobolev spaces and apply them to the Fourier expansion. General reference for Sobolev Spaces is from Bray (2018) and Jost (2013, p. 215). In addition to considering the boundary conditions surrounding $u(x,t)$ and their affects on our ability to calculate potential and kinetic energy, we must also consider the initial state for $u(x,t)$. What conditions on $f(x)$ must be present in order to have a solution? Clearly from (1.10) and (1.11), $\sum_{n=1}^{\infty} n^2 \hat{f}_s^2(n) < \infty$, for a solution in the clamped case to make sense. Also, the initial state $f(x)$ must possess a certain degree of continuity in order for it to have the required derivatives. These derivatives do not necessarily have to be the "strong" derivatives from calculus. The calculations can be done as long as a weak derivative exists. We define a function $u \in L^1_{loc}(a,b)$ to have a weak derivative $v \in L^1_{loc}(a,b)$ if for all $\phi \in C_c^{\infty}(a,b)$,

$$\int_a^b v\phi \, dx = -\int_a^b u\phi' dx. \tag{3.18}$$

Keeping this definition in mind, consider the following Sobolev space. Define:

$$W^{1,2}[a,b] = \{u \in L^2[a,b] \,|\, u' \in L^2[a,b]\}. \tag{3.19}$$

In this space suppose the inner product, $(u,v)_{W^{1,2}}$, is

$$(u,v)_{W^{1,2}} = \int_a^b uv \, dx + \int_a^b u'v' \, dx$$

and the norm is

$$||u||_{W^{1,2}} = (||u||^2_{L^2} + ||u'||^2_{L^2})^{\frac{1}{2}}.$$

Given these definitions for norm and inner product $W^{1,2}[a,b]$ is also a Hilbert space.

We begin with two lemmas.

LEMMA 3.1: Let $v \in C[a,b]$ and let $u(x)$ be any anti-derivative of v ($u' = v$). Then,

(i) for $a \leq x, y \leq b$, $\quad |u(x) - u(y)| \leq |x-y|^{\frac{1}{2}} ||u'||_{L^2[a,b]}$

(ii) for some constant c, $\quad ||u||_{C[a,b]} \leq c||u||_{W^{1,2}[a,b]}$.

Proof. (i) Let $a \leq x < y \leq b$. Then, using the Cauchy-Schwarz inequality,

$$|u(x) - u(y)| = |\int_x^y v\, dx| \leq |y-x|^{\frac{1}{2}} ||v||_{L^2[a,b]}.$$

(ii) Let $c \in [a,b]$ so that $u(c) = \frac{1}{b-a}\int_a^b u(s)ds$. Then,

$|u(c)| \leq (b-a)^{-\frac{1}{2}} ||u||_{L^2[a,b]}$ and from part (i), $|u(x) - u(c)| \leq |c-x|^{\frac{1}{2}} ||u'||_{L^2}$.

So,

$$\begin{aligned}
|u(x)| &\leq |u(c)| + |u(x) - u(c)| \\
&\leq (b-a)^{-\frac{1}{2}} ||u||_{L^2} + |b-a|^{\frac{1}{2}} ||u'||_{L^2} \\
&\leq \max\{(b-a)^{-\frac{1}{2}}, (b-a)^{\frac{1}{2}}\}(||u||_{L^2} + ||u'||_{L^2}) \\
&\leq \max\{(b-a)^{-\frac{1}{2}}, (b-a)^{\frac{1}{2}}\}(\sqrt{2}||u||_{W^{1,2}[a,b]}).
\end{aligned}$$

Note: The last line is due to the following inequality. For $s, t \geq 0$,

$$\begin{aligned}
(s+t)^2 &= s^2 + 2st + t^2 = s^2 + 2\sqrt{s^2t^2} + t^2 \\
&\leq s^2 + t^2 + 2(\frac{s^2+t^2}{2}) = 2(s^2+t^2) \\
(s+t) &\leq \sqrt{2}(s^2+t^2).
\end{aligned}$$

□

LEMMA 3.2: $C^1[a,b]$ is dense in $W^{1,2}[a,b]$.

Proof. Let $u \in W^{1,2}[a,b]$. Let $\{v_n\} \subset C[a,b]$ such that $||v_n - u'||_{L^2[a,b]} \to 0$, as

$n \to \infty$ (i.e. $v_n \to u'$).

Set $U_n(x) = \int_a^x v_n(s)ds$, in other words $U_n' = v_n$.

(a) Using (i) in Lemma 3.1 we have,

$$|(U_n(x) - U_m(x)) - (U_n(a) - U_m(a))| \leq |x-a|^{\frac{1}{2}} ||v_n - v_m||_{L^2[a,b]}$$

but since $U_n(a) = U_m(a) = 0$, this simplifies to

$$|(U_n(x) - U_m(x))| \leq |b-a|^{\frac{1}{2}} ||v_n - v_m||_{L^2[a,b]}.$$

Hence, $\{U_n(x)\}$ is Cauchy in $C[a,b]$ which means that $U_n \xrightarrow{C[a,b]} U \in C[a,b]$.

(b) Claim: $U = u$. To see this let $\phi \in C_c^\infty[a,b]$. Then,

$$\int_a^b U_n' \phi \, dx = -\int_a^b U_n \phi' \, dx \to -\int_a^b U \phi' \, dx = \int_a^b U' \phi \, dx.$$

Since $U_n' = v_n$,

$$\int_a^b U_n' \phi \, dx = \int_a^b v_n \phi \, dx \to \int_a^b u' \phi \, dx.$$

It follows that $u = U + k$, where k is a constant.

(c) Let $u_n = U_n + k$. Then $u_n \to u$ in $C[a,b]$ and in $L^2[a,b]$. Furthermore, $u_n' = v_n \to u'$ in $L^2[a,b]$. The sequence $\{u_n\} \subset C^1[a,b]$ with $u_n \xrightarrow{W[1,2]} u$.

□

The fundamental result regarding the structure of Sobolev spaces is as follows:

THEOREM 3.3: (Sobolev Embedding Theorem) $W^{1,2}[a,b] \subset C[a,b]$. Moreover, the Sobolev inequality holds:

$$||u||_{C[a,b]} \leq c||u||_{W^{1,2}[a,b]}. \tag{3.20}$$

Proof. Let $\{u_n\} \subset C^1[a,b]$ such that $u_n \overset{W[1,2]}{\longrightarrow} u$. Then from Lemma 3.1,

$$||u_n||_{C[a,b]} \leq c||u_n||_{W^{1,2}[a,b]}.$$

Let $n \to \infty$. $\square$

3.2 A Return to the Dirichlet Example

Now, let us return to the clamped string described in Section 1.2 and consider the following two spaces of functions.

$$C^1_{dd}[0,\pi] = \{f \in C^1[0,\pi] | f(0) = f(\pi) = 0\} \tag{3.21}$$

If a function, f, is in $C'_{dd}[0,\pi]$ then the following equality holds concerning the Fourier cosine and Fourier sine coefficients.

$$\begin{aligned}\hat{f'}_c(n) &= \frac{2}{\pi}\int_0^\pi f'(x)\cos nx\,dx \\ &= \frac{2}{\pi}\Big([f(x)\cos nx]_0^\pi + \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \\ &= n\hat{f}_s(n)\end{aligned}$$

Define $W^{1,2}_{dd}[0,\pi]$ to be the closure of $C^1_{dd}[0,\pi]$ in $W^{1,2}[0,\pi]$. By the density argument in Lemma 3.2, if $f \in W^{1,2}_{dd}[0,\pi]$, then $\hat{f'}_c(n) = n\hat{f}_s(n)$. Hence, $\sum_{n=1}^\infty n^2\hat{f}_s^2(n) < \infty$, satisfying the requirements discussed at the beginning of Section 3.1. This condition characterizes $W^{1,2}_{dd}[0,\pi]$, making it the right class of functions for performing the energy calculations in Section 1.3. Furthermore,

$$u(x,t) = \sum_n \hat{f}_s(n)\cos nt \sin nx$$

gives a distributional (or weak) solution of $u_{tt} = u_{xx}$: for $\phi = \phi(x,t) \in C_c^\infty([0,\pi] \times [0,\pi])$,

$$\int_0^\pi \int_0^\pi u(x,t)[\phi_{tt}(x,t) - \phi_{xx}(x,t)]dxdt = 0.$$

Details will be seen more generally in a later section.

3.3 Boundary Condition Variation of the Space $W^{1,2}[0,\pi]$

Let $k \in C^1[a,b]$, and $\rho \in C[a,b]$ with $k, \rho > 0$ on $[a,b]$. Separation of variables applied to the general wave equation (1.1) leads us to the boundary value problem

$$(kX')' + \mu\rho X = 0, \quad a < x < b$$

BC: D, N, M, or R

where the boundary conditions are D (Dirichlet), N (Neumann), M (mixed Dirichlet and Neumann), or R (Robin).

Classically, (Birkhoff & Rota, 1989) we know there is a sequence $\{\mu_n\}_1^\infty$ of eigenvalues, μ_n, increasing to infinity such that the corresponding eigenfunctions $\{X_n\}_1^\infty$, $(\subset C^1[a,b])$, form a complete orthogonal system in $L^2([a,b], \rho dx)$, with the inner product from (1.6).

Lemma 3.4: If the boundary conditions are D (Dirichlet), N (Neumann), or M (mixed), then $\{X_n'(x)\}_1^\infty$ forms an orthogonal system in $L^2([a,b], kdx)$.

Proof. For $n \neq m$,

$$\begin{aligned}\int_a^b X_n' X_m' k \, dx &= [X_n X_m' k]_a^b - \int_a^b X_n (kX_m')' \, dx \\ &= \mu_m \int_a^b X_n X_m \rho \, dx.\end{aligned}$$

The latter integral is zero for $n \neq m$. $\square$

Recall, firstly, if $w = w(x)$ is a positive function, the norm on $L^2([a,b], wdx)$ is denoted $||\cdot||_{2,w}$. From the above proof we get:

$$||X_n'||_{2,k}^2 = \mu_n ||X_n||_{2,\rho}^2. \tag{3.22}$$

Secondly, a zero eigenvalue occurs only if the boundary conditions are Neumann. Also, for the Sobolev space $W^{1,2}[a,b]$, it is useful to use the inner product and norm

$$(f,g) = \int_a^b f(x)g(x)\rho(x)dx + \int_a^b f'(x)g'(x)k'(x)dx, \tag{3.23}$$

$$||f||_{W^{1,2}} = [||f||_{2,\rho}^2 + ||f'||_{2,k}^2]^{\frac{1}{2}}. \tag{3.24}$$

As $0 < \rho_1 \leq \rho(x) \leq \rho_2$ and $0 < k_1 \leq k(x) \leq k_2$, this norm is equivalent to the usual Sobolev norm.

Given the above framework, we have two orthogonal systems: $\{X_n(x)\}_1^\infty$ with respect to ρdx and $\{X_n'(x)\}_1^\infty$ with respect to kdx. Given g on $[a,b]$ we write the Fourier coefficients of g with respect to the two systems as:

$$\begin{aligned}(\hat{g})_\rho(n) &= \frac{1}{||X_n||_{2,\rho}^2}\int_a^b g(x)X_n(x)\rho(x)dx \\ (\hat{g})_k(n) &= \frac{1}{||X_n'||_{2,k}^2}\int_a^b g(x)X_n'(x)k(x)dx.\end{aligned} \tag{3.25}$$

If $g \in C^1[a,b]$, then

$$\begin{aligned}(\hat{g'})_k(n) &= \frac{1}{||X_n'||_{2,k}^2}\int_a^b g'(x)[k(x)X_n'(x)]dx \\ &= \frac{1}{||X_n'||_{2,k}^2}[g(x)X_n'(x)k(x)]_a^b - \frac{1}{||X_n'||_{2,k}^2}\int_a^b g(x)[k(x)X_n'(x)]'dx \\ &= \text{boundary terms} + \frac{\mu_n}{||X_n'||_{2,k}^2}\int_a^b g(x)X_n(x)\rho(x)dx\end{aligned}$$

$$= \text{boundary terms} + (\hat{g})_\rho(n), \tag{3.26}$$

from (3.22). In the expression, the boundary terms are:

$$\text{Boundary Terms} = \frac{1}{||X_n'||_{2,k}^2}[g(b)X_n'(b)k(b) - g(a)X_n'(a)k(a)]. \tag{3.27}$$

To have a useful formula we need the boundary terms to be equal to zero. This spells out additional conditions on g dependent on the underlying boundary conditions. For now we restrict to boundary conditions of form D, N, or M.

BC	C^1 -class	$W^{1,2}$ -completion
dd	$C^1_{d,d}[a,b] \quad (g(a) = g(b) = 0)$	$W^{1,2}_{d,d}[a,b]$
nn	$C^1[a,b]$	$W^{1,2}[a,b]$
dn	$C^1_{d,}[a,b] \quad (g(a) = 0)$	$W^{1,2}_{d,}[a,b]$
nd	$C^1_{,n}[a,b] \quad (g(b) = 0)$	$W^{1,2}_{,n}[a,b]$

If g is in one of the C^1-classes of the table, the the boundary terms in (3.27) vanish.

Notation: Given boundary conditions of the type D, N, or M, we let $W^{1,2}_{bc}[a,b]$ be the entry in the right column of the table.

Proposition 3.5: Let $f \in W^{1,2}_{bc}[a,b]$. Then

$$(\hat{f'})_k(n) = (\hat{f})_\rho(n). \tag{3.28}$$

Proof. Let $\{g_m\} \subset C^1_{bc}[a,b]$ such that $||g_m - f||_{W^{1,2}_{bc}} \to 0$. We know such a function exists from Lemma 3.2. Then,

$$\begin{aligned}(\hat{f'})_k(n) &= (\widehat{f' - g_m'})_k(n) + (\widehat{g_m'})_k(n) \\ &= (\widehat{f' - g_m'})_k(n) + (\widehat{g_m})_\rho(n)\end{aligned}$$

$$= (\widehat{f' - g'_m})_k(n) + (\widehat{g_m - f})_\rho(n) + (\hat{f})_\rho(n). \tag{3.29}$$

However, by the Cauchy-Schwartz inequality, for each $n \in \mathbb{N}$,

$$|(\widehat{f' - g'_m})_k(n)| \leq ||f' - g'_m||_{2,k}||X'_n||_{2,k}^{-1} \to 0,$$

as $m \to \infty$, and

$$|(\widehat{g_m - f})_\rho(n)| \leq ||g_m - f||_{2,\rho}||X_n||_{2,\rho}^{-1} \to 0,$$

as $m \to \infty$. Then, using these inequalities along with (3.29),

$$|(\hat{f'})_k(n) - (\hat{f})_\rho(n)| \leq |(\widehat{f' - g'_m})_k(n)| + |(\widehat{g_m - f})_\rho(n)|.$$

As the quantities on the right hand side approach zero as $m \to \infty$,

$$(\hat{f'})_k(n) = (\hat{f})_\rho(n).$$

$\square$

REMARK 3.6: If $k = \rho = 1$ with Dirichlet boundary conditions, (3.28) is equivalent to the desired formula:

$$\hat{f'}_c(n) = n\hat{f}_s(n),$$

where $\hat{f'}_c(n)$ are the Fourier cosine coefficients of f'.

PROPOSITION 3.7: The collection $\{X_n\}$ is a complete orthogonal system in $W^{1,2}_{bc}[a, b]$.

Proof. Orthogonality is immediate:

$$(X_n, X_m)_{W^{1,2}} = (X_n, X_m)_\rho + (X'_n, X'_m)_k = 0$$

for $n \neq m$. To prove completeness, suppose $f \in W_{bc}^{1,2}[a,b]$ with $(f, X_n)_{W^{1,2}} = 0$ for all n. We must show $f = 0$. Now,

$$\begin{aligned}
(f, X_n)_{W^{1,2}} &= (f, X_n)_\rho + (f', X_n')_k \\
&= ||X_n||_{2,\rho}^2 (\hat{f})_\rho(n) + ||X_n'||_{2,k}^2 (\hat{f'})_k(n) \\
&= ||X_n||_{2,\rho}^2 (1 + \mu_n)(\hat{f})_\rho(n),
\end{aligned}$$

using (3.22) and (3.28). Hence if $(f, X_n)_W^{1,2} = 0$ for all n, then $(\hat{f})_\rho(n) = 0$ for all n and $f = 0$ since $\{X_n\}$ is complete in $L^2([a,b], \rho dx)$. □

Corollary 3.8: (Parseval) Let $f \in W_{bc}^{1,2}[a,b]$. Then,

$$||f||_{W^{1,2}}^2 = \sum_n (\hat{f})_\rho(n)^2 ||X_n||_{2,\rho}^2 + \sum_n \mu_n (\hat{f})_\rho(n)^2 ||X_n||_{2,\rho}^2.$$

Proof. From Lemma (3.4), the set X_n is a complete orthogonal system relative to the inner product on $W_{bc}^{1,2}[a,b]$. We have

$$\begin{aligned}
||X_n||_{W^{1,2}}^2 &= ||X_n||_{2,\rho}^2 + ||X_n'||_{2,k}^2 \\
&= ||X_n||_{2,\rho}^2 (1 + \mu_n).
\end{aligned}$$

The Fourier coefficients of f in $W_{bc}^{1,2}[a,b]$ are given by

$$\begin{aligned}
\hat{f}(n) &= \frac{(f, X_n)_{W^{1,2}}}{||X_n||_{W^{1,2}}^2} \\
&= \frac{1}{||X_n||_{2,\rho}^2 (1 + \mu_n)} [\int_a^b f(x) X_n(x) \rho(x) dx + \int_a^b f'(x) X_n'(x) k(x) dx] \\
&= \frac{1}{||X_n||_{2,\rho}^2 (1 + \mu_n)} [||X_n||_{2,\rho}^2 (\hat{f})_\rho(n) + ||X_n'||_{2,k}^2 (\hat{f'})_k(n)] \\
&= \frac{1}{||X_n||_{2,\rho}^2 (1 + \mu_n)} [||X_n||_{2,\rho}^2 (\hat{f})_\rho(n) + \mu_n ||X_n||_{2,\rho}^2 (\hat{f})_\rho(n)] \\
&= (\hat{f})_\rho(n)
\end{aligned}$$

using (3.28) and (3.22). Parseval's equality in $W^{1,2}_{bc}[a,b]$ is

$$||f||^2_{W^{1,2}} = \sum_n (\hat{f})(n)^2 ||X_n||^2_{W^{1,2}}.$$

Using the definition of $||f||_{W^{1,2}}$ and the above formulas concludes the proof.

□

Furthermore, since $||f||^2_{W^{1,2}} = ||f||^2_{2,\rho} + ||f'||^2_{2,k}$ we get

$$\begin{aligned} ||f'||^2_{2,k} &= \sum_n \mu_n \hat{f}_\rho(n)^2 ||X_n||^2_{2,\rho} \\ &= \sum_n \mu_n (\hat{f'})_k(n)^2 ||X'_n||^2_{2,k}, \end{aligned}$$

using (3.28).

A variation on the above completeness result is as follows.

PROPOSITION 3.9: (1) If the boundary conditions are D, then the collection $\{X'_n\}$ is complete in the Hilbert space

$$L^2_0[a,b] = \{g \in L^2([a,b], kdx) | \int_a^b g dx = 0\}.$$

(2) If the boundary conditions are N or M, the collection $\{X'_n\}$ is complete in $L^2([a,b], kdx)$.

Proof. (1) Let $g \in L^2_0$ and let $G(x) = \int_a^x g(s)ds$ (so $G' = g$ almost everywhere). Then

$$\begin{aligned} (g, X'_n)_k &= \int_a^b g(x) X'_n(x) k(x) dx \\ &= (G(x) X'_n(x) k(x)|_a^b + \mu_n \int_a^b G(x) X_n(x) \rho(x) dx. \end{aligned} \tag{3.30}$$

Hence, if $(g, X'_n)_k = 0$ for all n, then since $G(a) = G(b) = 0$, we get $(G, X_n)_\rho =$

0 for all n. It follows that

$$0 = G(x) = \int_a^x g(s)ds$$

for almost every x. From the real variables, $g(x) = 0$ almost everywhere.

(2) The argument based on (3.30) works for boundary conditions of type N and the mixed case d,n (without the extra condition on g). For boundary conditions of type n,d, replace $G(x)$ with $G(x) = -\int_x^b g(s)dx$.

□

4. GENERAL ENERGY CALCULATIONS

4.1 Form for Kinetic and Potential Energy

In an equation representing the transfer of energy through a vibrating string, one would expect that the equation would obey physical laws of motion, namely conservation of energy. In order to construct a conservation law for the wave equation, let us begin by deriving an expression including the terms for kinetic and potential energy. From our approach we will also derive, in general, an energy equipartition principle first noted for the Dirichlet problem earlier (Bray, 2012, p. 201). Here motivation for the form of kinetic and potential value is derived from the wave equation, assuming a smooth solution exists.

We begin with the generalized wave equation (1.1) on the interval $a < x < b$ and multiply both sides by u_t.

$$\rho u_{tt} = (u_x k)_x$$
$$u_t \cdot \rho u_{tt} = (u_x k)_x \cdot u_t$$

Now, the left-hand side can be rewritten using the chain rule.

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u_t^2\right) = (u_x k)_x \cdot u_t$$

Then, we integrate with respect to x, integrating by parts on the right-hand side.

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int_a^b \rho u_t^2(x,t)dx\right] = \int_a^b (u_x k)_x u_t dx$$
$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int_a^b \rho u_t^2(x,t)dx\right] = k u_x u_t|_a^b - \int_a^b k u_x u_{xt} dx$$

The right-hand side can also be rewritten.

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int_a^b \rho u_t^2(x,t)dx\right] = ku_xu_t|_a^b - \frac{\partial}{\partial t}\left[\frac{1}{2}\int_a^b ku_x^2(x,t)dx\right]$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\int_a^b \rho u_t^2(x,t)dx + \frac{1}{2}\int_a^b ku_x^2(x,t)dx\right] = ku_xu_t|_a^b$$

$$\frac{\partial}{\partial t}[KE(t) + PE_s(t)] = k(b)u_x(b,t)u_t(b,t) - k(a)u_x(a,t)u_t(a,t) \tag{4.31}$$

Conservation with Dirichlet, Neumann, or Mixed Conditions. Consider the wave equation with Dirichlet boundary conditions at $x = a$ and $x = b$,

$$\text{BC:} \begin{cases} u(a,t) = 0 \\ u(b,t) = 0. \end{cases}$$

In this case, the right-hand side of equation 4.31 simplifies to zero. Since this is a derivative, that means that $KE(t) + PE_s(t) = E$, the total energy, is constant, giving us conservation of energy for the Dirichlet case.

Similarly in the case with Neumann conditions at $x = a$ and $x = b$,

$$\text{BC:} \begin{cases} u_x(a,t) = 0 \\ u_x(b,t) = 0, \end{cases}$$

or for mixed conditions, conservation of energy is also attained. In all cases applying the initial conditions gives

$$E = \frac{1}{2}\int_a^b [f'(x)]^2 k(x)dx.$$

Conservation with Robin Conditions. Now, consider the case with Robin boundary conditions at $x = a$ and $x = b$.

$$\text{BC:} \begin{cases} b_1 u_x(a,t) - u(a,t) = 0, \\ b_2 u_x(b,t) + u(b,t) = 0, \quad b_1, b_2 \geq 0 \end{cases} \tag{4.32}$$

$$\text{IC:} \begin{cases} u(x,0) = f(x), \\ u_t(x,0) = 0 \end{cases} \tag{4.33}$$

Here, the boundary terms from (4.31) can be rewritten as follows:

$$\begin{aligned} u_x(b,t)u_t(b,t) - u_x(a,t)u_t(a,t) &= -\frac{u(b,t)u_t(b,t)}{b_2} - \frac{u(a,t)u_t(a,t)}{b_1} \\ &= -\frac{\partial}{\partial t}\left[\frac{u^2(b,t)}{2b_2} + \frac{u^2(a,t)}{2b_1}\right]. \end{aligned}$$

Then the conservation law becomes,

$$\frac{\partial}{\partial t}[KE(t) + PE_s(t) + \left[\frac{k(b)u^2(b,t)}{2b_2} + \frac{k(a)u^2(a,t)}{2b_1}\right]] = 0,$$

or

$$KE(t) + PE_s(t) + \left[\frac{k(b)u^2(b,t)}{2b_2} + \frac{k(a)u^2(a,t)}{2b_1}\right] = E,$$

where the potential energy is divided into potential energy originating from the string, $PE_s(t)$, and potential energy from the boundary conditions. Note that if b_1 or b_2 is equal to zero, the corresponding term in the conservation law would be dropped.

By substituting the initial conditions into the conservation law, we can find the value of the total energy, E. Since kinetic energy is zero when $t = 0$,

$$\frac{1}{2}\int_a^b k(x)[f'(x)]^2 dx + \frac{k(b)}{2b_2}f^2(b) + \frac{k(a)}{2b_1}f^2(a) = E. \tag{4.34}$$

It is now necessary to compute $\int_a^b k(x)[f'(x)]^2 dx$, but as it was shown in Section 2.4, $\{X_n'(x)\}$ (which includes f') is not necessarily orthogonal with respect to L^2-norm. However, physically, $f(x)$ should satisfy the boundary conditions, specifically suppose $f \in^2 [a, b]$ and satisfies the boundary conditions. Then, through integration by parts,

$$\begin{aligned}\int_a^b k(x)f'(x)f'(x)dx &= [k(x)f(x)f'(x)]_a^b - \int_a^b f(x)(k(x)f'(x))'dx \\ &= [k(b)f(b)f'(b) - k(a)f(a)f'(a)] - \int_a^b f(x)(k(x)f'(x))'dx \\ &= -\left[\frac{k(b)f^2(b)}{b_2} + \frac{k(a)f^2(a)}{b_1}\right] + \int_a^b f \cdot Lf\rho dx,\end{aligned}$$

where the operator L is defined by

$$Lf = \frac{-1}{\rho}(kf')'. \tag{4.35}$$

Noting that $LX_n = \frac{-1}{\rho}(kX_n')' = \frac{-1}{\rho}(-\mu_n\rho X_n) = \mu_n X_n$, f and Lf can be written as:

$$\begin{aligned}f &\sim \sum_n (\hat{f})_\rho(n)X_n(x) \\ Lf &\sim \sum_n (\hat{f})_\rho(n)LX_n(x) = \sum_n \mu_n(\hat{f})_\rho(n)X_n.\end{aligned}$$

Then the integral on the right in the integration by parts can be written as a sum.

$$\int_a^b f \cdot Lf\rho dx = \sum_n \mu_n^2(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2$$

Now, we substitute this information into (4.34):

$$E = -\left[\frac{k(b)f^2(b)}{2b_2} + \frac{k(a)f^2(a)}{2b_1}\right] + \frac{1}{2}\sum_{n=1}^{\infty}\mu^2|(\hat{f})_k(n)|^2||X_n||_2^2 + \frac{k(b)}{2b_2}f^2(b) + \frac{k(a)}{2b_1}f^2(a)$$

or,

$$E = \frac{1}{2}\sum_{n=1}^{\infty}\mu^2(\hat{f})_k(n)^2||X_n||_2^2.$$

So, the total energy, E, in the system is $\frac{1}{2}\sum_{n=1}^{\infty}\mu^2(\hat{f})_k(n)^2||X_n||_2^2$.

The solution of this problem is

$$u(x,t) = \sum_{n=1}^{\infty}\hat{f}(n)\cos\lambda_n t X_n(x).$$

Given this backdrop and similar calculations earlier in this section, Parseval's Equality may again be used to calculate the expression for kinetic energy. From (1.2),

$$\begin{aligned} KE(t) &= \frac{1}{2}\int_0^L u_t^2(x,t)\rho(x)dx \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n^2\hat{f}^2(n)||X_n||_2^2\sin^2\lambda_n t. \end{aligned}$$

Now, the total potential energy is then

$$\begin{aligned} PE(t) &= \frac{1}{2}\int_0^L u_x^2(x,t)\rho(x)dx + \left[\frac{k(b)u^2(b,t)}{2b_2} + \frac{k(a)u^2(a,t)}{2b_1}\right] \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\lambda_n^2\hat{f}^2(n)||X_n||_2^2\cos^2\lambda_n t. \end{aligned}$$

This value can be computed by subtracting the kinetic energy from the total energy.

4.2 The Wave Equation and Energy in Cases D, N, and M

We consider the following initial boundary value problem:

$$\rho u_{tt} = (ku_x)_x, \quad a < x < b,\ t > 0$$
$$BC: D \text{ or } N \text{ or } M \tag{4.36}$$
$$IC: \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0. \end{cases}$$

Here $\rho = \rho(x) \in C[a,b]$, $k = k(x) \in C^1[a,b]$ with $\rho > 0$ and $k > 0$ on $[a,b]$.

As before, let $\{\mu_n\}_0^\infty$ and $\{X_n(x)\}_0^\infty$ be the eigenvalues and eigenfunctions from the boundary value problem

$$(kX')' + \mu\rho X = 0, \quad a < x < b$$

with the boundary conditions from the above initial boundary value problem.

Separation of variables suggests a solution:

$$u(x,t) = \sum_n (\hat{f})_\rho(n) \cos\sqrt{\mu_n} t X_n(x). \tag{4.37}$$

Proposition 4.1: Let $f \in W_{bc}^{1,2}[a,b]$. Then the series (4.37) converges uniformly on $[a,b] \times [0,\infty)$, $x \to u(x,t)$ is in $W_{bc}^{1,2}[a,b]$ for all $t > 0$, and $u(x,t)$ is a weak solution to the wave equation, i.e.,

$$\int_0^\infty \int_a^b u(x,t)[\rho(x)\phi_{tt}(x,t) - (k(x)\phi_x(x,t))_x]dxdt = 0 \tag{4.38}$$

for all $\phi \in C_c^\infty([a,b] \times [0,\infty))$.

Proof. Since $f \in W^{1,2}_{bc}[a,b]$, we know $\sum_n (\hat{f})_\rho(n)^2 ||X_n||^2_{2,\rho} < \infty$ and

$$\sum_n (\hat{f})_k(n)^2 ||X'_n||^2_{2,k} = \sum_n \mu_n (\hat{f})_\rho(n)^2 ||X_n||^2_{2,\rho} < \infty.$$

Consequently the series (4.37) converges in $L^2([a,b], \rho dx)$ as does the series of term by term derivatives in the x-variables. The latter defines the weak derivative $u_x(x,t)$. Let

$$u^N(x,t) = \sum_{n=0}^{N} (\hat{f})_\rho(n) \cos \sqrt{\mu_n} t X_n(x).$$

Then $x \to u^N(x,t)$ is in $C^1_{bc}[a,b]$ and $||u^N - u||_{2,\rho} \to 0$, $||u^N_x - u_x||_{2,k} \to 0$ as $N \to \infty$. Thus, $u \in W^{1,2}_{bc}[a,b]$. By the Sobolev inequality (3.20)

$$\sup_x |u^N(x,t) - u(x,t)| \leq c||u^N - u||_{W^{1,2}[a,b]}. \tag{4.39}$$

Notice that

$$||u^N - u||^2_{W^{1,2}} \leq \sum_{n=N+1}^{\infty} (\hat{f})_\rho(n)^2 ||X_n||^2_{2,\rho} + \sum_{n=N+1}^{\infty} (\hat{f})_k(n)^2 ||X'_n||^2_{2,k},$$

a bound independent of t. Returning to (4.39) shows $u^N \to u$ uniformly on $[a,b] \times [0,\infty)$. Finally, $u(x,t)$ defines a weak solution of the wave equation is proved by substituting the series (4.37) into (4.38), integrating term by term, and using the fact that $\cos \sqrt{\mu_n} t X_n(x)$ are solutions of the wave equation. □

The above proof also shows that the weak derivative $u_t(x,t)$ is given by

$$u_t(x,t) = \sum_n \sqrt{\mu_n} (\hat{f})_\rho(n) \sin \sqrt{\mu_n} t X_n(x),$$

the series converges in $L^2([a,b], \rho dx)$ norm for all $t > 0$. Consequently, if $f \in$

$W_{bc}^{1,2}[a,b]$, the kinetic and potential energies are well defined:

$$KE(t) = \frac{1}{2}\int_a^b u_t^2(x,t)\rho(x)dx$$
$$PE(t) = \frac{1}{2}\int_a^b u_x^2(x,t)k(x)dx. \tag{4.40}$$

Corollary 4.2: Let $f \in W_{bc}^{1,2}[a,b]$. Then the kinetic and potential energies are given by

$$KE(t) = \frac{1}{2}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2 \sin^2\sqrt{\mu_n}t$$
$$PE(t) = \frac{1}{2}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2 \cos^2\sqrt{\mu_n}t. \tag{4.41}$$

Furthermore, conservation law holds:

$$KE(t) + PE(t) = \frac{1}{2}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2 = E.$$

Proof. Follows from $x \to u_x(x,t) \in W_{bc}^{1,2}[a,b]$ and the fact that $u_t \in L^2([a,b],\rho dx)$

□

Physically, the total energy should be balanced, in some sense, between its kinetic and potential forms. We introduce average kinetic and potential energy as:

$$A_{KE} = \lim_{L\to\infty}\frac{1}{L}\int_0^L KE(t)dt$$
$$\text{and} \quad A_{PE} = \lim_{L\to\infty}\frac{1}{L}\int_0^L PE(t)dt.$$

Corollary 4.3: Let the boundary conditions be of the type D, N, or M and $f \in W_{bc}^{1,2}[a,b]$. Then $A_{KE} = \frac{1}{2}E = A_{PE}$, where E is the total energy.

Proof. We use the formulas from Corollary 4.2. Since $\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2$ converges we may integrate term by term.

$$
\begin{aligned}
\int_0^L KE(t)dt &= \frac{1}{2}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2 \int_0^L \sin^2\sqrt{\mu_n}tdt \\
&= \frac{1}{2}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2 \int_0^L \frac{1-\cos 2\sqrt{\mu_n}t}{2}dt \\
&= \frac{L}{4}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2(1-\frac{\sin 2\sqrt{\mu_n}L}{L})
\end{aligned}
$$

We then obtain A_{KE} by dividing by L and taking the limit as $L \to \infty$.

$$
\begin{aligned}
\lim_{L\to\infty}\frac{1}{L}\int_0^L KE(t)dt &= \lim_{L\to\infty}\frac{1}{4}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2(1-\frac{\sin 2\sqrt{\mu_n}L}{L}) \\
&= \frac{1}{4}\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2
\end{aligned}
$$

Clearly, from the formula given in Corollary 4.2, $A_{KE} = \frac{1}{2}E$. A similar proof applied to A_{PE} yields $A_{PE} = \frac{1}{2}E$. $\square$

4.3 The Wave Equation and Energy in Case R

We now look at the initial boundary value problem (4.36) where the boundary conditions are from R:

$$
\text{BC:} \begin{cases} b_1u_x(a,t) - u(a,t) = 0, \\ b_2u_x(b,t) + u(b,t) = 0, \quad b_1, b_2 \geq 0 \end{cases} \tag{4.42}
$$

Returning to (4.34) it is now necessary to compute $\int_a^b k(x)[f'(x)]^2dx$, but as it was shown in Section 2.4, $\{X_n'(x)\}$ (which includes f') is not necessarily orthogonal with respect to L^2-norm. However, physically, $f(x)$ should satisfy the boundary conditions, specifically suppose $f \in C^2[a,b]$ and satisfies the boundary conditions.

Then, through integration by parts,

$$
\begin{aligned}
\int_a^b k(x)f'(x)f'(x)dx &= [k(x)f(x)f'(x)]_a^b - \int_a^b f(x)(k(x)f'(x))'dx \\
&= [k(b)f(b)f'(b) - k(a)f(a)f'(a)] - \int_a^b f(x)(k(x)f'(x))'dx \\
&= -\left[\frac{k(b)f^2(b)}{b_2} + \frac{k(a)f^2(a)}{b_1}\right] + \int_a^b f \cdot Lf\rho dx,
\end{aligned}
$$

where the operator L is defined by

$$
Lf = \frac{-1}{\rho}(kf')'. \tag{4.43}
$$

Noting that $LX_n = \frac{-1}{\rho}(kX_n')' = \frac{-1}{\rho}(-\mu_n\rho X_n) = \mu_n X_n$, f and Lf can be written as:

$$
\begin{aligned}
f &\sim \sum_n (\hat{f})_\rho(n)X_n(x) \\
Lf &\sim \sum_n (\hat{f})_\rho(n)LX_n(x) = \sum_n \mu_n(\hat{f})_\rho(n)X_n.
\end{aligned}
$$

Then the integral on the right in the integration by parts can be written as a sum.

$$
\int_a^b f \cdot Lf\rho dx = \sum_n \mu_n^2(\hat{f})_\rho(n)^2||X_n||_{2,\rho}^2
$$

Now, we substitute this information into (4.34):

$$
\begin{aligned}
E = -\left[\frac{k(b)f^2(b)}{2b_2} + \frac{k(a)f^2(a)}{2b_1}\right] &+ \frac{1}{2}\sum_{n=1}^{\infty} \mu^2|(\hat{f})_k(n)|^2||X_n||_2^2 \\
&+ \frac{k(b)}{2b_2}f^2(b) + \frac{k(a)}{2b_1}f^2(a)
\end{aligned}
$$

or,

$$E = \frac{1}{2} \sum_{n=1}^{\infty} \mu^2 (\hat{f})_k (n)^2 ||X_n||_2^2.$$

So, the total energy, E, in the system is $\frac{1}{2} \sum_{n=1}^{\infty} \mu^2 (\hat{f})_k (n)^2 ||X_n||_2^2$.

The solution of this problem is

$$u(x,t) = \sum_{n=1}^{\infty} \hat{f}(n) \cos \lambda_n t X_n(x).$$

Given this backdrop and similar calculations earlier in this section, Parseval's Equality may again be used to calculate the expression for kinetic energy. From (1.2),

$$\begin{aligned} KE(t) &= \frac{1}{2} \int_a^b u_t^2(x,t) \rho(x) dx \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n^2 \hat{f}^2(n) ||X_n||_2^2 \sin^2 \lambda_n t. \end{aligned}$$

Now, the total potential energy is then

$$\begin{aligned} PE(t) &= \frac{1}{2} \int_a^b u_x^2(x,t) \rho(x) dx + \left[\frac{k(b) u^2(b,t)}{2b_2} + \frac{k(a) u^2(a,t)}{2b_1} \right] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n^2 \hat{f}^2(n) ||X_n||_2^2 \cos^2 \lambda_n t. \end{aligned}$$

This value can be computed by subtracting the kinetic energy from the total energy.

We can now formulate the analog of Proposition (4.1) and its corollaries as follows. Let $C_{rr}^2[a,b]$ be the class of C^2-functions on $[a,b]$ that satisfy the Robin boundary conditions. Let $W_{rr}^{2,2}[a,b]$ be its completion relative to the norm

$$||f||_{W^{2,2}}^2 = ||f||_{2,\rho}^2 + ||f'||_{2,k}^2 + ||Lf||_{2,\rho}^2,$$

where L is defined in (4.35).

Proposition 4.4: Let $f \in W^{2,2}_{rr}[a,b]$. Then,

$$u(x,t) = \sum_n (\hat{f})_\rho(n) \cos \sqrt{\mu_n} t X_n(x) \tag{4.44}$$

is a weak solution of the initial boundary value problem (4.36) with boundary conditions replaced by (2.13). Furthermore the kinetic and potential energies are given by (4.41) and energy equipartition holds:

$$A_{KE} = \frac{1}{2}E = A_{PE}.$$

Proof. Because $\sum_n \mu_n (\hat{f})_\rho(n)^2 ||X_n||^2_{2,\rho}$ converges and using the notation from the proof of Proposition (4.1),

$$\sup_t ||U^N(\cdot,t) - u(\cdot,t)||_{2,\rho} \to 0$$

as $N \to \infty$. This suffices to justify term by term integration in (4.38) as in the proof of Proposition (4.1). The rest of the proof follows from our computations. □

5. CONCLUSION

The energy equipartition principle has now been proven for Dirichlet, Neumann, mixed, and Robin boundary conditions. This principle states that the kinetic energy and potential energy, when averaged over time, equally divide the total energy in the vibrating string. Essential in the proof of this principle is the convergence of the series $\sum_n \mu_n(\hat{f})_\rho(n)^2||X_n||^2_{2,\rho}$. This result was obtained by placing f in the Sobolev space $W^{1,2}_{bc}[a,b]$, a space that was specifically constructed to ensure that f would meet the necessary conditions for a solution to exist and for convergence to occur.

In the Robin case, many of the same calculations were completed. An energy conservation law was found and explicit forms for kinetic and potential energy were stated. Here, Parseval could not be used directly to calculate the potential energy, as, while $X_n'(x)$ is always orthogonal, $X_n'(x)$ was shown not to be an orthogonal system with respect to the L^2-norm. Remarkably, the values for kinetic and potential energy took the same form for the Robin case as they did for the Dirichlet, Neumann, and mixed cases. Should we have presumed to use Parseval's equality for the Robin case, the same result would have been reached.

To conclude, we defined a space in $W^{1,2}_{bc}[a,b]$ for f when the wave equation has Robin boundary conditions. This case required stricter conditions on f than the previous cases, but based on the energy forms, an energy equipartition principle could be established for the Robin case, as well. The calculation was almost identical to the calculations proving Corollary 4.3. So, despite significantly more complicated boundary terms, the same even separation of average kinetic and potential energy still holds.

6. REFERENCES

Birkhoff, G. & Rota, G. (1989) *Ordinary Differential Equations* (4th ed.). John Wiley & Sons.

Bray, W. O. (2012) *A Journey into Partial Differential Equations*. Sudbury, MA: Jones & Bartlett Learning.

Bray, W. O. (2018) *Research Seminar Notes, Spring 2018*. Unpublished manuscript.

Jost, J. (2013) *Partial Differential Equations* (3rd ed.). New York, NY: Springer-Verlag.