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The Average Measure of a k-Dimensional Simplex in an n-Cube

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THE AVERAGE MEASURE OF A K-DIMENSIONAL SIMPLEX IN AN N-CUBE

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

John A. Carter

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ABSTRACT

Within an n-dimensional unit cube, a number of k-dimensional simplices can be formed whose vertices are the vertices of the n-cube. In this thesis, we analyze the average measure of a k-simplex in the n-cube. We develop exact equations for the average measure when $k = 1, 2$, and 3. Then we generate data for these cases and conjecture that their averages appear to approach $n^{k/2}$ times some constant. Using the convergence of Bernstein polynomials and a k-simplex Bernstein generalization, we prove the conjecture is true for the 1-simplex and 2-simplex cases. We then develop a generalized formula for the average measure of the k-simplex in the n-cube and prove the average is asymptotic to $n^{k/2} \cdot \frac{\sqrt{k+1}}{2^k k!}$.

KEYWORDS: k-simplex, n-cube, Cayley-Menger determinants, Bernstein polynomials, Monte Carlo methods

This abstract is approved as to form and content

Dr. Les Reid
Chairperson, Advisory Committee
Missouri State University

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Approved:

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1. INTRODUCTION

A unit n-cube is the convex hull of 2^n points of the form $(x_1, \ldots, x_n)$, where $x_i \in \{0, 1\}$. A k-simplex is defined to be the convex hull of $k+1$ points. This paper will discuss the average measure of a k-dimensional simplex in an n-cube, and will show that the average measure approaches $n^{k/2} \cdot C$ for some constant C as n approaches infinity. The value of this constant is in fact the measure of a regular k-simplex of side length $\frac{1}{\sqrt{2}}$, which will be shown to be $\frac{\sqrt{k+1}}{2^k k!}$.

This research began in the spring of 2016 as part of an undergraduate research project. With a contribution by Dr. Xingping Sun, we were able to use Bernstein polynomials to prove the average measure of a 1-simplex approaches $\sqrt{n/2}$ as $n \to \infty$. As the project evolved into a thesis, we developed programs to collect data for the average measure of the 2-simplex and 3-simplex in the n-cube to support our conjecture and determine the value of C in each case. We then implemented the k-simplex Bernstein polynomial and developed a generalized proof for the convergence of the average measure of any k-simplex in the unit n-cube.

2. 1-SIMPLEX CASE

A 1-simplex is the convex hull of 2 points, or a line segment. We may analyze the total number of line segments in the 2-cube, including degenerate cases, by fixing a vertex at the origin of a 2-dimensional grid and considering the possible coordinates for a second vertex. This process is demonstrated in Figure 1.

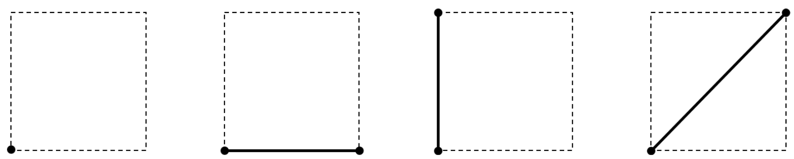

Figure 1: Forming all 1-simplices, including degenerates, in the 2-cube.

The pair of coordinates $((0,0),(c_1,c_2))$ consists of the coordinates that comprise a line segment, where $c_1, c_2 \in \{0,1\}$. Then $\{(0,0),(0,1),(1,0),(1,1)\}$ is the set of possible choices for (c_1,c_2). So for c_1 and c_2, we can choose these possibilities: two 1's and zero 0's, a single 1 and a single 0, or zero 1's and two 0's. That is, we're dealing with the number of ways to partition 2 into two parts.

(0)	2	0	1
(1)	0	2	1

Because there are 2^2 vertices in a 2-cube, and because there is one coordinate to choose to complete a 1-simplex, we should expect $(2^2)^1$ 1-simplices in the 2-cube. There are 2! ways to assign a particular number of 1's and 0's to c_1 and c_2. There are $2! \cdot 0!$ ways to choose two of the same component and zero of the other. Then we can pick either two 0's or two 1's, so in total there are $\left(\frac{2!}{2!0!} \cdot 2\right)$ ways to pick 1-simplices of this form. There are $1! \cdot 1!$ ways to choose one of a single component and one of the other, and in this situation we can only pick one 1 and one 0. So there are $\left(\frac{2!}{1!1!} \cdot 1\right)$ ways to pick 1-simplices of this form. In total we have

$$\binom{2}{2} \cdot 2 + \binom{2}{1} \cdot 1 = 4,$$

confirming that these forms cover all the 1-simplices in the 2-cube. Notice that in the case of the 1-simplex, the measure will be the length or distance from the fixed vertex at the origin to the second vertex. This distance will be the square root of the number of 1's chosen for c_1 and c_2. If we denote a to be the number of 0's and b to be the number of 1's chosen at any time, then the average measure can be represented as

$$\frac{1}{2^2} \sum_{a+b=2} \frac{2!}{a!b!} \sqrt{b} = \frac{1}{2^2} \sum_{a+b=2} \frac{2!}{(2-b)!b!} \sqrt{b} = \frac{1}{2^2} \sum_{k=0}^{2} \binom{2}{k} \sqrt{k} \approx 0.85355.$$

In a similar fashion, we can find the average measure of a 1-simplex in a 3-cube. This time we fix a vertex at the origin and consider the pair $((0,0,0),(c_1,c_2,c_3))$ of coordinates that comprise a line segment in the 3-cube, where $c_1, c_2, c_3 \in \{0,1\}$. So as before, the number of 0's and 1's we choose for c_1, c_2, and c_3 can be represented as partitions of 3 into two parts:

(0)	3	0	2	1
(1)	0	3	1	2

Because there are 2^3 vertices in a 3-cube, this time we should expect $(2^3)^1$ 1-simplices. There are 3! ways to assign a particular number of 0's and 1's to c_1, c_2 and c_3. There are $3! \cdot 0!$ ways to choose three of the same component and zero of the other. Then we can pick either three 0's (a degenerate case) or three 1's. In total there are $\left(\frac{3!}{3!0!} \cdot 2\right)$ ways to pick 1-simplices of this form. There are $2! \cdot 1!$ ways to choose two of the same component and one of the other. Then one may choose either two 0's and one

1 or two 1's and one 0. So there are $\left(\frac{3!}{2!1!} \cdot 2\right)$ ways to pick 1-simplices of this form. In total we have

$$\binom{3}{3} \cdot 2 + \binom{3}{2} \cdot 2 = 8,$$

confirming that these forms cover all the 1-simplices in the 3-cube. Again we can easily verify that the average measure is represented as

$$\frac{1}{2^3} \sum_{k=0}^{3} \binom{3}{k} \sqrt{k} \approx 1.12184.$$

Similarly, the average measure of a 1-simplex in an n-cube is

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sqrt{k}. \qquad (*)$$

The graph in Figure 2 below represents the frequency of lengths for the 1-simplices contained in the 15-cube.

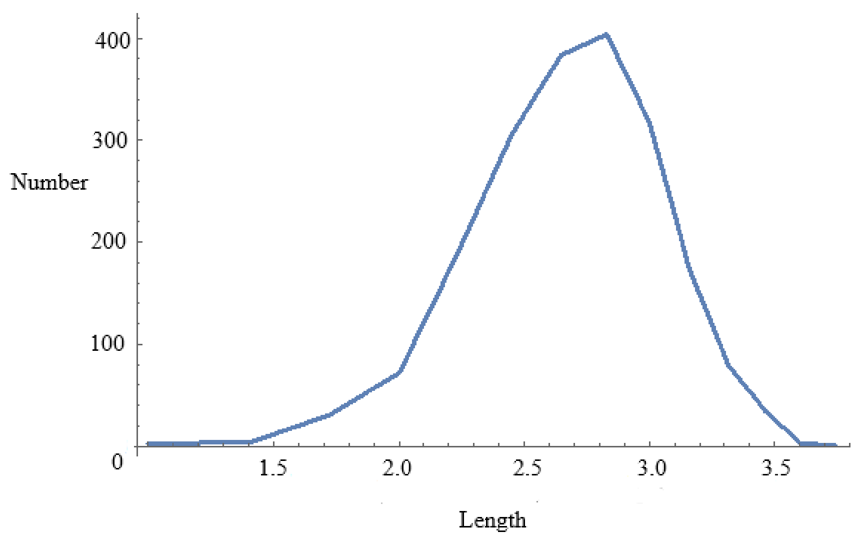

Figure 2: Frequency distribution of segment length for 15-dimensional cube.

This distribution is very smooth, as is the case for each graph representing the length distribution for many higher values of n. By analyzing many of these graphs, we find the peak of each curve is very close to $\sqrt{n/2}$. This leads us to believe that the average length of a 2-simplex in the n-cube is approaching $\sqrt{n/2}$ as n grows large.

Figure 3 represents the exact average measure of a 1-simplex in an n-cube for some values of n, calculated with formula *.

n	20	50	100	200	500
Average	3.14148	4.98726	7.06214	9.99372	15.8074
$\sqrt{n/2}$	3.16228	5	7.07107	10	15.81139

Figure 3: Average measure of a 1-simplex in the n-cube for several values of n.

The bottom row represents our conjecture for the average. The results compared with the actual averages appear to support our hypothesis. To prove the hypothesis, we will need an implementation of Bernstein polynomials.

3. BERNSTEIN POLYNOMIALS

The Bernstein polynomial is defined as follows: for any real valued function f defined and bounded on the interval $[0,1]$, $B_n(f)$ is the polynomial on $[0,1]$ given the value

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right).$$

This polynomial has a very useful property that we will utilize in showing the convergence of our function for the average measure of a 1-simplex in an n-cube.

THEOREM 3.1: If f is a real-valued function defined on the interval $[0,1]$ and bounded by M, then for every point x where f is continuous we have $B_n(f)_{\lim_{n\to\infty}} = f(x)$.

Proof. The following proof is from Kadison and Liu [2]. We have that

$$\begin{aligned} B_n(f)(x) - f(x) &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\tfrac{k}{n}\right) - f(x) \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left[f\left(\tfrac{k}{n}\right) - f(x) \right]. \end{aligned}$$

Thus we can say for each $x \in [0,1]$,

$$|B_n(f)(x) - f(x)| \leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \left| f\left(\tfrac{k}{n}\right) - f(x) \right|.$$

For any $\delta > 0$ we can consider the above sum in two parts, $\sum_1$ where $\left|\frac{k}{n} - x\right| < \delta$, and $\sum_2$ where $\left|\frac{k}{n} - x\right| \geq \delta$. So long as x is a point of continuity of f, for any $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x_1) - f(x)| < \frac{\epsilon}{2}$ whenever $|x_1 - x| < \delta$. Now for

$\sum_1$,

$$\begin{aligned}
\sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} x^k(1-x)^{n-k} \left|f\left(\tfrac{k}{n}\right)-f(x)\right| &< \sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} x^k(1-x)^{n-k}\frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2}\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

For the sum $\sum_2$, note that $\delta^2 \leq \left|\frac{k}{n}-x\right|^2$. Also note,

$$\begin{aligned}
\frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\right] &= \frac{d}{dx}\left[1\right], \\
\frac{1}{x}\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k} - \frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}(n-k)x^k(1-x)^{n-k} &= 0, \\
\frac{1}{x}\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k} &= \frac{1}{1-x}\left[n-\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k}\right], \\
\frac{1}{x(x-1)}\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k} &= \frac{n}{1-x}, \\
\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k} &= nx,
\end{aligned}$$

and

$$\frac{d}{dx}\left[\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k}\right]=\frac{d}{dx}\left[xn\right],$$

$$\frac{1}{x}\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k}-\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}k(n-k)x^k(1-x)^{n-k}=n,$$

$$\frac{1}{x}\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k}-\frac{n}{1-x}\sum_{k=0}^{n}\binom{n}{k}kx^k(1-x)^{n-k}$$
$$+\frac{1}{1-x}\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k}=n,$$

$$\frac{1}{x(x-1)}\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k}-\frac{n}{1-x}(nx)=n,$$

$$\sum_{k=0}^{n}\binom{n}{k}k^2x^k(1-x)^{n-k}=\frac{x^2(n-1)}{n}+\frac{x}{n}.$$

Thus we have

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{k}{n}-x\right)^2x^k(1-x)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\frac{k^2}{n^2}x^k(1-x)^{n-k}$$
$$-2x\sum_{k=0}^{n}\binom{n}{k}\frac{k}{n}x^k(1-x)^{n-k}+x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}$$
$$=\frac{x(1-x)}{n}.$$

Define $M = \max_{x \in [0,1]} f(x)$. Now,

$$\begin{aligned}
\delta^2 \sum_{\left|\frac{k}{n}-x\right| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \\
&\leq \sum_{\left|\frac{k}{n}-x\right| \geq \delta} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \\
&\leq \sum_{\left|\frac{k}{n}-x\right| \geq \delta} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} 2M \\
&\leq 2M \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k} \\
&= 2M \frac{x(1-x)}{n} \\
&\leq \frac{2M}{n}.
\end{aligned}$$

Ultimately we have

$$\sum_{\left|\frac{k}{n}-x\right| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \left|f\left(\tfrac{k}{n}\right) - f(x)\right| \leq \frac{2M}{n\delta^2}.$$

Considering δ, we can pick N large enough so that $\frac{2M}{\delta^2 n} < \frac{\epsilon}{2}$ when $n \geq N$. In this case we have

$$|B_n(f)(x) - f(x)| \leq \sum_1 + \sum_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

showing that $\lim_{n \to \infty} B_n(f)(x) = f(x)$ for each $x \in [0,1]$ where f is continuous. $\square$

4. 1-SIMPLEX (CONCLUSION)

To prove the convergence of the average, an implementation of Bernstein polynomials is used. Let $f(x) = \sqrt{x}$, then $f \in C[0,1]$. Notice that $f(1/2) = 1/\sqrt{2}$. We have

$$\begin{aligned} B_n(f)(1/2) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(1 - \frac{1}{2}\right)^{n-k} \sqrt{\frac{k}{n}} \\ &= \frac{\sum_{k=0}^{n} \binom{n}{k} \sqrt{k}}{2^n \sqrt{n}}. \end{aligned}$$

By Theorem 3.1,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} \sqrt{k}}{2^n \sqrt{n}} = \sqrt{\frac{1}{2}}.$$

This means

$$\frac{\sum_{k=0}^{n} \binom{n}{k} \sqrt{k}}{2^n} \sim \sqrt{\frac{n}{2}},$$

thus proving our conjecture.

5. 2-SIMPLEX CASE

In the non-degenerate case, a 2-simplex is the convex hull of 3 points, or a triangle. We may analyze the total number of 2-simplices in the 2-cube, including degenerate cases, by fixing a vertex at the origin of a 2-dimensional grid and considering the possible coordinates for a second and third vertex that will complete a 2-simplex. That is, $((0,0),(a_1,a_2),(b_1,b_2))$ consists of the coordinates that comprise the 2-simplex, where $a_i, b_i \in \{0,1\}$ for $i = 1,2$. Similar to the 1-simplex case, we can consider a certain number of pairs. One pair (a_1,b_1) will describe the first coordinate value of each vertex, assigning values to a_1 and b_1. A second pair (a_2,b_2) will describe the second coordinate value of each vertex, assigning values to a_2 and b_2. So there are 2 pairs in total used to form the two vertices of a 2-simplex that are not fixed at the origin. Of the choices for (a_1,b_1) and (a_2,b_2) we have $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. The number of ways to choose two of these, one for (a_1,b_1) and one for (a_2,b_2), is the number of ways to partition 2 into four distinct parts, as demonstrated below.

$(0,0)$	2	0	0	0	1	1	1	0	0	0
$(0,1)$	0	2	0	0	1	0	0	1	1	0
$(1,0)$	0	0	2	0	0	1	0	1	0	1
$(1,1)$	0	0	0	2	0	0	1	0	1	1

Because there are (2^2) vertices in a 2-cube, and because we need two vertices to complete a 2-simplex, we should expect $(2^2)^2 = 16$ 2-simplices. There are 2! ways to assign a particular number of pairs to (a_1,b_1) and (a_2,b_2). There are $2! \cdot 0! \cdot 0! \cdot 0!$ ways to choose two of the same pair and zero of the rest (which would be degenerate 2-simplices). Then we can pick two of the same pair out of four possible pairs, so in total there are $\left(\frac{2!}{2!0!0!0!} \cdot 4\right)$ ways to pick 2-simplices of this form. There are $1! \cdot 1! \cdot 0! \cdot 0!$ ways to choose two different pairs and zero of the rest. There are $\binom{4}{2}$ ways to choose one pair then another distinct pair out of four possible pairs. So there are $\left(\frac{2!}{1!1!0!0!} \cdot \binom{4}{2}\right)$ ways to pick 2-simplices of this form. In total we have

$$\binom{2}{2,0,0} \cdot 4 + \binom{2}{1,1,0} \cdot \binom{4}{2} = 16.$$

Recall we are dealing with the number of ways to partition 2 into four parts. For each of these partitions, we can assign a to represent the number of $(0,0)$ pairs, b the number of $(0,1)$ pairs, c the number of $(1,0)$ pairs, and d the number of $(1,1)$ pairs, where $a+b+c+d=2$.

We can invoke Heron's formula to help create an equation for the average measure of a 2-simplex in a 2-cube. The length of a side of the 2-simplex will be the distance between two of the three coordinates that comprise it. For example, the distance between the origin and the vertex (a_1, a_2) will be the magnitude $\sqrt{a_1^2 + a_2^2}$. However, since $a_1, a_2 \in \{0,1\}$ this distance is just $\sqrt{a_1 + a_2}$ and the values a_1 and a_2 each only increase this distance beyond zero if they are 1. So if we want to analyze this distance we should only be concerned with the number of $(1,0)$ and $(1,1)$ pairs chosen for a particular 2-simplex, since each of these would contribute a 1 to the vertex (a_1, a_2). So if we call the length of this side x, then $x = \sqrt{c+d}$. In a similar manner we can see that if we call y the distance between the origin and the vertex (b_1, b_2), then $y = \sqrt{b+d}$. If we define z to be the distance between the vertices (a_1, a_2) and (b_1, b_2), then the values a_1 and b_1 will only increase the distance beyond zero if they are each different. The same can be said for a_2 and b_2. Thus $z = \sqrt{b+c}$.

If we define V_2 to be the area of a 2-simplex, using Heron's formula we have

$$V_2 = \frac{1}{4}\sqrt{2x^2y^2 + 2x^2z^2 + 2y^2z^2 - x^4 - y^4 - z^4}.$$

We can write this as an equation where a, b, c, and d are parameters:

$$\begin{aligned} g(a,b,c,d) = \frac{1}{4}\Big\{ & 2(c+d)(b+d) + 2(c+d)(b+c) + 2(b+d)(b+c) \\ & - (c+d)^2 - (b+d)^2 - (b+c)^2 \Big\}^{1/2}. \end{aligned}$$

So now an equation for the average measure of a 2-simplex in a 2-cube is

$$\frac{1}{16} \sum_{a+b+c+d=2} \frac{2!}{a!b!c!d!} g(a,b,c,d).$$

In fact, since $d = 2 - a - b - c$ we can define the side lengths by using only a, b, and c as parameters. We can define a new function for the area using only these three variables like so,

$$\begin{aligned} f(a,b,c) &= g(a,b,c,2-a-b-c) \\ &= \frac{1}{4}\Big\{ 2(2-a-b)(2-a-c) + 2(2-a-b)(b+c) \\ &\quad + 2(2-a-c)(b+c) - (2-a-b)^2 - (2-a-c)^2 - (b+c)^2 \Big\}^{1/2}. \end{aligned}$$

So the equation for the average can be rewritten as

$$\frac{1}{16} \sum_{a+b+c \leq 2} \binom{2}{a,b,c} f(a,b,c) = \frac{1}{2}.$$

Now let's find an equation for the average measure of a 2-simplex in a 3-cube using the same approach. Here we analyze the total number of triangles in the 3-cube, including degenerate cases, by fixing a vertex at the origin of a 3-dimensional grid and considering the possible coordinates for a second and third vertex that will complete a 2-simplex. Let $((0,0,0),(a_1,a_2,a_3),(b_1,b_2,b_3))$ consist of the coordinates that comprise the 2-simplex, where $a_i, b_i \in \{0,1\}$ for $i = 1,2,3$. We can consider a pair (a_1, b_1) assigning values to the first entry of each vertex, a second

(a_2, b_2) assigning values to the second coordinate value of each vertex, and a third pair (a_3, b_3) assigning values to the third coordinate value of each vertex.

There are three pairs in total we may use to form the two vertices of the 2-simplex aside from the origin. In any given 2-simplex we have a certain number of $(0,0)$'s, $(0,1)$'s, $(1,0)$'s, and $(1,1)$'s pairs. To avoid the trouble of listing all possibilities, we will go through the number of cases analytically. There are $3!$ ways to choose a number of pairs for (a_1, b_1), (a_2, b_2), and (a_3, b_3). There are $3!\cdot 0!\cdot 0!\cdot 0!$ ways to choose three of the same pair and zero of the rest (which are degenerate cases). Then we can pick three of the same pair out of four possible pairs, so in total there are $(\frac{3!}{3!0!0!0!} \cdot 4)$ ways to pick 2-simplices of this form. There are $2! \cdot 1! \cdot 0! \cdot 0!$ ways to choose two of the same pair, one of another pair, and zero of the two remaining pairs. There are $\binom{4}{2}$ ways to choose zero of two pairs, and we multiply this by two since from the remaining pairs we can pick two of the first and one of the second, or one of the first and two of the second. Thus there are $\left(\frac{3!}{2!1!0!0!} \cdot 2\binom{4}{2}\right)$ ways to pick 2-simplices of this form. Then we can pick three distinct pairs out of four. In total we have

$$\binom{3}{3,0,0} \cdot 4 + \binom{3}{2,1,0} \cdot 2\binom{4}{2} + \binom{3}{1,1,1} \cdot 4 = 64,$$

which is $(2^3)^2$, the number of 2-simplices we would expect in the 3-cube. Using the same approach as before, we find the average to be

$$\frac{1}{64} \sum_{a+b+c \leq 3} \binom{3}{a,b,c} f(a,b,c) \approx 0.641,$$

where

$$f(a,b,c) = g(a,b,c,3-a-b-c),$$

again an application of Heron's formula.

Similarly, the average measure of a 2-simplex in an n-cube is

$$\frac{1}{4^n} \sum_{a+b+c \le n} \binom{n}{a, b, c} f(a, b, c), \tag{**}$$

where

$$\begin{aligned} f(a,b,c) &= g(a,b,c,n-a-b-c) \\ &= \frac{1}{4}\Big\{ 2(n-a-b)(n-a-c) + 2(n-a-b)(b+c) \\ &\quad + 2(n-a-c)(b+c) - (n-a-b)^2 - (n-a-c)^2 - (b+c)^2 \Big\}^{1/2}. \end{aligned}$$

The graph in Figure 4 displays the distribution for the average area of the 2-simplex in the 15-cube.

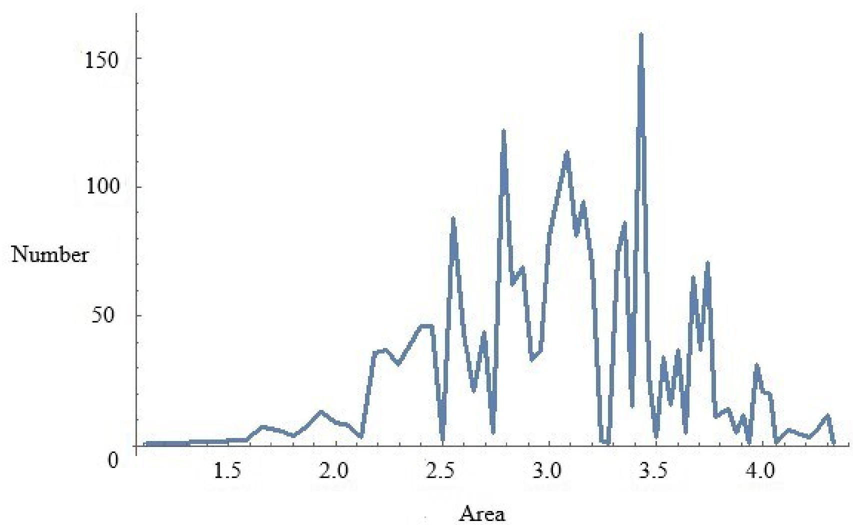

Figure 4: Frequency distribution of triangle area for 15-dimensional cube.

This distribution is not as even as the distribution for the 1-simplex case. It's not so obvious from the graph that the average is converging to any particular value. Nonetheless, we conjecture that the average length of the 2-simplex in the n-cube approaches $n \cdot C$ where C is some constant.

Figure 5 represents the exact average area of a 2-simplex in an n-cube for some values of n, derived from formula **.

n	20	50	100	200	500
Average	4.18051	10.67899	21.50532	43.15645	108.10865
Average/n	0.20903	0.21358	0.21505	0.21578	0.21622

Figure 5: Average measure of a 2-simplex in the n-cube for several values of n.

The bottom row represents the average divided by n. The data indicates this ratio does appear to approach some constant. This encouragement prompts us to apply a method similar to that used for the 1-simplex case, involving k-simplex Bernstein polynomials to find the convergence.

6. k-SIMPLEX BERNSTEIN THEOREM

Let $I = [0,1]$ and $k \in \mathbb{N}$. Define a k-simplex $\triangle_k$ as follows,

$$\triangle_k = \{\vec{x} = (x_1, \ldots, x_k) \in I^k : x_1 + \cdots + x_k \leq 1\}.$$

Define $\vec{v} = (v_1, \ldots, v_k) \in \mathbb{N}_0^k$ to be a multi-index such that

$$|\vec{v}| = v_1 + \cdots + v_k \in \{0, 1, \ldots, n\}.$$

We then define the following notation for any $\vec{x} \in \triangle_k$,

$$\begin{aligned} \vec{x}^{\vec{v}} &= \prod_{i=1}^{k} x_i^{v_i}, \\ \vec{v}\,! &= v_1! \cdots v_k!, \\ \binom{n}{\vec{v}} &= \frac{n!}{\vec{v}\,!(n-|\vec{v}|)!}. \end{aligned}$$

Similar to a regular n-th degree Bernstein polynomial, we set

$$B_{\vec{v},n}(\vec{x}) = \binom{n}{\vec{v}} \vec{x}^{\vec{v}} (1-|\vec{x}|)^{n-|\vec{v}|}.$$

For any f defined on $\triangle_k$ we define

$$\mathbb{B}_n(f)(\vec{x}) = \sum_{|\vec{v}| \leq n} B_{\vec{v},n}(\vec{x}) f\left(\tfrac{\vec{v}}{n}\right).$$

THEOREM 6.1: If $f : \triangle_k \to \mathbb{R}$ is continuous, then $\mathbb{B}_n(f) \to f$ uniformly on $\triangle_k$ as $n \to \infty$.

Proof. The structure of the following proof is from A. Bayad, T. Kim and S.-H. Rim [1] and is very similar to the proof of the regular Bernstein polynomial convergence. We first note that

$$\begin{aligned}\sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) &= \sum_{|\vec{v}|\le n} \binom{n}{v_1 v_2 \cdots v_k} x_1^{v_1} x_2^{v_2} \cdots x_k^{v_k} (1 - x_1 - \cdots - x_k)^{n-v_1-\cdots-v_k} \\ &= [x_1 + x_2 + \cdots + x_k + (1 - x_1 - x_2 - \cdots - x_k)]^n \\ &= 1.\end{aligned}$$

Since f is a continuous function on $\triangle_k$, for any $\frac{\epsilon}{2} > 0$ there exists a $\delta > 0$ such that for $\vec{x} = (x_1, \ldots, x_k)$ and $\vec{y} = (y_1, \ldots, y_k)$, when $|x_i - y_i| < \delta$ for all $i = 1, 2, \ldots, k$, then $|f(\vec{x}) - f(\vec{y})| < \frac{\epsilon}{2}$. To indicate an upper bound for two vectors, we define a measure on $\triangle_k$ as $d(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. Now similar to the regular Bernstein case, we have

$$\begin{aligned}|\mathbb{B}_n(f)(\vec{x}) - f(\vec{x})| &= \left| \sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) f\left(\tfrac{\vec{v}}{n}\right) - \sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) f(\vec{x}) \right| \\ &= \left| \sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| \\ &\le \sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right|.\end{aligned}$$

We may break the sum into two parts,

$$\begin{aligned}\sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| &= \sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) < \delta} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| \\ &\quad + \sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) \ge \delta} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right|.\end{aligned}$$

The first summation satisfies

$$\sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) < \delta} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| < \sum_{|\vec{v}|\le n} B_{\vec{v},n}(\vec{x}) \cdot \frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

Let $v_{k+1} = n - v_1 - \cdots - v_k$ and note that for any x_m,

$$\frac{\partial}{\partial x_m}\left[\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{kk}}\right]$$
$$= \frac{\partial}{\partial x_m}\left[\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^n\right],$$
$$\frac{1}{x_m}\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}v_m{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{kk}}$$
$$= n\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^{n-1},$$
$$\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}v_m{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{kk}}$$
$$= x_m n\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^{n-1},$$

and

$$\frac{\partial}{\partial x_m}\left[\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}v_m{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{k+1}}\right]$$
$$= \frac{\partial}{\partial x_m}\left[x_m n\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^{n-1}\right],$$
$$\frac{1}{x_m}\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}{v_m}^2{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{k+1}}$$
$$= x_m n(n-1)\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^{n-2},$$
$$\sum_{v_1+\cdots+v_{k+1}=n}\binom{n}{v_1!v_2!\cdots v_k!}{v_m}^2{x_1}^{v_1}\cdots{x_m}^{v_m}\cdots{x_k}^{v_k}\left(x_{k+1}\right)^{v_{k+1}}$$
$$= {x_m}^2 n(n-1)\left(x_1+\cdots+x_m+\cdots+x_{k+1}\right)^{n-2}.$$

If we let $x_{k+1} = 1 - x_1 - \cdots - x_k$, then from the equalities above

$$\sum_{v_1+\cdots+v_k \leq n} \binom{n}{v_1!v_2!\cdots v_k!} v_m {x_1}^{v_1} \cdots {x_m}^{v_m} \cdots {x_k}^{v_k} \left(1 - x_1 - \cdots - x_k\right)^{n-v_1-\cdots-v_k}$$

$$= x_m n \left(x_1 + \cdots + x_k + 1 - x_1 - \cdots - x_k\right)^{n-1}$$

$$= x_m n \left(1\right)^{n-1}$$

$$= x_m n,$$

and

$$\sum_{v_1+\cdots+v_k \leq n} \binom{n}{v_1!v_2!\cdots v_k!} {v_m}^2 {x_1}^{v_1} \cdots {x_m}^{v_m} \cdots {x_k}^{v_k} \left(1 - x_1 - \cdots - x_k\right)^{n-v_1-\cdots-v_k}$$

$$= {x_m}^2 n(n-1) \left(x_1 + \cdots + x_k + 1 - x_1 - \cdots - x_k\right)^{n-2}$$

$$= {x_m}^2 n(n-1) \left(1\right)^{n-2}$$

$$= {x_m}^2 n(n-1).$$

Thus we have

$$\begin{aligned}
&\sum_{|\vec{v}| \leq n} B_{\vec{v},n}(\vec{x}) \left(x_m - \frac{v_m}{n}\right)^2 \\
&\qquad = x_m^2 \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \vec{x}^{\vec{v}} (1 - |\vec{x}|)^{n-|\vec{v}|} - 2x_m \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \frac{v_m}{n} \vec{x}^{\vec{v}} (1 - |\vec{x}|)^{n-|\vec{v}|} \\
&\qquad\quad + \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \frac{{v_m}^2}{n^2} \vec{x}^{\vec{v}} (1 - |\vec{x}|)^{n-|\vec{v}|} \\
&\qquad = \frac{x_m(1 - x_m)}{n}.
\end{aligned}$$

Now, to analyze the second summation first note that

$$\delta^2 \leq \left(x_1 - \frac{v_1}{n}\right)^2 + \left(x_1 - \frac{v_1}{n}\right)^2 + \cdots + \left(x_k - \frac{v_k}{n}\right)^2.$$

Define $M = \max_{x \in I^k} f(\vec{x})$ so that for any $\vec{x}, \vec{y} \in [0,1]$,

$$|f(\vec{x}) - f(\vec{y})| \leq |f(\vec{x})| + |f(\vec{y})| \leq 2M.$$

Then we have

$$\begin{aligned}
&\delta^2 \sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) \geq \delta} B_{\vec{v},n}\left(\vec{x}\right) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| \\
&\qquad \leq 2M \sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) \geq \delta} B_{\vec{v},n}(\vec{x}) \left[\left(x_1 - \frac{v_1}{n}\right)^2 + \left(x_1 - \frac{v_1}{n}\right)^2 + \cdots + \left(x_k - \frac{v_k}{n}\right)^2 \right] \\
&\qquad \leq 2M \sum_{|\vec{v}| \leq n} B_{\vec{v},n}(\vec{x}) \left[\left(x_1 - \frac{v_1}{n}\right)^2 + \left(x_2 - \frac{v_2}{n}\right)^2 + \cdots + \left(x_k - \frac{v_k}{n}\right)^2 \right] \\
&\qquad = 2M \left(\frac{x_1(1-x_1) + \cdots + x_k(1-x_k)}{n} \right).
\end{aligned}$$

Here we have that $x_i(1-x_i) \leq \frac{1}{4}$ for all $i = 1, \ldots, k$. Thus

$$2M \left(\frac{x_1(1-x_1) + \cdots + x_k(1-x_k)}{n} \right) \leq 2M \left(\frac{k \cdot \frac{1}{4}}{n} \right) = \frac{Mk}{2n}.$$

From the above we have

$$\sum_{d\left(\frac{\vec{v}}{n}, \vec{x}\right) \geq \delta} B_{\vec{v},n}(\vec{x}) \left| f\left(\tfrac{\vec{v}}{n}\right) - f(\vec{x}) \right| \leq \frac{Mk}{2n\delta^2}.$$

Again by considering δ, we can pick N large enough so that $\frac{2M}{\delta^2 n} < \frac{\epsilon}{2}$ when $n \geq N$. So we have

$$|\mathbb{B}_n(f)\,(\vec{x}) - f(\vec{x})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

7. 2-SIMPLEX (CONCLUSION)

Notice that

$$f(a,b,c) = \frac{1}{4}\Big\{2(n-a-b)(n-a-c) + 2(n-a-b)(b+c)$$
$$+ 2(n-a-c)(b+c) - (n-a-b)^2 - (n-a-c)^2 - (b+c)^2\Big\}^{1/2}$$

is continuous on $\triangle_k$. Also note that when f is the formula above,

$$\begin{aligned}
\frac{1}{n}f(a,b,c) &= \frac{1}{n}g(a,b,c,n-a-b-c)\\
&= \frac{1}{n}\cdot\frac{1}{4}\Big\{2(n-a-b)(n-a-c) + 2(n-a-b)(b+c)\\
&\quad + 2(n-a-c)(b+c) - (n-a-b)^2 - (n-a-c)^2 - (b+c)^2\Big\}^{1/2}\\
&= \frac{1}{4}\Big\{\frac{1}{n^2}\Big[2(n-a-b)(n-a-c) + 2(n-a-b)(b+c)\\
&\quad + 2(n-a-c)(b+c) - (n-a-b)^2 - (n-a-c)^2 - (b+c)^2\Big]\Big\}^{1/2}\\
&= \frac{1}{4}\Big\{2\left(1-\frac{a}{n}-\frac{b}{n}\right)\left(1-\frac{a}{n}-\frac{c}{n}\right) + 2\left(1-\frac{a}{n}-\frac{b}{n}\right)\left(\frac{b}{n}+\frac{c}{n}\right)\\
&\quad + 2\left(1-\frac{a}{n}-\frac{c}{n}\right)\left(\frac{b}{n}+\frac{c}{n}\right) - \left(1-\frac{a}{n}-\frac{b}{n}\right)^2\\
&\quad - \left(1-\frac{a}{n}-\frac{c}{n}\right)^2 - \left(\frac{b}{n}+\frac{c}{n}\right)^2\Big\}^{1/2}\\
&= g\left(\frac{a}{n},\frac{b}{n},\frac{c}{n},1-\frac{a}{n}-\frac{b}{n}-\frac{c}{n}\right)\\
&= \bar{f}\left(\frac{a}{n},\frac{b}{n},\frac{c}{n}\right),
\end{aligned}$$

where $\bar{f}$ is a new function that still defines the area of the 2-simplex requiring only $\frac{a}{n}$, $\frac{b}{n}$, and $\frac{c}{n}$ to form the parameters. At this point we can apply the generalization of Bernstein polynomials on simplices, where

$$B_n(f)(\vec{x}) = \sum_{|\vec{v}|\le n}\binom{n}{\vec{v}}\vec{x}^{\,\vec{v}}(1-|\vec{x}|)^{n-|\vec{v}|}f\left(\tfrac{\vec{v}}{n}\right)$$

and $\lim_{n\to\infty} B_n(f)(\vec{x}) = f(\vec{x})$. By choosing $\vec{x} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ we have

$$\begin{aligned}
B_n\left(\bar{f}\right)\left(\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right)\right) &= \sum_{a+b+c\leq n} \binom{n}{a,b,c} \left(\frac{1}{4}\right)^{a+b+c} \left(\frac{1}{4}\right)^{n-a-b-c} \bar{f}\left(\frac{a}{n}, \frac{b}{n}, \frac{c}{n}\right) \\
&= \sum_{a+b+c\leq n} \binom{n}{a,b,c} \left(\frac{1}{4}\right)^{n} \frac{1}{n} f\left(a,b,c\right) \\
&= \frac{1}{n} \cdot \frac{1}{4^n} \sum_{a+b+c\leq n} \binom{n}{a,b,c} f\left(a,b,c\right).
\end{aligned}$$

Therefore from Theorem 6.1,

$$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{4^n} \sum_{a+b+c\leq n} \binom{n}{a,b,c} f\left(a,b,c\right) = \bar{f}\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right).$$

It's easily found that $\bar{f}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = \frac{\sqrt{3}}{8}$, so ultimately

$$\frac{1}{4^n} \sum_{a+b+c\leq n} \binom{n}{a,b,c} f\left(a,b,c\right) \sim n\frac{\sqrt{3}}{8}.$$

Note that $\sqrt{3}/8 \approx 0.216506$ agrees with the collected data from Figure 5 for the 2-simplex case.

8. CAYLEY-MENGER DETERMINANTS

Let V_k be the measure of a k-simplex. Note that any vertex of the simplex can be defined by $(x_{1m}, x_{2m}, \cdots, x_{km})$, where m can be from 1 to $k+1$. Every side length of the k-simplex can be defined by

$$S_{ij} = \sqrt{(x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2 + \cdots + (x_{ki} - x_{kj})^2},$$

where $(x_{1i}, x_{2i}, \cdots, x_{ki})$ and $(x_{1j}, x_{2j}, \cdots, x_{kj})$ for $i, j = 1, 2, \ldots, k+1$ are two vertices of the k-simplex and S_{ij} is the distance between them. Define

$$|M_k| = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & {S_{12}}^2 & \cdots & {S_{1(k+1)}}^2 \\ 1 & {S_{21}}^2 & 0 & \cdots & {S_{2(k+1)}}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & {S_{(k+1)1}}^2 & {S_{(k+1)2}}^2 & \cdots & 0 \end{vmatrix}$$

Theorem 8.1: The measure of a k-dimensional simplex is defined by

$$(V_k)^2 = -\frac{\left|{M_k}^2\right|}{(-2)^k (n!)^2}.$$

Proof. The following proof for this theorem is influenced directly by [3]. Consider a k-dimensional simplex with $k+1$ vertices. Choose any one of these vertices and regard it as the apex of a k-dimensional pyramid above a $(k-1)$-dimensional base. Then we can call the height of this apex from this base h_k and the measure of the base V_{k-1}. If we consider a $(k-1)$-dimensional slice perpendicular to the k-dimensional plane, when we are a distance of h from the apex, the $(k-1)$-dimensional measure of that slice will be $V_{k-1}\left(\frac{h}{h_k}\right)^{k-1}$. Thus the measure V_k for the k-dimensional simplex is

$$V_k = \int_{h=0}^{h_k} V_{k-1} \left(\frac{h}{h_k} \right)^{k-1} dh = V_{k-1} \left(\frac{h_k}{k} \right).$$

We can continue this process by defining h_i for $i = 1, 2, \ldots, k$ to be the height of the $(i+1)$th vertex above the plane containing the i-dimensional simplex base. The measure can then be written as

$$V_k = \frac{1}{k!} h_k h_{k-1} h_{k-2} \cdots h_1.$$

To determine the heights in the above equation, we can rigidly translate the vertices of a k-dimensional simplex without affecting its measure so that the $(k+1)$th vertex is positioned at the origin. We then arrange the following matrix,

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \ldots & x_{kk} \end{bmatrix}$$

Where $(x_{i1}, x_{i2}, \ldots, x_{ik})$ with $i = 1, 2, \ldots, k$ are the coordinates of each vertex not positioned at the origin. Any rigid rotation of the vertices about the origin will not affect the determinant of the matrix. Thus we can rotate the simplex in k-dimensional space in such a way that $k-1$ vertices are contained in $(k-1)$-dimensional space, orthogonal to one of the axes. In doing this, we can obtain a matrix with the same determinant such that the last entry is zero for every coordinate except the kth coordinate.

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \ldots & x_{kk} \end{vmatrix} = \begin{vmatrix} x'_{11} & x'_{12} & \cdots & 0 \\ x'_{21} & x'_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x'_{k1} & x'_{k2} & \ldots & x'_{kk} \end{vmatrix}$$

Above the coordinate x'_{kk} is the height h_k of the vertex $(x'_{k1}, x'_{k2}, \ldots, x'_{kk})$ above the $(k-1)$-dimensional space containing the other k vertices (including the origin). So by cofactor decomposition of the last column we have

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \ldots & x_{kk} \end{vmatrix} = \begin{vmatrix} x'_{11} & x'_{12} & \cdots & 0 \\ x'_{21} & x'_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x'_{k1} & x'_{k2} & \ldots & x'_{kk} \end{vmatrix} = h_k \begin{vmatrix} x'_{11} & x'_{12} & \cdots & x'_{1(k-1)} \\ x'_{21} & x'_{22} & \cdots & x'_{2(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ x'_{(k-1)1} & x'_{(k-1)2} & \cdots & x'_{(k-1)(k-1)}. \end{vmatrix}$$

We can again rotate the vertices in such a way that the last coordinate in this matrix lies above the $(k-2)$-dimensional plane containing the other vertices and apply the same procedure, i.e.

$$h_k h_{k-1} \left| x''_{ij} \right|, \qquad i, j = 1, \ldots, k-2.$$

Ultimately, we have

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \ldots & x_{kk} \end{vmatrix} = h_k h_{k-1} h_{k-2} \cdots h_1.$$

Thus,

$$V_k = \frac{1}{k!} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \dots & x_{kk} \end{vmatrix}$$

We can alter the above equation to consider a simplex with $k + 1$ arbitrary vertices. If the vertices are located anywhere without requiring one vertex to be at the origin, we can call the $(k + 1)$th vertex $\left(x_{(k+1)1}, x_{(k+1)2}, \dots, x_{(k+1)k}\right)$ and write the content of the k-simplex as

$$V_k = \frac{1}{k!} \begin{vmatrix} \left(x_{11} - x_{(k+1)1}\right) & \left(x_{12} - x_{(k+1)2}\right) & \cdots & \left(x_{1k} - x_{(k+1)k}\right) \\ \left(x_{21} - x_{(k+1)1}\right) & \left(x_{22} - x_{(k+1)2}\right) & \cdots & \left(x_{2k} - x_{(k+1)k}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(x_{k1} - x_{(k+1)1}\right) & \left(x_{k2} - x_{(k+1)2}\right) & \dots & \left(x_{kk} - x_{(k+1)k}\right) \end{vmatrix}$$

By once again using cofactor decomposition, we can write the above as the determinant of a higher dimensional matrix like so

$$V_k = \frac{1}{k!} \begin{vmatrix} 1 & \left(x_{11} - x_{(k+1)1}\right) & \cdots & \left(x_{1k} - x_{(k+1)k}\right) \\ 1 & \left(x_{21} - x_{(k+1)1}\right) & \cdots & \left(x_{2k} - x_{(k+1)k}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \left(x_{k1} - x_{(k+1)1}\right) & \cdots & \left(x_{kk} - x_{(k+1)k}\right) \\ 1 & 0 & \cdots & 0 \end{vmatrix}$$

Since adding a multiple of any row or column to another doesn't affect determinants, we can add $x_{(k+1)1}$ times the first column to the second one, $x_{(k+1)2}$ times the first column to the third, and so on to get

$$V_k = \frac{1}{k!} \begin{vmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k1} & \cdots & x_{kk} \\ 1 & x_{(k+1)1} & \cdots & x_{(k+1)k} \end{vmatrix}$$

Now we wish to alter the equation further to write everything in terms of side lengths instead of single vertices. Note that since the transpose of a matrix has the same determinant as that matrix, we have

$$(V_k)^2 = \left(\frac{1}{k!} \begin{vmatrix} 1 & x_{11} & \cdots & x_{1k} \\ 1 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k1} & \cdots & x_{kk} \\ 1 & x_{(k+1)1} & \cdots & x_{(k+1)k} \end{vmatrix} \right) \left(\frac{1}{k!} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{(k+1)1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{12} & x_{22} & \cdots & x_{(k+1)2} \\ x_{1k} & x_{2k} & \cdots & x_{(k+1)k} \end{vmatrix} \right)$$

$$= \frac{1}{(k!)^2} \begin{vmatrix} 1 + m_1 \cdot m_1 & 1 + m_1 \cdot m_2 & \cdots & 1 + m_1 \cdot m_{k+1} \\ 1 + m_2 \cdot m_1 & 1 + m_2 \cdot m_2 & \cdots & 1 + m_2 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 + m_k \cdot m_1 & 1 + m_k \cdot m_2 & \cdots & 1 + m_k \cdot m_{k+1} \\ 1 + m_{k+1} \cdot m_1 & 1 + m_{k+1} \cdot m_2 & \cdots & 1 + m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

Where we have $m_i \cdot m_j = x_{i1}x_{j1} + x_{i2}x_{j2} + \cdots + x_{ik}x_{jk}$. We can again express this as a matrix of one higher dimension,

$$(V_k)^2 = \frac{1}{(k!)^2} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 + m_1 \cdot m_1 & \cdots & 1 + m_1 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 + m_k \cdot m_1 & \cdots & 1 + m_k \cdot m_{k+1} \\ 0 & 1 + m_{k+1} \cdot m_1 & \cdots & 1 + m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

We may subtract the first row from every other row without changing the determinant,

$$(V_k)^2 = \frac{1}{(k!)^2} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -1 & m_1 \cdot m_1 & \cdots & m_1 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & m_k \cdot m_1 & \cdots & m_k \cdot m_{k+1} \\ -1 & m_{k+1} \cdot m_1 & \cdots & m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

Notice that

$$0 = \begin{vmatrix} 0 & x_{11} & \cdots & x_{1k} \\ 0 & x_{21} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{k1} & \cdots & x_{kk} \\ 0 & x_{(k+1)1} & \cdots & x_{(k+1)k} \end{vmatrix} \begin{vmatrix} 0 & 0 & \cdots & 0 \\ x_{11} & x_{21} & \cdots & x_{(k+1)1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{12} & x_{22} & \cdots & x_{(k+1)2} \\ x_{1k} & x_{2k} & \cdots & x_{(k+1)k} \end{vmatrix}$$

$$= \begin{vmatrix} m_1 \cdot m_1 & m_1 \cdot m_2 & \cdots & m_1 \cdot m_{k+1} \\ m_2 \cdot m_1 & m_2 \cdot m_2 & \cdots & m_2 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_k \cdot m_1 & m_k \cdot m_2 & \cdots & m_k \cdot m_{k+1} \\ m_{k+1} \cdot m_1 & m_{k+1} \cdot m_2 & \cdots & m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

Thus the cofactor of the top left-hand element in the determinant for V_k is zero and this element has no impact on the determinant. So we may set it to zero,

$$(V_k)^2 = \frac{1}{(k!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ -1 & m_1 \cdot m_1 & \cdots & m_1 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ -1 & m_k \cdot m_1 & \cdots & m_k \cdot m_{k+1} \\ -1 & m_{k+1} \cdot m_1 & \cdots & m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

Because multiplying a single row or a column of a determinant by a constant is the same as multiplying the whole determinant by that constant, we can multiply the first column by -1 to get

$$(V_k)^2 = -\frac{1}{(k!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & m_1 \cdot m_1 & \cdots & m_1 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & m_k \cdot m_1 & \cdots & m_k \cdot m_{k+1} \\ 1 & m_{k+1} \cdot m_1 & \cdots & m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

We can multiply every column except the first by -2 and multiply the first row by $-\frac{1}{2}$ to obtain

$$(V_k)^2 = -\frac{(-2)}{(-2)^{k+1}(k!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & -2m_1 \cdot m_1 & \cdots & -2m_1 \cdot m_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -2m_k \cdot m_1 & \cdots & -2m_k \cdot m_{k+1} \\ 1 & -2m_{k+1} \cdot m_1 & \cdots & -2m_{k+1} \cdot m_{k+1} \end{vmatrix}$$

From here, for each $i = 1, 2, \ldots, k+1$ we can add the first column multiplied by $m_i \cdot m_i$ to the $(i+1)$th column. Then we can add the first row multiplied by $m_i \cdot m_i$

to the $(i+1)$th row, obtaining the result

$$(V_k)^2 = -\frac{1}{(-2)^k(k!)^2}\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & m_1^2 - 2m_1m_1 + p_1^2 & \cdots & m_1^2 - 2m_1m_{k+1} + p_{k+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & m_k^2 - 2m_km_1 + p_1^2 & \cdots & m_k^2 - 2m_km_{k+1} + p_{k+1}^2 \\ 1 & m_{k+1}^2 - 2m_{k+1}m_1 + p_1^2 & \cdots & m_{k+1}^2 - 2m_{k+1}m_{k+1} + p_{k+1}^2 \end{vmatrix}$$

where $p_1p_j = p_i \cdot p_j$. The square of the distance between these two is

$$p_i \cdot p_i - 2p_i \cdot p_j + p_j \cdot p_j = (x_{1i} - x_{1j})^2 + (x_{2i} - x_{2j})^2 + \cdots + (x_{ki} - x_{kj})^2 = {S_{ij}}^2,$$

where ${S_{ij}}^2$ is the square of a side length of the k-simplex. So we ultimately have

$$(V_k)^2 = -\frac{|M_k|}{(-2)^k(k!)^2}.$$

Thus we can write an equation for the measure of any dimensional simplex using the side lengths of the simplex as parameters, of which there are $\binom{k+1}{2}$. $\square$

9. k-SIMPLEX CASE

Figure 6 displays data collected for the average measure of a 3-simplex in an n-cube for several values of n,

n	20	50	100	200	300	400	500
Average	3.41	14.26	40.90	116.94	215.33	331.92	464.32
(Average/$n^{3/2}$)	0.0381	0.0403	0.0409	0.04134	0.04144	0.04149	0.04153

Figure 6: Average measure of a 3-simplex in the n-cube for several values of n, with Monte Carlo approximations.

Since the hypothesis is that the average is approaching $n^{3/2} \cdot C$ for some constant C, the bottom row lists the average divided by $n^{3/2}$. If there is any convergence here, for the first four values of n in the figure it is quite slow. Also, computationally these exact values are time-consuming to derive. Thus the last three columns of the figure, divided by a double bar, represent a utilization of the Monte Carlo method implemented to reduce the number of computations needed and approximate values for these higher dimensions. The approximated data does seem to indicate a convergence. Now we apply general methods for determining the average measure of any k-simplex in the n-cube.

In any n-cube, we can consider the coordinates that define each vertex of a particular k-dimensional simplex in that n-cube by fixing the first vertex at the origin and choosing the remaining k vertices that form it. To create the vertices aside from the origin, we can consider n seperate k-tuples. One k-tuple will assign values for the first coordinate value of each vertex. Another k-tuple will assign values to the second coordinate value of each vertex, and so on. For each k-tuple we can choose a 0 or 1 for its k values. So we can expect 2^k possible k-tuples to choose from. One side length of the k-simplex can be defined by the first entry in each k-

tuple, another side length by the second entry in each k-tuple, etc.. For each side length defined this way there are 2^{k-1} corresponding k-tuples because if we fix a 1 in a particular position when considering k-tuples, there are 2^{k-1} ways to position the other values of the k-tuple.

Another type of side length corresponds to the instance when the ith and jth entry in the k-tuples differ. For example, consider two different k-tuples where every entry is zero except the first two.

$$(1, 0, 0, \cdots, 0)$$
$$(0, 1, 0, \cdots, 0)$$

The k-tuples above correspond to a distance between the first and second vertices. For each side length defined this way there are also 2^{k-1} corresponding k-tuples since when considering these k-tuples, one may fix a 1 in the ith position and a 0 in the jth position, leaving 2^{k-2} possible placements for the other entries. Alternatively, one may fix a 0 in the ith position and a 1 in the jth position, also leaving 2^{k-2} possible arrangements. So there are $2^{k-2} + 2^{k-2} = 2^{k-1}$ k-tuples corresponding to the side length.

For every possible form for the k-tuples, let the list $P = \{a_1, a_2, \ldots, a_{2^k}\}$ contain the number of k-tuples of that form chosen to define any k-simplex, where $a_1 + a_2 + \cdots + a_{2^k} = n$. From these we can derive any side length of the k-simplex. For example, the first side length may be of the form $S_1 = \sqrt{a_{11} + a_{12} + \cdots + a_{1(2^{k-1})}}$ where each a_{1j} is a distinct member of the list P.

Note there are $\binom{k+1}{2}$ sides for the k-simplex. We define $g(a_1, a_2, \ldots, a_{2^k})$ to be the corresponding Cayley-Menger formula for the measure of a k-simplex, taking members of the list P as parameters to form the side lengths, i.e.,

$$\begin{aligned} g(a_1, a_2, \ldots, a_{2^k}) &= V_k\left(a_{11} + \cdots + a_{1(2^k-1)}, \ldots, a_{\binom{k+1}{2}1} + \cdots + a_{\binom{k+1}{2}(2^k-1)}\right) \\ &= V_k\left({S_1}^2, \ldots, {S_{\binom{k+1}{2}}}^2\right), \end{aligned}$$

where each sum in the parameters for V contains specific parameters of g and comprises a side length squared.

Note that because V_k is a function defining k-dimensional area, any scalar multiplied with the side lengths used to calculate V_k will be affected by a degree of k. That is,

$$\begin{aligned} g\left(\lambda a_1, \lambda a_2, \ldots, \lambda a_{2^k}\right) &= V_k(\lambda a_{11} + \cdots + \lambda a_{1(2^k-1)}, \ldots, \lambda a_{\binom{k+1}{2}1} + \cdots + \lambda a_{\binom{k+1}{2}(2^k-1)}) \\ &= V_k\left(\lambda {S_1}^2, \ldots, \lambda {S_{\binom{k+1}{2}}}^2\right) \\ &= V_k\left(\left(\sqrt{\lambda} S_1\right)^2, \ldots, \left(\sqrt{\lambda} S_{\binom{k+1}{2}}\right)^2\right) \\ &= \left(\sqrt{\lambda}\right)^k V_k\left({S_1}^2, \ldots, {S_{\binom{k+1}{2}}}^2\right) \\ &= \lambda^{k/2} g(a_1, a_2, \ldots, a_{2^k-1}). \end{aligned}$$

Since $a_{2^k} = n - a_1 - \cdots - a_{2^k-1}$, we can define another function,

$$f((a_1, a_2, \ldots, a_{2^k-1})) = g(a_1, a_2, \ldots, a_{2^k-1}, n - a_1 - \cdots - a_{2^k-1}),$$

to describe the measure. Note,

$$\begin{aligned} \frac{1}{n^{k/2}} f((a_1, a_2, \ldots, a_{2^k-1})) &= \left(\frac{1}{n}\right)^{k/2} g(a_1, a_2, \ldots, n - a_1 - \cdots - a_{2^k-1}) \\ &= g\left(\frac{a_1}{n}, \frac{a_2}{n}, \ldots, 1 - \frac{a_1}{n} - \cdots - \frac{a_{2^k-1}}{n}\right) \\ &= \bar{f}\left(\left(\frac{a_1}{n}, \frac{a_2}{n}, \ldots, \frac{a_{2^k-1}}{n}\right)\right), \end{aligned}$$

where $\bar{f}$ is another function defining measure using $\frac{a_1}{n}, \ldots, \frac{a_{2^k-1}}{n}$ as parameters.

If $\vec{v}$ consists of every member of P, we can define the average measure of the k-simplex in the n-cube with the following equation,

$$\frac{1}{(2^k)^n} \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} f\left(\vec{v}\right).$$

Next we will find a convergence for this average. To utilize the k-simplex Bernstein polynomial convergence, we take $\vec{x} = \left(\frac{1}{2^k}, \ldots, \frac{1}{2^k}\right)$.

$$\begin{aligned}
&\mathbb{B}_n(\bar{f})\left(\left(\tfrac{1}{2^k}, \ldots, \tfrac{1}{2^k}\right)\right) = \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \vec{x}^{\,\vec{v}} (1 - |\vec{x}|)^{n - |\vec{v}|} \bar{f}\left(\frac{\vec{v}}{n}\right) \\
&\quad = \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \left(\frac{1}{2^k}\right)^{a_1} \cdots \left(\frac{1}{2^k}\right)^{a_{2^k-1}} \left(1 - \left(\frac{1}{2^k}\right) - \cdots - \left(\frac{1}{2^k}\right)\right)^{n - a_1 - \cdots - a_{2^k-1}} \bar{f}\left(\frac{\vec{v}}{n}\right) \\
&\quad = \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \left(\frac{1}{2^k}\right)^{a_1} \cdots \left(\frac{1}{2^k}\right)^{a_{2^k-1}} \left(1 - \left(\frac{1}{2^k}\right) - \cdots - \left(\frac{1}{2^k}\right)\right)^{n - a_1 - \cdots - a_{2^k-1}} \frac{1}{(2^k)^n} f\left(\vec{v}\right) \\
&\quad = \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} \left(\frac{1}{2^k}\right)^{n} \frac{1}{n^{k/2}} f\left(\vec{v}\right) \\
&\quad = \frac{1}{n^{k/2}} \cdot \frac{1}{(2^k)^n} \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} f\left(\vec{v}\right).
\end{aligned}$$

Thus by convergence,

$$\lim_{n \to \infty} \frac{1}{n^{k/2}} \cdot \frac{1}{(2^k)^n} \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} f\left(\vec{v}\right) = \bar{f}\left(\left(\tfrac{1}{2^k}, \ldots, \tfrac{1}{2^k}\right)\right)$$

and

$$\frac{1}{(2^k)^n} \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} f\left(\vec{v}\right) \sim n^{k/2} \cdot \bar{f}\left(\left(\tfrac{1}{2^k}, \ldots, \tfrac{1}{2^k}\right)\right).$$

To analyze the value of our constant term above, note that

$$\begin{aligned}
\bar{f}\left(\left(\tfrac{1}{2^k},\ldots,\tfrac{1}{2^k}\right)\right) &= g\left(\left(\tfrac{1}{2^k},\ldots,1-\tfrac{2^k-1}{2^k}\right)\right)\\
&= g\left(\left(\tfrac{1}{2^k},\ldots,\tfrac{1}{2^k}\right)\right)\\
&= V_k\left(\left(\tfrac{2^{k-1}}{2^k},\ldots,\tfrac{2^{k-1}}{2^k}\right)\right)\\
&= V_k\left(\tfrac{1}{2},\ldots,\tfrac{1}{2}\right)\\
&= V_k\left(\left(\tfrac{1}{\sqrt{2}}\right)^2,\ldots,\left(\tfrac{1}{\sqrt{2}}\right)^2\right).
\end{aligned}$$

This is the measure of a regular k-simplex in the unit n-cube $\left(\text{of side length } \tfrac{1}{\sqrt{2}}\right)$. Recall that

$$V_k = \frac{1}{k!}h_k h_{k-1} h_{k-2} \cdots h_1.$$

Any height h_i for an i-simplex can be considered as the distance between the $(i+1)$th vertex and the centroid of its $(i-1)$-simplex base. For a $(i-1)$-simplex in n-dimensional space with coordinates $(x_{11},\ldots,x_{1n}),\ldots,(x_{i1},\ldots,x_{in})$, the centroid is located at

$$\left(\frac{x_{11}+x_{21}+\cdots+x_{i1}}{i},\ldots,\frac{x_{1n}+x_{2n}+\cdots+x_{in}}{i}\right).$$

Consider a regular i-simplex embedded in $(i+1)$-dimensional space with the vertices

$$\begin{gathered}
\left(\frac{1}{2},0,0,\ldots,0,0\right)_1\\
\left(0,\frac{1}{2},0,\ldots,0,0\right)_2\\
\left(0,0,\frac{1}{2},\ldots,0,0\right)_3\\
\vdots\\
\left(0,0,0,\ldots,\frac{1}{2},0\right)_i\\
\left(0,0,0,\ldots,0,\frac{1}{2}\right)_{i+1}
\end{gathered}$$

The sides of this regular i-simplex all clearly have a length of $\frac{1}{\sqrt{2}}$. For the $(i-1)$-simplex base of the $(i+1)$th vertex, its centroid is at

$$\left(\frac{1}{2i}, \frac{1}{2i}, \frac{1}{2i}, \ldots, \frac{1}{2i}, 0\right).$$

The distance between this vertex and the $(i+1)$th vertex is

$$h_i = \sqrt{i\left(\frac{1}{2i}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{\frac{1}{i}+1} = \frac{1}{2}\sqrt{\frac{i+1}{i}}.$$

So now for a regular k-simplex of side length $\frac{1}{\sqrt{2}}$,

$$\begin{aligned} V_k &= \frac{1}{k!} \cdot \frac{1}{2}\sqrt{\frac{k+1}{k}} \cdot \frac{1}{2}\sqrt{\frac{k}{k-1}} \cdot \frac{1}{2}\sqrt{\frac{k-1}{k-2}} \cdots \frac{1}{2}\sqrt{\frac{2}{1}} \\ &= \frac{1}{2^k k!}\sqrt{\frac{(k+1)!}{k!}} \\ &= \frac{1}{2^k k!}\sqrt{k+1}. \end{aligned}$$

Main Theorem: The average measure of a k-dimensional simplex in an n-dimensional cube is asymptotically,

$$\frac{1}{(2^k)^n} \sum_{|\vec{v}| \leq n} \binom{n}{\vec{v}} f(\vec{v}) \sim n^{k/2} \cdot \frac{\sqrt{k+1}}{2^k k!}.$$

Note that when $k = 3$, we have $\frac{\sqrt{3+1}}{2^3 3!} \approx 0.04167$, which correlates with the collected data from Figure 6 for the 3-simplex case.

10. CONCLUSION

To begin this research, we developed an equation for the average measure of a 1-simplex in a unit n-cube. We then analyzed the distribution of the 1-simplex lengths in the n-cube and noticed the smoothness of this distribution. This allowed us to form a conjecture for the average measure, that it approaches $\sqrt{n/2}$. Results from our formula for multiple values of n appeared to coincide with this approximation and reinforced the conjecture. We then were able to apply Bernstein polynomials to prove the conjecture.

Using Cayley-Menger determinants, we developed equations for the average measure for higher values of k. The measure distribution for higher-dimensional simplices in the n-cube appeared less promising in finding a trend for the average measure. However, we were able to implement the Monte Carlo method in our calculations and collect enough data to determine a generalized conjecture, that the average measure approaches $n^{k/2} \cdot C$ for some constant C, did appear to be true. This conjecture was then proven using the convergence of k-simplex Bernstein polynomials. With this convergence, we were also able to determine the value of the constant C in the conjecture to be the measure of a regular k-simplex with side length $\frac{1}{\sqrt{2}}$.

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