BearWorks

Spring 2019

Survey of Lebesgue and Hausdorff Measures

Jacob N. Oliver
Missouri State University, Jacob65@live.missouristate.edu

Follow this and additional works at: https://bearworks.missouristate.edu/theses

Part of the Analysis Commons

Recommended Citation

SURVEY OF LEBESGUE AND HAUSDORFF MEASURES

A Master's Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

Jacob Oliver

May 2019

SURVEY OF LEBESGUE AND HAUSDORFF MEASURES

Mathematics

Missouri State University, May 2019

Master of Science

Jacob Oliver

ABSTRACT

Measure theory is fundamental in the study of real analysis and serves as the basis for more robust integration methods than the classical Riemann integrals. Measure theory allows us to give precise meanings to lengths, areas, and volumes which are some of the most important mathematical measurements of the natural world. This thesis is devoted to discussing some of the major proofs and ideas of measure theory. We begin with a study of Lebesgue outer measure and Lebesgue measurable sets. After a brief discussion of non-measurable sets, we define Lebesgue measurable functions and the Lebesgue integral. In the last chapter we discuss general outer measures and give two specific examples of measures based on an outer measure and Carathéodory's definition of measurable sets. Lebesgue-Stieltjes measures are important because they are not limited to the identity function that gives rise to Lebesgue measure, and Hausdorff measure is often able to distinguish a variety of sets whose Lebesgue measure are all zero. The goal of this thesis is to present the proofs and ideas of fundamental measure theory in a way that is accessible and helpful to senior level undergraduates and beginning graduates in their quest to lay a solid foundation for further training in analysis.

KEYWORDS: Hausdorff measure, Lebesgue measure, Cantor set, outer measure, Lebesgue-Stieltjes, measurable sets, Dirichlet function

SURVEY OF LEBESGUE AND HAUSDORFF MEASURES

By

Jacob Oliver

A Masters Thesis
Submitted to The Graduate College
Of Missouri State University
In Partial Fulfillment of the Requirements
For the Degree of Master of Science, Mathematics

May 2019

Approved:

William Bray, Ph.D., Thesis Committee Co-Chair

Xingping Sun, Ph.D., Thesis Committee Co-Chair

Matthew Wright, Ph.D., Committee Member

Julie J. Masterson, Ph.D., Graduate College Dean

ACKNOWLEDGEMENTS

I would like to thank all of my professors for their encouragement and support throughout my graduate studies. I would like to specifically thank Dr. Sun and Dr. Bray for their time and knowledge in advising me on this thesis and Dr. Wright for being my graduate advisor during my time at Missouri State.

Finally, I would like to dedicate this thesis to my wife, for her support and patience during the process of graduate school and especially in the time spent writing and researching for this work.

TABLE OF CONTENTS

1. INTRODUCTION

Some of the most important ideas in mathematics are those of length, area, and volume and the methods for finding them. Much of introductory calculus is devoted to studying the integral as a way of calculating area. In many college and high school classes, students are introduced to the Riemann integral as the main definition for integration. The Riemann integral is based on sums of areas of rectangles and is generally a good introduction into integration theory. There are numerous applications of the Riemann integral and in many cases it is sufficient for evaluating and approximating values associated with areas and volumes. However, when it comes to more abstract functions the Riemann integral becomes less powerful.

A good example of a function that is not Riemann integrable is the Dirichlet function $D(x)$, which is defined on $[0, 1]$ by

$$
D(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}
$$

This function is not Riemann integrable because it is nowhere continuous on $[0, 1]$, but the function is, as we will show later, Lebesgue integrable. In fact, we will see that the Lebesgue integral of the function is 0. Some other drawbacks of the Riemann integration method are that it depends too much on the continuity of the underlying function and on uniform convergence of a limit process. These drawbacks of the Riemann integral are the motivation for finding a more powerful and flexible integration method.

In working towards such an integration method, we come to the study of measures. Measures are generalizations of length, area, and volume. A measure is essentially a set function that assigns a numerical value for a given set satisfying some desirable conditions. With these measures we can define a more robust method of integration. One of the main topics of this thesis will be on systematically establishing

1

Lebesgue measure theory and outlining the Lebesgue integral. Later on, we will focus our discussion on other types of measures. As a survey thesis, much of the proofs and theorems will be based on well-established works. Sometimes a specific text with exclusive ideas will be cited. Unless otherwise stated the general theory will be sourced from [1], [2], [3], [4]. We arrange the thesis as follows.

In Chapter 2 we discuss Lebesgue measure theory which entails constructing the Lebesgue outer measure. Next we discuss Lebesgue measurable sets and the existence of non-measurable sets. We then study measurable functions and briefly overview the Lebesgue integral.

In Chapter 3 we discuss other important measures. This requires that we define a general outer measure. Finally, we address some specific measures based on Carathéodory's definition of measurable sets. These measures are the Lebesgue-Stieltjes measures and Hausdorff measures.

2. LEBESGUE MEASURE

We will first give some definitions that serve as a foundation for Lebesgue measure theory. The first definition is that of a σ-algebra.

DEFINITION 2.1 Given a set X, a collection S of subsets of X is a σ-algebra if it has the following properties:

1. $\emptyset \in S$ (the empty set is in S).

2. If $T \in S$, then $T^C \in S$ where T^C is the complement of T.

3. If $V_n \in S$ for $n \in \mathbb{N}$, then $\bigcup_{k=1}^{\infty} V_k \in S$ (countable union).

Simply put, a σ-algebra is a collection of subsets of X which is closed under the set operations as specified above. Most of our discussion will focus on the real line $\mathbb{R}$ that has a well-established order and topology. An important property that follows from the definition of a σ-algebra and De Morgan's law is that if $V_n \in S$ for $n \in \mathbb{N}$, then $\bigcap_{k=1}^{\infty} V_k \in S$. We can say that a σ-algebra is closed under countable intersection.

DEFINITION 2.2 Let $E \subset \mathbb{R}$. If $E \cap (a, b) \neq \emptyset$ for any given open interval $(a, b) \in \mathbb{R}$, then we say E is dense in $\mathbb{R}$.

In other words, E is said to be *dense* in $\mathbb{R}$ if between any two elements of $\mathbb{R}$ there lies an element of E. For example, both the rationals and irrationals are dense in $\mathbb{R}$ since between any two real numbers we can find a rational or an irrational number.

DEFINITION 2.3 A nonempty set $E \subset \mathbb{R}$ is said to be **bounded above** if there exists a real number b such that $x \leq b$ for all $x \in E$. Such a number b is called an **upper bound** of E.

Axiom 2.4 (Completeness axiom) For a nonempty set of real numbers that is bounded above there exists among the set of upper bounds a **least upper bound** p where for any upper bound b, we have $p \leq b$. The least upper bound is often called the **supremum**.

DEFINITION 2.5 (Open sets) A set O is said to be open if for each $x \in O$ there exists an $r > 0$ such that $(x - r, x + r) \subseteq O$.

THEOREM 2.6 Every nonempty open set is the disjoint union of a countable collection of open intervals, meaning that for an open set $O \neq \emptyset$ there exists a countable collection of disjoint open intervals $\{I_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_n = O$.

A collection of sets $\{E_\lambda\}_{\lambda \in \Lambda}$ is said to be a *cover* of a set E if $E \subseteq \bigcup_{\lambda \in \Lambda} E_\lambda$. If each E_λ is open, then we say $\{E_\lambda\}_{\lambda \in \Lambda}$ is an *open cover*. We say $\{E_\lambda\}_{\lambda \in \Lambda}$ is a finite cover if $|\Lambda| < \infty$. We will mostly use the definition of open cover with respect to open intervals on the real line. This definition is also related to the following theorem.

THEOREM 2.7 (Heine-Borel Theorem) Let F be a closed and bounded set of real numbers. Then every open cover of F has a finite sub-cover.

2.1. Lebesgue Outer Measure

A **set function** is a function that assigns a real value to each set from a collection of sets. We now define a set function that generalizes the concept of length of an interval. For a given interval I we denote the **length** by $\ell(I)$ and define it as follows,

$$
\ell(I) = \begin{cases} b - a & \text{if } I = (a, b), \\ \infty & \text{if } I \text{ is unbounded.} \end{cases}
$$

DEFINITION 2.8 For a set A of real numbers, let $\{I_k\}_{k=1}^{\infty}$ be an open cover for A, meaning that the union of the I_k's contains A. We define the set function **Lebesgue outer measure**, sometimes called an exterior measure, as

$$
m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \mid A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.
$$

Here the infimum is taken over all possible open covers of E. From the defini-

tion, we note that $m^*(\emptyset) = 0$. It is also clear that the outer measure of a countable set is zero, which we will use in a later proposition. Outer measure is also *monotone*; meaning if $A \subseteq B$, then $m^*(A) \le m^*(B)$. It is straight forward to verify the following excision property:

$$m^*(A) = m^*(A \backslash B) \text{ where } B \text{ is a countable set.}$$

Some often used properties of Lebesgue outer measure are as follows.

PROPOSITION 2.9 The Lebesgue outer measure of an interval is its length. This simply means that, $\ell(I) = m^*(I)$.

PROPOSITION 2.10 Lebesgue outer measure is translation invariant. Let

$$\mathcal{S}(\mathbb{R}) = \{s \mid s \subset \mathbb{R}\}.$$

If $r \in \mathbb{R}$ and $A \in \mathcal{S}(\mathbb{R})$, then $m^*(A) = m^*(A + r)$. Where $A + r = \{a + r : a \in A\}$.

Proof. If $\{I_k\}_{k=1}^\infty$ is an open cover for A, then $\{I_k + r\}_{k=1}^\infty$ is an open cover for $A + r$. Since the length of an interval is translation invariant, i.e. $\ell(I_k) = \ell(I_k + r)$, we have $\sum_{k=1}^\infty \ell(I_k) = \sum_{k=1}^\infty \ell(I_k + r)$. Therefore $m^*(A) = m^*(A + r)$. $\qquad\square$

Another property we want outer measure to have is **countable additivity** in the following sense:

$$m^* \left(\bigcup_{k=1}^\infty E_k \right) = \sum_{k=1}^\infty m^*(E_k).$$

However, this is not to be the case. In fact, resorting to some ingenious ways of construction, we can show the existence of two disjoint sets A and B such that[1]

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

---

[1]We will later show that these sets A, B thus constructed are not Lebesgue measurable.

As such, we can only prove the weaker property for outer measure, which is countable subadditivity.

PROPOSITION 2.11 For a given countable collection of sets $\{E_k\}_{k=1}^{\infty}$, the following subadditivity holds true for outer measure:

$$m^*(\bigcup_{k=1}^{\infty} E_k) \le \sum_{k=1}^{\infty} m^*(E_k).$$

Outer measure becomes countably additive when it is restricted to what we will define as measurable sets.

2.2. Lebesgue Measurable Sets

To do Lebesgue integrals, countable additivity is a necessary property. As such, we settle upon a smaller σ-algebra $\mathcal{L}$ consisting of Lebesgue measurable sets. There are several ways to restrict the outer measure m^* on $\mathcal{L}$ so that countable additivity is satisfied. We will use the method first proposed by the Greek mathematician Constantine Carathéodory; see [3, pp. 35].

DEFINITION 2.12 A set E is **Lebesgue measurable** if for any set A, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C). \tag{2.1}$$

For brevity, we will often shorten Lebesgue measure as simply measure.

Using this definition for Lebesgue measurable sets is helpful in many ways. For one, we see that the empty set and $\mathbb{R}$ are measurable. We can show that if A and B are Lebesgue measurable with $A \cap B = \emptyset$, then

1. $A \cup B$ is measurable.

2. $m^*(A \cup B) = m^*(A) + m^*(B).$

We will defer our proof of the first part until Proposition 2.17. For the second part, we write

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^C)$$

$$= m^*(A) + m^*(B) \text{ (using the absorption law).}$$

Another interesting observation from this definition is that if E is measurable, then so is E^C, as can be shown by interchanging the places of E and E^C in Equation 2.1. Therefore, measurable sets are closed under operations of taking complements. This means that measurable sets already fulfill the second property of a σ-algebra. We will show later that the collection of measurable sets forms a σ-algebra.

Carathéodory's definition of measurable sets, though arguably not one that flows naturally, is very useful when it comes to showing a set is measurable in a simple and abstract form. To be clear, there are alternate definitions of what constitutes a Lebesgue measurable set. Another commonly used definition is as follows,

DEFINITION 2.13 A subset E of $\mathbb{R}$ is said to be *Lebesgue measurable* if given $\epsilon > 0, \exists$ an open set G such that

$$E \subset G \text{ and } m^*(G \backslash E) < \epsilon.$$

In the future, we will mainly consider Carathéodory's definition.

DEFINITION 2.14 If E is a measurable set then its Lebesgue outer measure is defined as **Lebesgue measure** which will be denoted $m(E)$. That is,

$$m^*(E) = m(E) \text{ if } E \text{ is a measurable set.}$$

This brings us to our next set of propositions as they relate to this idea of a Lebesgue measurable set.

PROPOSITION 2.15 If $m^*(E) = 0$, then E is measurable.

Proof. Let $m^*(E) = 0$ for a set E. By the subadditivity of outer measure, we have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^C).$$

We need to show the opposite inequality. Since outer measure is monotone,

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \cap E^C) \leq m^*(A).$$

Thus,

$$m^*(A) \geq m^*(A \cap E^C) = 0 + m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C).$$

Therefore, E is measurable. $\square$

Since a countable set has outer measure zero, it follows from Proposition 2.15 that a countable set is measurable.

At this point we know measurable sets are closed with respect to complements. To show that $\mathcal{L}$ is a σ-algebra, it suffices to show that the union of a countable collection of measurable sets is measurable. This requires two preliminary results.

PROPOSITION 2.16 The union of a finite number of measurable sets is measurable. In other words, if E_k is measurable for each $k = 1, ..., n$, then $\cup_{k=1}^{n} E_k$ is measurable.

Proof. We will proceed by induction. First, suppose E_1 and E_2 are measurable sets and let A be any set. By the subadditivity of outer measure,

$$m^*(A) \leq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C).$$

We want to show the other direction. Since E_1 and E_2 are measurable we have,

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C)$$
$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C).$$

Applying De Morgan law gives

$$m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*([A \cap E_1^C] \cap E_2^C)$$
$$= m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^C).$$

Then using the subadditivity of outer measure and the fact that

$$[A \cap E_1] \cup [A \cap E_1^C \cap E_2] = A \cap [E_1 \cup E_2],$$

we have

$$m^*(A) = m^*(A \cap E_1) + m^*([A \cap E_1^C] \cap E_2) + m^*(A \cap [E_1 \cup E_2]^C)$$
$$\geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C).$$

Thus, we have both directions and $m^*(A) = m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^C)$.
That is, $E_1 \cup E_2$ is measurable. □

PROPOSITION 2.17 The **finite additive property** of measurable sets holds.
That is for a finite collection of disjoint measurable sets E_k and any set A we have,

$$m^*(A \cap [\bigcup_{k=1}^{n} E_k]) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

We can use mathematical induction to show that the result in Proposition 2.16
is true for a finite collection of $\{E_k\}_{k=1}^{n}$ of measurable sets. Suppose the $(n-1)$ case

9

is true. Since we proved the union of two measurable sets is measurable and we know n and the union of all E_k for $k \leq n - 1$ are measurable, we can say their union is also measurable. With Propositions 2.17 and 2.16, we can prove the following,

PROPOSITION 2.18 The union of a countable collection of measurable sets is measurable.

Proof. Let $E = \cup_{k=1}^{\infty} E_k$, where each E_k is measurable and $E_i \cap E_j = \emptyset$ if $i \neq j$. Define $F_n = \cup_{k=1}^{n} E_k$ for $n \in \mathbb{N}$. From the previous proposition we know that the union of a finite number of measurable sets is measurable. Then using the additive property of finite measurable sets, we have for some set A,

$$m^*(A \cap F_n) = \sum_{k=1}^{n} m^*(A \cap E_k).$$

Since $F_n \subseteq E$, we have $E^C \subseteq F_n^C$. Then using the monotonicity of outer measure,

$$
\begin{aligned}
m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^C) \\
&\geq m^*(A \cap F_n) + m^*(A \cap E^C) \\
&= m^*(A \cap [\cup_{k=1}^{n} E_k]) + m^*(A \cap E^C) \\
&= \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^C).
\end{aligned}
$$

Since $m^*(A)$ does not depend on n, the inequality holds for all the E_k's and we can take the sum over infinity. We know by subadditivity that

$$m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Then, using the fact that $E = \bigcup_{k=1}^{\infty} E_k$, we get

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^C)$$

$$\geq m^*(A \cap E) + m^*(A \cap E^C).$$

Thus E is measurable. $\qquad\square$

Thus the measurable sets are closed under countable union. We already know that the empty set is measurable and that measurable sets are closed under complements. With this information, we can now say that the collection of measurable sets is a σ-algebra.

PROPOSITION 2.19 Every interval is a measurable set.

Proof. Since we know that measurable sets are a σ-algebra, our next goal is to show that if a σ-algebra of $\mathbb{R}$ contains intervals of the form (a, ∞), then it contains all intervals. Let S be a σ-algebra that contains such intervals. Since S is a σ-algebra, the complement of (a, ∞) is in S. Note that $(a, \infty)^C = (-\infty, a]$.

We need to show $[b, \infty), (a, b), [a, b), (a, b], [a, b] \in S$. We know $(b - \frac{1}{n}, \infty) \in S$ for all $n \in \mathbb{N}$ since it is of the form (a, ∞), so we have $[b, \infty) = \cap_{n=1}^{\infty}(b - \frac{1}{n}, \infty)$. Thus the complement $(-\infty, b) \in S$. Then $(a, b) = (-\infty, b) \cap (a, \infty)$. Finally, $(a, b] = (-\infty, b] \cap (a, \infty)$, $[a, b) = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b)$, $[a, b] = [a, \infty) \cap (-\infty, b]$.

Let $I = (a, \infty)$. Then for any set A we can split it into parts $A_1 = A \cap I^C$ and $A_2 = A \cap I$ such that $A = A_1 \cup A_2$. By subadditivity of outer measure we have,

$$m^*(A) = m^*(A_1 \cup A_2) \leq m^*(A_1) + m^*(A_2).$$

We want to show the opposite inclusion. By definition of $m^*(A)$ as an infimum, for

any given $\epsilon > 0$, there exists an open cover $\{I_k\}_{k=1}^{\infty}$ for A such that

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(A) + \epsilon.$$

Let $I_k' = I_k \cap I^C$ and $I_k'' = I_k \cap I$ such that $\ell(I_k) = \ell(I_k') + \ell(I_k'')$. Since the countable sets $\{I_k'\}_{k=1}^{\infty}$ and $\{I_k''\}_{k=1}^{\infty}$ form open covers for A_1 and A_2, then by the definition of outer measure we have

$$m^*(A_1) \leq \sum_{k=1}^{\infty} \ell(I_k') \text{ and } m^*(A_2) \leq \sum_{k=1}^{\infty} \ell(I_k'').$$

So then

$$
\begin{aligned}
m^*(A_1) + m^*(A_2) &\leq \sum_{k=1}^{\infty} \ell(I_k') + \sum_{k=1}^{\infty} \ell(I_k'') \\
&= \sum_{k=1}^{\infty} \ell(I_k') + \ell(I_k'') \\
&= \sum_{k=1}^{\infty} \ell(I_k) < m^*(A) + \epsilon.
\end{aligned}
$$

Since this holds for each $\epsilon > 0$, it also holds for $\epsilon = 0$. Thus we have

$$m^*(A) = m^*(A \cap I^C) + m^*(A \cap I).$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.3.   Non-measurable Sets

So far we have gained a basic understanding of measurable sets, we now discuss the existence of non-measurable sets. Our goal is to prove that any measurable set with positive measure contains a non-measurable subset.

DEFINITION 2.20 For a nonempty family X of nonempty sets and for every set $A \in X$, a **choice function** is a function $f : X \to \bigcup_{A \in X} A$ that has the property that

12

$f(A) \in A$. [i.e. a choice function is a function f on X defined as above such that for all $A \in X$ we have $f(A) \in A$].

For example, if we had a set $X = \{\{1, 2\}, \{3, 4\}\{5, 6\}\}$ then we can define a choice function f such that $f(\{3, 4\}) \in \{3, 4\}$. If we want the choice function f to return the max of the set, then $f(\{3, 4\}) = 4$ and so on.

THEOREM 2.21 (Axiom of Choice) For a nonempty family X of nonempty sets there exists a choice function.

In many cases the axiom is not necessary, such as in the example above where a choice function can be clearly defined. It is essential, however, in cases where there is not a clear distinction between elements in a set that would yield a choice function. The axiom is then necessary in order to make an arbitrary choice in such sets. Another important definition is that of cosets of the rationals.

DEFINITION 2.22 Let E be a set real numbers and $x, y \in \mathbb{R}$. Then x, y are called **rationally equivalent** if the difference between them is rational. That is, $x \sim y$ iff $x - y \in \mathbb{Q}$. This set forms a rational equivalence relation on E in the sense that it is reflexive, symmetric, and transitive.

DEFINITION 2.23 A coset of $\mathbb{Q}$ in $\mathbb{R}$ is any set of the form

$$x + Q = \{x + q \mid q \in \mathbb{Q}\}$$

for a chosen $x \in \mathbb{R}$.

From the two definitions we see that if two numbers are rationally equivalent, then they are in the same coset of $\mathbb{Q}$ in $\mathbb{R}$. We will now prove a lemma that is needed to show there exists non-measurable sets.

LEMMA 2.24 Let $E \subset \mathbb{R}$ be a measurable set with positive outer measure. Then the set of differences $\{d \mid d = x - y \text{ where } x, y \in E\}$ contains an interval centered at the origin.

Proof. Given $\epsilon > 0$, it follows from the definition of outer measure that there exists an open set G such that $E \subset G$ and

$$m(G) < (1 + \epsilon) \cdot m(E).$$

By Theorem 2.6, we know every nonempty open set is the disjoint union of a countable collection of open intervals, so for some intervals I_k we can write $G = \bigcup I_k$. Let $E_k = E \cap I_k$. Then,

$$E = \bigcup (E \cap I_k) = \bigcup E_k.$$

It follows that each E_k is measurable. Then by the additive property of measurable sets, $m(G) = \sum m(I_k)$ and $m(E) = \sum m(E_k)$. Since $m(G) < (1 + \epsilon) \cdot m(E)$, we have $m(I_j) < (1 + \epsilon) \cdot m(E_j)$ for some j. Let $\epsilon = \frac{1}{3}$. Then we have $I_j \subset E_j$ and simplifying gives $m(E_j) > \frac{3}{4} \cdot m(I_j)$. Now, if we translate E_j by some d such that $m(d) < \frac{1}{2} m(I_j)$, then the translated set has points in common with E_j. If not, then $E_j \cup (E_j + d)$ is contained in an interval of length $m(I_j) + m(d)$. Since measure is translation invariant we have $m(E_j) + m(E_j + d) \leq m(I_j) + m(d)$. but this is false if $m(d) < \frac{1}{2} m(I_j)$ since $m(E_j) > \frac{3}{4} \cdot m(I_j)$. Thus we have proved our claim. $\square$

THEOREM 2.25 There exists non-measurable sets

Proof. For some $x \in \mathbb{R}$, let $E_x = \{x + q \mid q \in \mathbb{Q}\}$ the cosets of $\mathbb{Q}$ with $\mathbb{R}$. By the Axiom of Choice, define the C_E as the choice set of exactly one element from each equivalence class of E_x. Note, the number of distinct equivalence classes is uncountable and the classes themselves are countable since $\mathbb{Q}$ is countable, so the union of all classes is uncountable. In other words, $\bigcup_{x \in \mathbb{R}} E_x = \mathbb{R}$. Since any element in C_E must differ by an irrational number, the set

$$\{d \mid d = x - y \text{ where } x, y \in C_E\}$$

cannot contain an interval since $\mathbb{Q}$ is dense in $\mathbb{R}$. It follows from the lemma then that C_E is not measurable or $m(C_E) = 0$. Since $\bigcup_{x \in \mathbb{R}} E_x = \mathbb{R}$, $m(C_E)$ is not zero. Thus C_E is non-measurable. $\qquad\square$

COROLLARY 2.26 Any set E of real numbers with positive outer measure contains a non-measurable subset.

Proof. Let E be a set of real numbers with positive outer measure. Let C be the non-measurable set in the previous theorem. For $r \in \mathbb{R}$, let C_r denote the translate of C by r. Then each C_r is disjoint and the union is the real line. That is $E = \cup(E \cap C_r)$. From this we have $m^*(C) \leq \sum m^*(C \cap E_R)$. If $m^*(C) > 0$, then there exists some r such that $C \cap E_r$ is not measurable. $\qquad\square$

2.4. Lebesgue Measurable Functions

In this subsection, we introduce the definitions of measurable functions and prove some basic properties of measurable functions.

DEFINITION 2.27 Let f be a function defined on a measurable domain E taking values in the extended real line. We say f is a Lebesgue measurable function (or simply a measurable function) if for every real number c the set

$$\{x \in E \mid f(x) > c\}$$

is measurable.

Essentially, for a function to be measurable it must have a domain E that is measurable and for all $c \in \mathbb{R}$ the set $\{x \in E \mid f(x) > c\}$ is measurable.

THEOREM 2.28 Let f be a real-valued function defined on a measurable set E. Then f is measurable if and only if any of the following equivalent statements holds. For each $c \in \mathbb{R}$,

1. The set $\{x \in E \mid f(x) > c\}$ is measurable.

2. The set $\{x \in E \mid f(x) \geq c\}$ is measurable.

3. The set $\{x \in E \mid f(x) < c\}$ is measurable.

4. The set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Proof. From the definition of measurable sets, we know that the complement of a measurable set is measurable. Note that the set in Statements 1 and 4 are complements and so are those in Statements 2 and 3. It suffices to show that Statement 1 implies Statement 2 and vice versa. We also know that the intersection of a countable collection of measurable sets and the union of a countable collection of measurable sets are measurable. Thus we have for $x \in E$,

$$\{x \mid f(x) \geq c\} = \bigcap_{n=1}^{\infty} \left\{ x \mid f(x) > c - \frac{1}{n} \right\},$$

which shows Statement 1 implies Statement 2. Moreover, we write

$$\{x \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \left\{ x \mid f(x) \geq c + \frac{1}{n} \right\},$$

which shows Statement 2 implies Statement 1. Thus, all four statements are equivalent. $\square$

Note that Statement 1 in Theorem 2.28 is our definition for a function being measurable. Thus, the result of the Theorem 2.28 allows us to interchangeably use any statement to show a function is measurable.

COROLLARY 2.29 Let f be a real valued function defined on a measurable set E. If f is a measurable function, then for any $c \in \mathbb{R}$ the sets

$$\{x \in E \mid f(x) = c\} \text{ and } \{x \in E \mid f(x) = \infty\}$$

16

are measurable.

Proof. We know that the countable intersection of measurable sets is measurable. Thus,

$$\{x \in E \mid f(x) \leq c\} \cap \{x \in E \mid f(x) \geq c\} = \{x \in E \mid f(x) = c\}.$$

We also have

$$\{x \in E \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > c\}.$$

This completes the proof. $\qquad\square$

PROPOSITION 2.30 Let f be a real valued function defined on a measurable set E. If f is measurable, then $|f|$ is measurable.

Proof. Let $x \in E$. For any $c \in \mathbb{R}$ we have the following,

$$\{x \mid |f(x)| < c\} = \{x \mid f(x) < c\} \cup \{x \mid f(x) > -c\}.$$

Thus $|f|$ is measurable. $\qquad\square$

DEFINITION 2.31 Let f be a real valued function defined on a set E. If $S \subset \mathbb{R}$, then the inverse image of S under f is defined by

$$f^{-1}(S) = \{x \in E \mid f(x) \in S\}.$$

THEOREM 2.32 A function f defined on a measurable domain E is measurable if and only if for every open set $O \in \mathbb{R}$, the inverse image $f^{-1}(O)$ is measurable.

Proof. The sufficiency is obvious. To prove the necessity, suppose that f is measurable, and O is an open set. We want to show $f^{-1}(O)$ is measurable. By Theorem 2.6, the set O can be written as $O = \bigcup_{k=1}^{\infty}(a_k, b_k)$ with $a_k, b_k \in \mathbb{R}$ and $a_k < b_k$. However, we

know $f^{-1}((a_k, b_k)) = \{x \in E \mid f(x) \in (a_k, b_k)\}$ is measurable by Theorem 2.28. Since

$$f^{-1}(O) = \bigcup_{k=1}^{\infty} \{x \in E \mid f(x) \in (a_k, b_k)\},$$

it again follows from Theorem 2.6 that $f^{-1}(O)$ is measurable. $\qquad\square$

DEFINITION 2.33 A property is said to hold **almost everywhere** (a.e.) on a measurable set E provided it holds on $E \backslash E_0$ where $E_0 \subset E$ for which $m(E_0) = 0$.

THEOREM 2.34 If f is measurable and if $g = f$ a.e., then g is measurable, and for any $a \in \mathbb{R}$,

$$m(\{x \in E \mid g > a\}) = m(\{x \in E \mid f > a\}).$$

Proof. Let $Z = \{x \in E \mid f \neq g\}$, then

$$\{x \in E \mid g > a\} \cup Z = \{x \in E \mid f > a\} \cup Z.$$

Since f is measurable, $\{x \in E \mid g > a\} \cup Z$ is measurable. Since this set differs from $\{x \in E \mid g > a\}$ by a set of measure zero, g is also measurable. Because Z is a set of measure zero, we have

$$m(\{x \in E \mid g > a\}) = m(\{x \in E \mid g > a\} \cup Z)$$
$$= m(\{x \in E \mid g > a\}).$$

This completes the proof. $\qquad\square$

It is important to note that if f is measurable on E, then it is measurable on any measurable set $E_1 \subset E$. This is true since

$$\{x \in E_1 \mid f > a\} = \{x \in E \mid f > a\} \cap E_1.$$

THEOREM 2.35 If f and g are measurable functions on E, then so is $f + g$.

Proof. Since g is measurable, for any fixed $a \in \mathbb{R}$ we have $a - g$ is measurable. Let $\{r_k\}$ be the set of rational numbers. Then,

$$\{x \in E \mid f > g\} = \cup_k \{x \in E \mid f > r_k > g\}$$
$$= \cup_k \{x \in E \mid f > r_k\} \cap \{x \in E \mid g < r_k\}.$$

So $\{x \in E \mid f > g\}$ is measurable. Then we have

$$\{x \in E \mid f + g > a\} = \{x \in E \mid f > a - g\}.$$

Thus $f + g$ is measurable. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

COROLLARY 2.36 If f and g are measurable functions on E and λ_1 and λ_2 are real numbers, then $\lambda_1 f + \lambda_2 g$ is measurable. This is the so called linearity of measurable functions.

THEOREM 2.37 If f and g are measurable functions on E, then so is $f \cdot g$.

Proof. Observe that
$$f \cdot g = \frac{1}{2}[(f + g)^2 - f^2 - g^2].$$

Since linear combinations of measurable functions are measurable, it suffices to show that the square of a measurable function is measurable. Let $c \in \mathbb{R}$. We will consider two cases for c. For $c \geq 0$,

$$\{x \in E \mid f^2 > c\} = \{x \in E \mid f > \sqrt{c}\} \cup \{x \in E \mid f < -\sqrt{c}\}.$$

Since a finite union of measurable sets is measurable, we have proven our first case.

Now if $c < 0$, then our inequality will always be true for all $x \in E$. So,

$$\{x \in E \mid f^2 > c\} = E.$$

Thus f^2 is measurable and the desired result follows. □

THEOREM 2.38 Let ϕ be a continuous function on $\mathbb{R}$ and let f be finite a.e. in E. If f is measurable, then so is $\phi(f)$.

Proof. Let f be finite almost everywhere on E. Since ϕ is continuous, for any open set G we have $\phi^{-1}(G)$ is open (a proof of this can be found in [1, pp. 25-26]). With Theorem 2.32, it suffices to show that for every open set $G \in \mathbb{R}$,

$$\{x \in E \mid \phi(f(x)) \in G\} \tag{2.2}$$

is measurable. So $\{x \in E \mid \phi(f(x)) \in G\} = f^{-1}(\phi^{-1}(G))$. Since $\phi^{-1}(G)$ is open and f is measurable we can then apply Theorem 2.32 to conclude that Equation 2.2 is measurable. □

We have now shown that the composition of a continuous function with a measurable function is measurable. It is interesting to note that if we were to remove the continuous requirement from ϕ, the composition may not be measurable. In fact, there exists two measurable functions f and g such that $f \circ g$ is not measurable. We refer interested readers to [1, pp. 59].

In defining the Lebesgue integral later on we will need to first talk about the idea of max and min as well as the limit process. For a finite set of functions $\{f_k\}_{k=1}^n$ with a domain E, we can define the max as

$$\max\{f_1, ..., f_n\} = \max\{f_1(x), ..., f_n(x)\} \text{ for each given } x \in E.$$

The min function would be defined similarly. In the following, we show that the max function is measurable.

PROPOSITION 2.39 For a finite family of measurable functions $\{f_k\}_{k=1}^n$ with a common domain E, the $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable.

Proof. For any $c \in \mathbb{R}$, we have

$$\{x \in E \mid \max\{f_1, ..., f_n\} > c\} = \bigcup_{k=1}^n \{x \in E \mid f_k(x) > c\}.$$

By Proposition 2.16, the finite union of measurable sets is measurable. The proof of the other part is similar. $\qquad\square$

Some other important functions that will be used later are defined as follows,

$$f^+(x) = \max\{f(x), 0\},$$
$$f^-(x) = \max\{-f(x), 0\}.$$

We can see that if f is measurable on E, then so are f^+ and f^-. These two functions play an important role in Lebesgue integration. The function f can be expressed as the difference of the two functions, that is $f = f^+ - f^-$. The next step in our study of measurable functions is in convergence, which plays an important role in analysis. There are multiple ways to consider whether a sequence of functions converges. In developing our theory towards Lebesgue integration, we follow many of the same steps taken in an introductory analysis course.

DEFINITION 2.40 Let $\{f_n\}$ be a sequence of functions with a common domain E. For a function f on E and $A \subset E$, we have the following,

1. $\{f_n\}$ converges to f pointwise on A provided

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in E.$$

21

2. $\{f_n\}$ converges to f pointwise a.e. on A provided it converges to f pointwise on $A\backslash B$, where $m(B) = 0$.

3. $\{f_n\}$ converges to f uniformly on A provided for each $\epsilon > 0$, there exists an N such that
$$|f(x) - f_n(x)| < \epsilon \text{ on } A \text{ for all } n \geq N, \text{ and all } x \in A.$$

For more information on pointwise convergence, see [5, pp. 204-211].

LEMMA 2.41 For a measurable subset E_0 of E, f is measurable on E if and only if the restrictions of f to E_0 and $E\backslash E_0$ are measurable.

Proof. For any $c \in \mathbb{R}$,

$$\{x \in E \mid f(x) > c\} = \{x \in E_0 \mid f(x) > c\} \cup \{x \in E\backslash E_0 \mid f(x) > c\}.$$

Since measurable sets are closed under countable union, the above equation shows both directions. This completes the proof. $\square$

PROPOSITION 2.42 Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f. Then f is measurable.

Proof. Let $E_0 \subset E$ where $m(E_0) = 0$ and $\{f_n\}$ converges to f pointwise on $E\backslash E_0$. Since $m(E_0) = 0$, it follows from Lemma 2.41 that we can replace E by $E\backslash E_0$. Without loss of generality, we assume $f_n(x)$ converges to $f(x)$ for all $x \in E$. Let $c \in \mathbb{R}$. We want to show that $\{x \in E \mid f(x) < c\}$ is measurable. For $x \in E$, $f(x) < c$ if and only if there exists natural numbers n and k such that $f_j(x) < c - 1/n$ for all $j \geq k$. However since each f_j is measurable, then so is $\{x \in E \mid f_j(x) < c - 1/n\}$. Since a countable intersection of measurable sets is measurable, for any k, the set

$$\bigcap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\}$$

22

is also measurable. Furthermore, since a countable union of measurable sets is measurable we have

$$\{x \in E \mid f(x) < c\} = \bigcup_{1 \leq k,n < \infty} \left[\bigcap_{j=k}^{\infty} \{x \in E \mid f_j(x) < c - 1/n\} \right]$$

is measurable. Thus, f is measurable. $\square$

We can see, in the context of pointwise limits of sequences, measurable functions are more robust than continuous functions; in the case of continuous functions, it is not guaranteed that the pointwise limit of the sequence is continuous. For example, consider the piecewise function

$$f_n(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 < x < 1/n, \\ 1 & \text{if } x \geq 1/n. \end{cases}$$

The function f_n is continuous on $\mathbb{R}$ for each $n \in \mathbb{N}$. However, for each $x \in \mathbb{R}$, we have $\lim_{n \to \infty} f_n(x) = f(x)$, where

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases} \tag{2.3}$$

which is discontinuous at $x = 0$. This example shows that a sequence of continuous functions can converge pointwise to a discontinuous function.

DEFINITION 2.43 Let E be any set. The **characteristic function** of E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

It is interesting to note that the characteristic function has the property that

for a set E, χ_E is measurable if and only if E is measurable. This also implies that there exists non-measurable functions.

DEFINITION 2.44 A real valued function s defined on a measurable set X is called **simple** provided it is measurable and takes only a finite number of values.

It follows from the definition that finite linear combinations and finite products of simple functions are again simple functions. In notation we can say that if s is a simple function taking on distinct values $a_1, ..., a_n$ on disjoint measurable sets $E_1, ..., E_n$, then we can write s as a linear combination of characteristic functions as follows

$$s(x) = \sum_{k=1}^{n} a_k \cdot \chi_{E_k}(x).$$

Simple functions play an important role in the Lebesgue integration method, we will show that any measurable function can be represented as a pointwise limit of a sequence of simple functions. Before we show that any measurable function can be represented as the limit of a sequence of simple functions, we show that any measurable function can be approximated arbitrarily well by simple functions as shown in the following lemma.

LEMMA 2.45 Let f be a measurable real-valued function on domain E and suppose that f is bounded on E. Then for each $\epsilon > 0$, there exists simple functions ϕ_ϵ and ψ_ϵ defined on E such that

$$\phi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \phi_\epsilon < \epsilon.$$

Proof. Since f is bounded on E, there exists an $M \geq 0$ such that $|f| \leq M$. Let (c, d) be an open bounded interval containing $f(E)$. Let $y_0 < y_1 < ... < y_{n-1} < y_n$ be a partition of the closed bounded interval $[c, d]$ given by

$$c = y_0 < y_1 < ... < y_n = d$$

where $y_k - y_{k-1} < \epsilon$ for $1 \le k \le n$. In other words, we want the distance between each pair of adjacent y_{k-1}, y_k in our partition to be less than the given ϵ. We then define

$$I_k := [y_{k-1}, y_k) \text{ and } E_k := f^{-1}(I_k) \text{ for } 1 \le k \le n.$$

Since each I_k is an interval and f is measurable, from Proposition 2.32 and Lemma 2.41 we see that each E_k is measurable. We then define the simple functions ϕ_ϵ and ψ_ϵ on E as follows,

$$\phi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k} \text{ and } \psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k},$$

where χ_{E_k} is the characteristic function for E_k. Let $x \in E$. Since $f(E)$ is contained in $[c, d]$, there exists a unique $1 \le k \le n$ such that $y_{k-1} \le f(x) < y_k$ and therefore

$$\phi_\epsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\epsilon(x).$$

Since $y_k - y_{k-1} < \epsilon$, we have our approximation property. $\qquad\square$

THEOREM 2.46 (The Simple Approximation Theorem) An extended real-valued function f on a measurable set E is measurable if and only if there exists a sequence of simple functions $\{f_k\}$ on E that converges pointwise on E to f and has the property that

$$|f_k| \le |f| \text{ on } E \text{ for all } n.$$

If f is non-negative, we may choose $\{f_k\}$ to be an increasing sequence.

Proof. Suppose that there is a sequence $\{f_k\}$ of simple functions on E that converges pointwise on E to f. Since simple functions are measurable, from Proposition 2.42, we see that f is measurable.

Now suppose f is measurable. Lets first show that if $f \ge 0$ then we may choose $\{f_k\}$ to be an increasing sequence. If $f \ge 0$, then for each $k \in \mathbb{N}$ we subdivide the

values of f between $[0, k]$ by partitioning the interval into subintervals

$$\left[\frac{j-1}{2^k}, \frac{j}{2^k}\right] \text{ for } j = 1, ..., k2^k.$$

Now let

$$f_k(x) := \begin{cases} \frac{j-1}{2^k} & \text{if } f(x) \in [\frac{j-1}{2^k}, \frac{j}{2^k}) \text{ for } j = 1, ..., k2^k, \\ k & \text{if } f(x) \geq k. \end{cases}$$

Each f_k is a simple function defined in the domain of f with $f_k \leq f_{k+1}$. This is true since increasing from f_k to f_{k+1} each time divides the subinterval in half. If we have $f(x) < \infty$, then there exists a k such that $f(x) < k$. It follows that

$$0 \leq f(x) - f_k(X) \leq \frac{1}{2^k}.$$

If $f(x) = \infty$, then $f_k(x) = k$, which approaches infinity. We can apply our increasing function to f^+ and f^- denoting those sequences of simple functions as $\{f_k'\}$ and $\{f_k''\}$ where $f_k' \to f^+$ and $f_k'' \to f^-$. Then $f_k' - f_k''$ is simple with

$$f_k' - f_k'' \to f^+ - f^- = f.$$

Let $I_{k,j} = [\frac{j-1}{2^k}, \frac{j}{2_k})$. Thus for $f \geq 0$,

$$f_k = \sum_{j=1}^{k2^k} \frac{j-1}{2^k} \cdot \chi_{I_{k,j}} + k \cdot \chi_{(f \geq k)}.$$

This equation defines a measurable function since each f_k is measurable. Thus we have our sequence of simple functions converging pointwise on E to f. $\qquad\square$

Next we study two important theorems: Egorov's theorem and Lusin's theorem. The basic idea of Egorov's theorem is that if a sequence of measurable functions converges pointwise to a real-valued function on a domain E of finite measure, then for

26

any $\epsilon > 0$, there exists a measurable set $E_1 \subset E$ such that

1. $m(E \backslash E_1) < \epsilon$,

2. the sequence converges to its limit function uniformly on E_1.

Before we go into the formal statement and proof of Egoroff's theorem, it will be helpful to establish a lemma.

LEMMA 2.47 Suppose E has finite measure. Let $\{f_k\}$ be a sequence of measurable functions on E that converges pointwise on E to a real-valued function f. Then for any given $\epsilon, \delta > 0$, there exists a set $A \subset E$ with an integer N where

$$|f_n - f| < \epsilon \text{ on } A \text{ for all } n \geq N \text{ and } m(E \backslash A) < \delta.$$

Proof. Let $\epsilon, \delta > 0$ be given. Since f is a real-valued function and is measurable, $|f_k - f|$ is defined for each k. Then the set $\{x \in E \mid |f(x) - f_k(x)| < \epsilon\}$ is measurable. Since the intersection of a countable collection of measurable sets is measurable, the set

$$E_n = \bigcap_{k \geq n} \{x \in E \mid |f(x) - f_k(x)| < \epsilon\}$$

$$= \{x \in E \mid |f(x) - f_k(x)| < \epsilon \text{ for all } k \geq n\}$$

is also measurable. From this we see that $E_n \subset E_{n+1}$, so $\{E_n\}_{n=1}^{\infty}$ is an ascending collection of measurable sets with $E = \bigcup_{n=1}^{\infty} E_n$. Then by the continuity of measure, for which one can find a proof at [1, pp. 44-45], $m(E) = \lim_{n \to \infty} m(E_n)$. Since $m(E) < \infty$, we may choose a sufficiently large N such that $m(E_N) > m(E) - \delta$, which implies that $m(E) - m(E_N) < \delta$. Let $A := E_N$. Then we have

$$|f_n(x) - f(x)| < \epsilon, \ x \in A, \ n \geq N.$$

Then by the excision property, $m(E \backslash A) = m(E) - m(A) < \delta$. This proves our last

27

property and thus the lemma. □

THEOREM 2.48 (Egoroff's Theorem) Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to a real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in E for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \backslash F) < \epsilon.$$

Proof. Let $\epsilon > 0$. We will use Lemma 2.47 to create a closed set $F_m \subset E$ for $m \geq 1$ and an integer $K_{m,\epsilon}$ such that

$$|f_k - f| < \frac{1}{m} \text{ on } F_m \text{ if } k > K_{m,\epsilon} \text{ and } m(E \backslash F_m) < \frac{\epsilon}{2^m}.$$

The set $F := \bigcap_{m=1}^{\infty} F_m$ is closed. Since $F \subset F_m$ for all m, we can say that f_k converges uniformly to f on F. Finally, $E \backslash F = E \backslash \bigcap_{m=1}^{\infty} F_m = \bigcup_{m=1}^{\infty} (E \backslash F_m)$. Then using the countable subadditivity of measure and the geometric series, we have

$$m(E \backslash F) \leq \sum m(E \backslash F_m)$$
$$= \sum \frac{\epsilon}{2^m} < \epsilon.$$

This completes the proof. □

We know that continuous functions are measurable. Lusin's theorem seeks to give some clarification to the converse of this statement; its goal is to give a continuity property that we can apply to measurable functions. We will start by proving the special case for simple functions, and then use the special case to prove the general result, which is Lusin's Theorem.

PROPOSITION 2.49 Let f be a simple function defined on a measurable domain E. Then for each $\epsilon > 0$, there exists a continuous function g on $\mathbb{R}$ and a closed set

$F \subset E$ such that

$$f = g \text{ on } F \text{ and } m(E \backslash F) < \epsilon.$$

Proof. Suppose f is a simple function on E. Let $a_1, a_2, ..., a_n$ be a finite number of distinct values from f taken on measurable subsets of E denoted $E_1, E_2, ..., E_n$ respectively. Then $\{E_k\}_{k=1}^n$ is disjoint since each a_k is distinct. Let $\epsilon > 0$ be given. Now choose closed sets $F_1, F_2, ..., F_n$ such that for each $1 \leq k \leq n$, we have

$$F_k \subseteq E_k \text{ and } m(E_k \backslash F_k) < \frac{\epsilon}{n}. \tag{2.4}$$

Then the set $F := \bigcup_{k=1}^n F_k$ is closed, since it is a union of a finite collection of closed sets. Since $\{E_k\}_{k=1}^n$ is disjoint and by the finite additivity of measurable sets,

$$m(E \backslash F) = m \left(\bigcup_{k=1}^n [E_k \backslash F_k] \right) = \sum_{k=1}^n m(E_k \backslash F_k) < \epsilon.$$

Define a function g on F to take the values of a_k on F_k for $1 \leq k \leq n$. Since $\{F_k\}_{k=1}^n$ is disjoint, g is well defined. Also, g is continuous on F since for $x \in F_i$ there exists an open interval containing x that is disjoint from the closed set

$$\bigcup_{k \neq i} F_k.$$

Thus on the intersection of this open interval with F the function g is constant. To extend g to be a continuous function on $\mathbb{R}$, we first note that the set $\mathbb{R} \backslash F$ is open, and therefore can be written as a union of at most countably many disjoint open intervals whose end points are in F. We then use linear interpolation to complete this extension. $\square$

THEOREM 2.50 (Lusin's Theorem) Let f be a real-valued measurable function on E. Then for each $\epsilon > 0$, there exists a continuous function g on $\mathbb{R}$ and a closed set

$F \subset E$ such that

$$f = g \text{ on } F \text{ and } m(E \backslash F) < \epsilon.$$

Proof. Let $m(E) < \infty$. The case $m(E) = \infty$ can be solved using the extension argument we employed in the proof of Proposition 2.49. By Theorem 2.46, there exists a sequence $\{f_n\}$ of simple functions defined on E that converge pointwise to f on E. Let n be a natural number. We want to use Proposition 2.49, so let f be replaced by f_n and ϵ replaced by $\frac{\epsilon}{2^{n+1}}$. Then choose a function g_n on $\mathbb{R}$ and a closed set $F_n \subset E$ such that

$$f_n = g_n \text{ on } F_n \text{ and } m(E \backslash F_n) < \frac{\epsilon}{2^n}.$$

By Egoroff's Theorem (Theorem 2.48) there exists a closed set $F_0 \subset E$ such that $\{f_n\}$ converges to f uniformly on F_0 and $m(E \backslash F_n) < \frac{\epsilon}{2^n}$. Define $F := \bigcap_{n=0}^{\infty} F_n$. By De Morgan's law and the countable subadditivity of measure,

$$m(E \backslash F) = m\left([E_0 \backslash F] \cup \bigcup_{n=0}^{\infty} [E \backslash F_n] \right) \leq \frac{\epsilon}{2} + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \epsilon.$$

The set F is closed since it is the intersection of closed sets. Each f_n is continuous on F since $F \subseteq F_n$ and $f_n = g_n$ on F_n. Finally, $\{f_n\}$ converges to f uniformly on F since $F \subseteq F_0$. Furthermore, the uniform limit of a sequence of continuous functions is continuous, so the restriction of f to F is continuous on F. So there is a continuous function g defined on $\mathbb{R}$ whose restriction to F is f. Thus g is the desired function. $\square$

2.5. Lebesgue Integral

We begin our study of the Lebesgue integral by reviewing some basic terms from the Riemann integral. This will also help us when we make comparison of the two integrals later on.

Let f be a bounded real-valued function defined on a closed bounded interval

$[a, b]$. We then define a **partition** of $[a, b]$ as $P = \{x_0, x_1, ..., x_n\}$ with

$$a = x_0 < x_1 < ... < x_n = b.$$

Define $\Delta x_i = x_i - x_{i-1}$. For $1 \leq i \leq n$, let $M_i = \sup\{f(x) \mid x_i \leq x \leq x_{i-1}\}$ and $m_i = \inf\{f(x) \mid x_i \leq x \leq x_{i-1}\}$. We define the upper and lower Darboux sums, see [3, pp. 120-121], for f with respect to P as the following:

$$U(f, P) = \sum_{1=1}^{n} M_i \cdot \Delta x_i,$$
$$L(f, P) = \sum_{1=1}^{n} m_i \cdot \Delta x_i.$$

At this point we define the upper and lower Riemann integrals of f over $[a, b]$ in terms of the upper and lower Darboux sums respectively by

$$(R)\overline{\int_a^b} f = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\},$$
$$(R)\underline{\int_a^b} f = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

We use the (R) to distinguish it from the Lebesgue integral to be defined. Since f is bounded and $[a, b]$ has finite length, the upper and lower integrals are also finite.

DEFINITION 2.51 If f is a bounded real-valued function over the interval $[a, b]$ and the upper and lower Riemann integrals are equal, then we say f is **Riemann integrable** over $[a, b]$. The common value of the Riemann integral of f over $[a, b]$ is denoted by

$$(R) \int_a^b f.$$

DEFINITION 2.52 A real-valued function ψ defined on $[a, b]$ is called a **step function** if there exists a partition $P = \{x_0, x_1, ..., x_n\}$ and numbers $c_1, c_2, ..., c_n$ such that

for $1 \leq i \leq n$,

$$\psi(x) = c_i \text{ if } x \in (x_{i-1}, x_i).$$

We see that a property of a step function ψ is that

$$L(\psi, P) = U(\psi, P).$$

This means that the step function ψ thus defined is Riemann integrable, which serves as a starting point for our study of the Lebesgue integral. We will denote the Lebesgue integral using the integral sign without the (R). The Lebesgue integral is usually defined by stages, see [4, pp. 49-50]. Though the purpose of this work is not to cover the Lebesgue integral in depth, we give an overview of its construction for simple functions. Let ψ be a simple function on E. Then ψ takes on the distinct values $a_1, ..., a_n$ on E. Since ψ is measurable, so is $\psi^{-1}(a_i) = \{x \in E \mid \psi(x) = a_i\}$. Let $E_i = \psi^{-1}(a_i)$. We can represent ψ as the following:

$$\psi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E.$$

This is known as the **canonical form**, which is a unique decomposition of ψ.

DEFINITION 2.53 If ψ is a simple function defined on a measurable set E of finite measure with $\psi(x) = \sum_{i=1}^{n} a_i \cdot \chi_{E_i}$ being its canonical form, we define the **integral** of ψ by

$$\int_E \psi(x) = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

LEMMA 2.54 Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subsets of a set of finite measure E. For $1 \leq i \leq n$, let a_i be a real number.

$$\text{If } \phi = \sum_{i=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ then } \int_E \phi = \sum_{i=1}^{n} a_i \cdot m(E_i).$$

Lemma 2.54 is needed to show the linearity and monotonicity of integration given in the following proposition.

PROPOSITION 2.55 Let ϕ and ψ be simple functions defined on a measurable set E of finite measure. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\int_E (\alpha\phi + \beta\psi) = \alpha \int_E \phi + \beta \int_E \psi.$$

Moreover, if $\phi \geq \psi$ on E, then $\int_E \phi \leq \int_E \psi$.

DEFINITION 2.56 Let f be a bounded measurable function defined on a measurable set E of finite measure. We define the **lower and upper Lebesgue integral** respectively by

$$\mathcal{L}_\ell(f) := \sup\left\{ \int_E \phi \;\middle|\; \phi \text{ is simple and } \phi \leq f \text{ on } E \right\},$$

$$\mathcal{L}_u(f) := \inf\left\{ \int_E \psi \;\middle|\; \psi \text{ is simple and } \psi \geq f \text{ on } E \right\}.$$

From this definition we see that the lower and upper integrals are finite with the property that $\mathcal{L}_\ell(f) \leq \mathcal{L}_u(f)$.

DEFINITION 2.57 A bounded function f on a domain E of finite measure is said to be **Lebesgue integrable** over E provided $\mathcal{L}_\ell(f) = \mathcal{L}_u(f)$. The common value of $\mathcal{L}_\ell(f)$ and $\mathcal{L}_u(f)$ is called the **Lebesgue integral** and is denoted $\int_E f$.

For an example of the usefulness of the Lebesgue integral over the Riemann integral we can examine the **Dirichlet function**. Define D on $[0, 1]$ by

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is not Riemann integrable because it is nowhere continuous. However, the set E of rational numbers on $[0, 1]$ is a measurable set of measure zero. Thus, the Dirichlet function D is the function χ_E. Thus, D is Lebesgue integrable over $[0, 1]$ with

$$\int_{[0,1]} D = \int_{[0,1]} 1 \cdot \chi_E = 1 \cdot m(E) = 0.$$

The next result we want to prove is related to bounded measurable functions.

THEOREM 2.58 Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.

Proof. Let $n \in \mathbb{N}$. By Lemma 2.45 with $\epsilon = \frac{1}{n}$, there exist two simple functions ϕ_n and ψ_n defined on E for which

$$\phi_n \leq f \leq \psi_n \text{ on } E, \text{ and } 0 \leq \psi_n - \phi_n \leq \frac{1}{n} \text{ on } E.$$

By Proposition 2.55,

$$\int_E \psi_n - \int_E \phi_n = \int_E [\psi_n - \phi_n] \leq \frac{1}{n} \cdot m(E).$$

With $\int_E \psi_n - \int_E \phi_n \geq 0$. But we have,

$$
\begin{aligned}
0 &\leq \inf \left\{ \int_E \psi \,\middle|\, \psi \text{ is simple, } \psi \geq f \right\} - \sup \left\{ \int_E \phi \,\middle|\, \phi \text{ is simple, } \phi \leq f \right\} \\
&\leq \int_E \psi_n - \int_e \phi_n \\
&\leq \frac{1}{n} \cdot m(E).
\end{aligned}
$$

The inequality holds for every $n \in \mathbb{N}$ and $m(E)$ finite. Thus the upper and lower Lebesgue integrals are equal and thus the function f is integrable on E, proving our theorem. $\qquad \square$

3.   OTHER MEASURES

The process we will use for constructing the following measures will be based on an outer measure, the development of these measures follows the same general steps as Lebesgue measure. A measure constructed this way is known as a **Carathéodory measure**. In the case of Lebesgue measure, we first established a primitive set function for the length of an interval $\ell(I)$. We then extended the length function to a set function for the outer measure, m^*, that was defined for every subset of real numbers. Once m^* was established, we restricted the outer measure to a collection of Lebesgue measurable sets $\mathcal{L}$, which were a σ-algebra. Then m^* restricted to measurable sets was called a Lebesgue measure and denoted by m.

3.1.   Outer Measure

We begin by giving some some properties that must be fulfilled in order for a set function to be an outer measure.

DEFINITION 3.1 A function $\Gamma = \Gamma(A)$ which is defined for every subset A of our space P is an **outer measure** if it satisfies the following:

1. $\Gamma(A) \geq 0$, $\Gamma(\emptyset) = 0$.

2. $\Gamma(A_1) \leq \Gamma(A_2)$ if $A_1 \subseteq A_2$.

3. $\Gamma(\bigcup A_k) \leq \sum \Gamma(A_k)$ for any countable collection of sets $\{A_k\}$.

We can see from this definition that Lebesgue outer measure is an outer measure. This definition gives flexibility when it comes to defining a class of measurable sets and the corresponding measure. We have at this point only talked about the space of $\mathbb{R}$, and the focus of our discussion will continue to be on $\mathbb{R}$. For measurable sets, we will use a similar definition to Carathéodory's in Definition 2.12. We can restate Carathéodory's definition here in terms of Γ.

DEFINITION 3.2 Given an outer measure Γ, $E \in P$ is $\Gamma - measurable$, or simply measurable, if for every $A \subset P$,

$$\Gamma(A) = \Gamma(A \cap E) + \Gamma(A \cap E^C).$$

THEOREM 3.3 Let Γ be an outer measure on the subsets of our space P.

1. The family of Γ-measurable subsets of P forms a σ-algebra.

2. If $\{E_k\}$ is a countable collection of disjoint measurable sets, then

$\Gamma(\bigcup E_k) = \sum \Gamma(E_k)$ and $\Gamma(A) = \sum \Gamma(A \cap E_k) + \Gamma(A \backslash \bigcup E_k).$

Interested readers can find a proof of Theorem 3.3 at [2, pp. 194-195].

DEFINITION 3.4 A **Borel σ-algebra** $\mathcal{B}$ is the smallest σ-algebra generated by the open sets in $\mathbb{R}^n$. The sets of $\mathcal{B}$ are called **Borel sets** and a measure is called a **Borel measure** if every Borel set is measurable.

We will also want to define the distance between sets. For two sets A_1 and A_2, the distance between the two is defined by

$$d(A_1, A_2) = \inf\{d(x, y) \mid x \in A_1, y \in A_2\}.$$

DEFINITION 3.5 For an outer measure Γ on some space P, we say that Γ is a **Carathéodory outer measure** if whenever we have two sets A_1 and A_2 such that $d(A_1, A_2) > 0$, we also have

$$\Gamma(A_1 \cup A_2) = \Gamma(A_1) + \Gamma(A_2). \tag{3.5}$$

A measure is also a Borel measure provided the the above property holds. Equation 3.5 is the familiar additive property from Lebesgue measure.

3.2. Lebesgue-Stieltjes Measure

We now consider a case of Carathéodory outer measure known as Lebesgue-Stieltjes measure. One of the benefits of this measure is that it clears the connection between measures and monotone functions. We will again restrict ourselves to $\mathbb{R}$ when discussing this measure.

To construct this measure, consider a reasonable function f which is finite and monotonically increasing on $\mathbb{R}$. For each half open interval of the form (a,b], we define a primitive set function λ as

$$\lambda(a, b] := \lambda_f((a, b]) = f(b) - f(a).$$

Note that $\lambda \geq 0$ since f is increasing. Already we see a distinction from Lebesgue measure; for Lebesgue our primitive set function was length $\ell(I)$ for an interval I, which again returned ∞ if I is unbounded and if $I = (a, b)$, it returned $b - a$. The function ℓ can be thought of as the set function for the identity function, namely $f(x) = x$. We now extend λ to an outer measure. If A is a nonempty subset of $\mathbb{R}$ and f is defined as above, let

$$\Lambda^*(A) := \Lambda_f^*(A) = \inf \sum \lambda(a_k, b_k],$$

where the inf is taken over all countable collections $\{(a_k, b_k]\}$ such that we have $A \subset \bigcup (a_k, b_k]$, with $\Lambda^*(\emptyset) = 0$.

THEOREM 3.6 Λ^* is a Carathéodory outer measure on $\mathbb{R}$.

Proof. We want to show that Λ^* fulfills the three properties given in Definition 3.1 as well as Definition 3.5. From the definition of Λ^* and since f is increasing we have $\Lambda^*(\emptyset) = 0$ and $\Lambda^* \geq 0$ respectively. We want to show that if $A_1 \subset A_2$, then

$$\Lambda^*(A_1) \leq \Lambda^*(A_2).$$

It is clear if either $A_1 = \emptyset$ or $\Lambda^*(A_2) = \infty$. Otherwise, choose $\{(a_k, b_k]\}$ a cover for A_2, that is $A_2 \subset \bigcup(a_k, b_k]$ and $\sum \lambda(a_k, b_k] < \Lambda^*(A_2) + \epsilon$. Then $A_1 \subset \bigcup(a_k, b_k]$ since $A_1 \subset A_2$. Thus $\Lambda^*(A_1) \leq \sum \lambda^*(a_k, b_k]$ and

$$\Lambda^*(A_1) < \Lambda^*(A_2) + \epsilon.$$

The result follows from letting $\epsilon \to 0$.

We want to show Λ^* is subadditive. Let $\{A_j\}_{j=1}^{\infty}$ be a collection of nonempty subsets of $\mathbb{R}$ and let $A := \bigcup A_j$. Assume that $\Lambda^*(A_j) < +\infty$ for each j. Choose $\{(a_k^j, b_k^j]\}$ as a cover for each A_j. That is,

$$A_j \subset \bigcup_k (a_k^j, b_k^j], \text{ with } \sum_k \lambda(a_k^j, b_k^j] < \Lambda^*(A_j) + \frac{\epsilon}{2^j}.$$

Since $A = \bigcup A_j$, we have $A \subset \bigcup_{j,k}(a_k^j, b_k^j]$ and

$$\Lambda^*(A) \leq \sum_{j,k} \lambda(a_k^j, b_k^j] < \sum_j \left[\Lambda^*(A_j) + \frac{\epsilon}{2^j}\right] = \sum_j \Lambda^*(A_j) + \epsilon.$$

It follows then that $\Lambda^*(A) \leq \sum \Lambda^*(A_j)$, and therefore Λ^* is an outer measure. Lastly, we want to show that Λ^* is a Carathéodory outer measure. To begin, if $a = a_0 < a_1 < \ldots < a_N = b$, then

$$\lambda(a, b] = f(b) - f(a) = \sum_{k=1}^{N}[f(a_k) - f(a_{k-1})] = \sum_{k=1}^{N} \lambda(a_{k-1}, a_k].$$

From the definition of Λ^*, we can always choose an arbitrarily small $(a_k, b_k]$. Thus, if A_1 and A_2 satisfy $d(A_1, A_2) > 0$, then given $\epsilon > 0$, choose $\{(a_k, b_k]\}$ such that each $(a_k, b_k]$ has length less than $d(A_1, A_2)$ with

$$A_1 \cup A_2 \subset \bigcup(a_k, b_k] \text{ and } \sum \lambda(a_k, b_k] \leq \Lambda^*(A_1 \cup A_2) + \epsilon.$$

Therefore we can split $\{(a_k, b_k]\}$ into coverings for A_1 and A_2 respectively. Thus, we have $\Lambda^*(A_1) + \Lambda^*(A_2) \leq \sum \lambda(a_k, b_k]$. Since ϵ is arbitrary, we have

$$\Lambda^*(A_1) + \Lambda^*(A_2) \leq \Lambda^*(A_1 \cup A_2).$$

From the definition of Λ^*, we always have

$$\Lambda^*(A_1) + \Lambda^*(A_2) \geq \Lambda^*(A_1 \cup A_2).$$

Thus we have equality and Λ^* is a Carathéodory outer measure. $\qquad\square$

We call Λ_f^* the **Lebesgue-Stieltjes outer measure** corresponding to f. Similar to our construction Lebesgue measure, Λ_f^* restricted to the sets which are Λ_f^*-measurable is called the **Lebesgue-Stieltjes measure**, which is denoted by Λ_f or simply Λ. An interesting property of Lebesgue-Stieltjes measure is that Λ_x^* which corresponds to the identity function $f(x) = x$ is the same as ordinary Lebesgue outer measure, namely

$$\lambda_f((a, b]) = f(b) - f(a) = b - a = \ell((a, b]).$$

DEFINITION 3.7 An outer measure Γ defined on $\mathbb{R}^n$ is said to be **regular** if for every $E \subset \mathbb{R}^n$, there exists a Γ-measurable set B such that $E \subset B$ and that $\Gamma(A) = \Gamma(E)$.

THEOREM 3.8 Let Λ^* be a Lebesgue-Stieltjes outer measure. If A is a subset of $\mathbb{R}$, there is a Borel set B with $A \subset B$ such that $\Lambda^*(A) = \Lambda(B)$.

The proof of Theorem 3.8 as well as further information on the subject of Lebesgue-Stieltjes measure can be found at [2, pp. 199-200]. The importance of this theorem comes in the fact that Lebesgue-Stieltjes measure is regular, which is a nice characteristic when it comes to constructing an integral based on Λ.

3.3. Hausdorff Measure

Hausdorff measure has many uses, in particular as a mechanism to distinguish sets of Lebesgue measure zero. The most common example is the Cantor set between $[0, 1]$ has Hausdorff dimension of $\frac{log(2)}{log(3)}$ and a measure of 1. For readers not familiar with the Cantor set, a construction of the Cantor set as well as some properties of the Cantor-Lebesgue function can be found at [1, pp. 51-53]. We now set about constructing Hausdorff measure.

DEFINITION 3.9 The **diameter** of a set E on $\mathbb{R}^n$ is defined as

$$\text{diam}(E) := \sup\{|x - y| \mid x, y \in E\}.$$

To define Hausdorff outer measure, fix $\alpha > 0$, and let A be any subset of $\mathbb{R}^n$. Given $\epsilon > 0$, let

$$H_\alpha^{(\epsilon)}(A) = \inf \left\{ \sum_{k=1}^{\infty} \text{diam}(A_k)^\alpha \ \middle| \ A \subset \bigcup_{k=1}^{\infty} A_k \text{ and } \text{diam}(A_k) < \epsilon \right\}. \tag{3.6}$$

We may assume that each A_k in a given covering are disjoint and that $A = \bigcup A_k$. If $\epsilon' < \epsilon$, each covering of A by sets with diameters less than ϵ' is also such a cover for ϵ. Hence, as ϵ decreases, the collection of coverings decreases, and $H_\alpha^{(\epsilon)}(A)$ increases. That is, $f : \epsilon \to H_\alpha^{(\epsilon)}(A)$ is non-decreasing. Hence, we can define

$$H_\alpha(A) := \lim_{\epsilon \to 0} H_\alpha^{(\epsilon)}(A). \tag{3.7}$$

Other texts use the more general definition of Hausdorff that requires a normalization constant

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma(\frac{s}{2} + 1)} \text{ for } s \geq 0$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$, see [6]. Define $H_\delta^s(E)$ as follows,

$$H_\delta^s(E) := \inf \left\{ \sum_{n=1}^\infty \alpha(s) \left(\frac{\text{diam}(c_n)}{2} \right)^s \; \middle| \; E \subset \bigcup_{n=1}^\infty c_n, \text{diam}(c_n) \leq \delta \right\}.$$

This notation is necessary for higher dimensions, see [7], but when the dimension $s = 1$ we get $\alpha(1) = \frac{\pi^{1/2}}{\Gamma(\frac{1}{2}+1)} = 2$ and $H_\delta^s(E)$ becomes what we see in Equation 3.6. With our main focus on the real line, we will use Equation 3.6 and 3.7 as our primary definition for Hausdorff outer measure.

THEOREM 3.10 For $\alpha > 0$, H_α is a Carathéodory outer measure on $\mathbb{R}^n$.

Proof. From the definition of H_α, we have $H_\alpha \geq 0$ and $H_\alpha(\emptyset) = 0$. If $A_1 \subset A_2$, then any covering of A_2 is also a covering of A_1. So, $H_\alpha^{(\epsilon)}(A_1) \leq H_\alpha^{(\epsilon)}(A_2)$. Letting $\epsilon \to 0$, we have $H_\alpha(A_1) \leq H_\alpha(A_2)$. We next want to show H_α is subadditive. Let $A = \bigcup A_k$, and choose a cover of A_k for each k. Now the union of the covers for each A_k is also a cover for A. Then,

$$H_\alpha^{(\epsilon)}(A) \leq \sum H_\alpha^{(\epsilon)}(A_k) \leq \sum H_\alpha(A_k).$$

Letting $\epsilon \to 0$, we get $H_\alpha \leq \sum H_\alpha(A_k)$. Finally, we need to show if $d(A_1, A_2) > 0$, then $H_\alpha(A_1 \cup A_2) = H_\alpha(A_1) + H_\alpha(A_2)$. From subadditivity we have

$$H_\alpha(A_1 \cup A_2) \leq H_\alpha(A_1) + H_\alpha(A_2).$$

We need to show $H_\alpha(A_1 \cup A_2) \geq H_\alpha(A_1) + H_\alpha(A_2)$. Fix $\epsilon > 0$ with $\epsilon < d(A_1, A_2)$. Let $\{B_k\}$ be a cover for $A_1 \cup A_2$ with the property $\text{diam}(B_k) < \delta$ where $\delta < \epsilon$, let

$$B_k' = A_1 \cap B_k \text{ and } B_k'' = A_2 \cap B_k.$$

Then the sets $\{B_k'\}$ and $\{B_k''\}$ are covers for A_1 and A_2 respectively with $\{B_k'\}$ and

41

$\{B_k''\}$ being disjoint. Hence,

$$\sum_j \operatorname{diam}(B_j')^\alpha + \sum_k \operatorname{diam}(B_k'')^\alpha \leq \sum_i \operatorname{diam}(B_i)^\alpha.$$

To get to H_α from our inequality, what remains to do is to take the infimum over the coverings and the limit as $\delta \to 0$. This gives us our inequality and thus,

$$H_\alpha(A_1 \cup A_2) \geq H_\alpha(A_1) + H_\alpha(A_2).$$

So H_α is a Carathéodory outer measure on $\mathbb{R}^n$. $\qquad\square$

We say H_α is the **Hausdorff outer measure** of dimension α on $\mathbb{R}^n$. One way to think of $H_\alpha(E)$, see [4, pp. 324], is as the α-dimensional mass of E among the sets of dimension α. Hausdorff measure is denoted the same as Hausdorff outer measure. We can also see that $0 \leq H_\alpha(E) \leq \infty$ for all subsets $E \subset \mathbb{R}^n$.

PROPOSITION 3.11 H_α is regular.

Proof. To show that H_α is regular, without loss of generality, assume the covering sets in the definition of $H_\alpha^{(\epsilon)}$ are closed. Let $E \subset \mathbb{R}^n$ with $H_\alpha < \infty$. Then $H_\alpha^{(\epsilon)} < \infty$ for all $\epsilon > 0$. For each $k \in \mathbb{N}$, let $\{C_n^k\}_{n=1}^\infty$ be a cover of E such that $\operatorname{diam}(C_n^k) \leq \frac{1}{k}$ and

$$\sum_{n=1}^\infty \operatorname{diam}(C_n^k)^s < H_{\frac{1}{k}}^s(E) + \frac{1}{k}.$$

Let $E_k = \bigcup_{n=1}^\infty E_n^k$ and $B = \bigcap_{k=1}^\infty E_k$. Then B is a Borel set with $E \subset B$. $\qquad\square$

THEOREM 3.12 For $H_\alpha(A)$ we have the following properties:

1. If $0 \leq H_\alpha(A) < +\infty$, then $H_\beta(A) = 0$ for $\beta > \alpha$.

2. If $H_\alpha(A) > 0$, then $H_\beta(A) = +\infty$ for $\beta < \alpha$.

42

Proof. Let $\epsilon > 0$. Property (2) is the contrapositive of (1), so the properties are equivalent. To prove (1), let $A = \bigcup A_k$ with $\text{diam}(A_k) < \epsilon$. If $\beta > \epsilon$, then we have

$$H_\beta^{(\epsilon)}(A) \leq \sum \text{diam}(A_k)^\beta < \epsilon^{\beta - \alpha} \sum \text{diam}(A_k)^\alpha.$$

Therefore,

$$H_\beta^{(\epsilon)}(A) \leq \epsilon^{\beta - \alpha} H_\beta^{(\epsilon)}(A).$$

Letting $\epsilon \to 0$, we get $H_\beta(A) = 0$ if $H_\alpha(A) < +\infty$. Thus completing our proof. $\square$

We have already shown that H_α is a Borel measure in Theorem 3.10 by showing H_α was a Carathéodory outer measure. Hausdorff measure also has the property that if $\{E_j\}$ is a countable family of disjoint Borel sets, and $E = \bigcup_{j=1}^\infty$, then

$$H_\alpha(E) = \sum_{j=1}^\infty H_\alpha(E_j).$$

PROPOSITION 3.13 Hausdorff measure has the following properties:

1. the quantity $H_0(A)$ counts the number of points in A. This is known as a **counting measure**.

2. $H_1(A) = m(A)$ on $\mathbb{R}$ where $m(A)$ is the Lebesgue measure.

3. if $\alpha > n$, then $H_\alpha(A) = 0$ for all $A \subset \mathbb{R}^n$.

Proof. For (1), if $\alpha = 0$ we have

$$H_0(A) = \lim_{\epsilon \to 0} \inf \left\{ \sum_k \text{diam}(A_k)^0 \ \middle| \ A \subset \bigcup_k A_k \text{ and } \text{diam}(A_k) < \epsilon \right\}$$

$$= \lim_{\epsilon \to 0} \inf \left\{ \sum_k 1 \ \middle| \ A \subset \bigcup_k A_k \text{ and } \text{diam}(A_k) < \epsilon \right\}.$$

For (2), if $\alpha = 1$,

$$m(A) = \inf \left\{ \sum_k \mathrm{diam}(A_k) \;\middle|\; A \subset \bigcup_k A_k \right\}$$

$$\leq \inf \left\{ \sum_k \mathrm{diam}(A_k) \;\middle|\; A \subset \bigcup_k A_k \text{ and } \mathrm{diam}(A_k) < \epsilon \right\}$$

$$= H_1^{(\epsilon)}.$$

To show the opposite of the inequality, let $I_k = [\epsilon k, \epsilon(k+1)]$. Then we have $\bigcup_{k \in \mathbb{Z}} I_k = \mathbb{R}$ and for any collection $\{c_n\}_{n=1}^\infty$, $\mathrm{diam}(c_n \cap I_k) \leq \epsilon$. Further,

$$\mathrm{diam}(c_n) \geq \sum_k \mathrm{diam}(c_n \cap I_k).$$

Now,

$$m(A) = \inf \left\{ \sum_n \mathrm{diam}(c_n) \;\middle|\; A \subset \bigcup_n c_n \right\}$$

$$\geq \inf \left\{ \sum_n \sum_k \mathrm{diam}(c_n \cap I_k) \;\middle|\; A \subset \bigcup_n c_n \right\}$$

$$\geq H_1^{(\epsilon)} \text{ (since } A \subset \bigcup_{k,n} (c_n \cap I_k).$$

It follows that $m(A) \geq H_1(A)$. Thus we have equality.

For (3), let $\alpha > n$. We show $H_\alpha(C) = 0$ for the unit cube C in $\mathbb{R}^n$. The result follows as any set can be covered by unit cubes. Divide C into m^n cubes of side length $\frac{1}{m}$ and diameter $\frac{\sqrt{n}}{m}$. Then

$$H_\alpha^{\left(\frac{\sqrt{n}}{m}\right)}(C) \leq \sum_{n=1}^{m^n} \left(\frac{\sqrt{n}}{m} \right)^\alpha = n^{\frac{\alpha}{2}} m^{n-\alpha}.$$

We can see that $n^{\frac{\alpha}{2}} m^{n-\alpha} \to 0$ as $m \to \infty$ if $\alpha > n$. This implies $H_\alpha(C) = 0$. $\qquad \square$

DEFINITION 3.14 Let $E \subset \mathbb{R}^n$. the **Hausdorff dimension** of E is defined as

$$H_{dim}(E) = \inf\{0 \leq \alpha < \infty \mid H_\alpha(E) = 0\}$$

$$= \sup\{0 \leq \alpha < \infty \mid H_\alpha(E) = +\infty\}.$$

From the definition of Hausdorff dimension and Theorem 3.12 we have that there exists a unique α such that

$$H_\beta(E) = \begin{cases} \infty & \text{if } \beta < \alpha, \\ 0 & \text{if } \beta > \alpha. \end{cases}$$

In other words, there exists a critical value of α at which $H_\alpha(E)$ jumps from ∞ to 0. We say that the set E has Hausdorff dimension α and write it as $\alpha = dimE$. At this value for α, we can say that $H_\alpha(E)$ in general satisfies $0 \leq H_\alpha(E) \leq \infty$. The final proposition we want to prove is that the Cantor set has Hausdorff dimension of $\frac{log2}{log3}$. In order to find the Hausdorff dimension of the Cantor set, we will use the scaling property for Hausdorff measure, see [8, pp. 27]. If $F \subset \mathbb{R}^n$ and $\lambda > 0$, then

$$H_\alpha(\lambda F) = \lambda^\alpha H_\alpha(F). \tag{3.8}$$

Where $\lambda F = \{\lambda x \mid x \in F\}$. We will follow the proof presented in [8, pp. 31-32].

PROPOSITION 3.15 For the Cantor set F, $H_{dim}(F) = \frac{log2}{log3}$.

Proof. We will consider two different approaches to finding $H_{dim}(F)$. We start with a heuristic calculation. The Cantor set F splits into a left part $F_L = F \cup [0, \frac{1}{3}]$ and a right part $F_R = F \cap [\frac{2}{3}, 1]$ with both parts being disjoint. Both F_R and F_L are geometrically similar to F and scaled by a ratio $\frac{1}{3}$ with $F = F_L \cup F_R$. Thus, for any α we have

$$H_\alpha(F) = H_\alpha(F_L) + H_\alpha(F_R) = \left(\frac{1}{3}\right)^\alpha H_\alpha(F) + \left(\frac{1}{3}\right)^\alpha H_\alpha(F),$$

by the scaling property in Equation 3.8. Assuming that at the critical value we have $\alpha = dimF$, then $0 < h_\alpha(F) < \infty$. Dividing both sides of Equation 3.3 by $H_\alpha(F)$ gives $1 = 2(\frac{1}{3})^\alpha$. Finally, taking the *log* of both sides and solving for α gives $\alpha = \frac{log2}{log3}$.

For the more rigorous calculation, we call the intervals of length 3^{-k} for $k = 0, 1, 2, ...$ that make up the sets E_k in the construction of F **basic intervals**. The covering $\{U_i\}$ of F consisting of the 2^k intervals of E_k of length 3^{-k} gives that $H_\alpha^{3^{-k}}(F) \leq \sum \text{diam}(U_i)^\alpha = 2^k 3^{-k\alpha} = 1$ if $\alpha = \frac{log2}{log3}$. Letting $k \to \infty$ gives $H_\alpha(F) \leq 1$. To prove $H_\alpha(F) \geq \frac{1}{2}$, we need to show that

$$\sum \text{diam}(U_i)^\alpha \geq \frac{1}{2} = 3^{-\alpha} \tag{3.9}$$

for any cover $\{U_i\}$ of F. We may assume that the $\{U_i\}$ are intervals. By expanding them and using the compactness of F, we only have to show Equation 3.9 if $\{U_i\}$ is a finite collection of closed subintervals of [0,1]. For each U_i, let $k \in \mathbb{Z}$ such that

$$3^{-(k+1)} \leq \text{diam}(U_i)^{-\alpha}. \tag{3.10}$$

Then U_i can intersect at most one basic interval of E_k since the separation of these basic intervals is at least 3^{-k}. If $j \geq k$ then, by construction and using Equation 3.10, U_i intersects at most $2^{j-k} = 2^j 3^{-\alpha k} \leq 2^j 3^\alpha \text{diam}(U_i)^\alpha$ basic intervals of E_j. If we choose a large j such that $3^{-(j+1)} \leq \text{diam}(U_i)^\alpha$ for all U_i, then since the $\{U_i\}$ intersects all 2^j basic intervals of length 3^{-j}, counting intervals gives

$$2^j \leq \sum 2^j 3^\alpha \text{diam}(U_i)^\alpha.$$

This equation reduces to Equation 3.9. This completes the proof. $\square$

This concludes our brief introduction to Hausdorff measure and dimension. Hausdorff measure is fundamental in many disciplines including fractal geometry, in-

terested readers can refer to [8, pp. 25-34] for more information. Hausdorff measure is an extensive subject. We have hopefully given a solid foundation to preface further study of the theory.

4. CONCLUSION

In this thesis we have illustrated some of the fundamental theorems that showcase the power and versatility of measure theory with specific examples in Lebesgue, Lebesgue-Stieltjes, and Hausdorff. We constructed Lebesgue outer measure, defined Lebesgue measurable sets, and gave a basic construction of the Lebesgue integral. We saw how the Lebesgue integral fixed some of the problems with the Riemann integral. For example, the Dirichlet function is measurable with Lebesgue measure even though the function is not Riemann integrable. In the last chapter, we defined abstract outer measure and a way of creating a measure from an outer measure. We also discussed two very important measures based on the definition of measurable sets given by Carathéodory. The measures we discussed were the Lebesgue-Stieltjes measure and Hausdorff measure which built on the measure theory given by Lebesgue. We have shown that measure theory is of core importance to the continual development of integration theory.

REFERENCES

[1] Royden, H. L. and Fitzpatrick, P. M. *Real Analysis.* Pearson, 2010.

[2] Wheeden, R. L. and Zygmund, A. *Measure and Integral: an Introduction to Real Analysis.* M. Dekker, 1977.

[3] Rudin, W. *Principles of Mathematical Analysis.* McGraw Education Private Limited, 2017.

[4] Stein, E. M. and Shakarchi, R. Princeton Lectures in Analysis. Princeton Univ. Press, 2005.

[5] Zorn, P. (2010). Understanding real analysis. Natick, Mass: A K Peters.

[6] Weisstein, E. W. *Gamma Function.* From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/GammaFunction.html.

[7] *Hausdorff Measure.* From Encyclopedia of Mathematics. https://www.encyclopediaofmath.org/index.php/Hausdorff_measure.

[8] Falconer, K. Fractal Geometry: Mathematical Foundations and Its Applications. John Wiley and Sons, 1990.