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Elements of Functional Analysis and Applications

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ELEMENTS OF FUNCTIONAL ANALYSIS AND APPLICATIONS

A Master's Thesis

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Master of Science, Mathematics

By

Chengting Yin

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ABSTRACT

Functional analysis is a branch of mathematical analysis that studies vector spaces with a limit structure (such as a norm or inner product), and functions or operators defined on these spaces. Functional analysis provides a useful framework and abstract approach for some applied problems in variety of disciplines. In this thesis, we will focus on some basic concepts and abstract results in functional analysis, and then demonstrate their power and relevance by solving some applied problems under the framework. We will give the definitions and provide some examples of some different spaces (such as metric spaces, normed spaces and inner product spaces), and some useful operators (such as differential operators, integral operators, contractions, completely continuous operators, etc.). Specifically, we will use the Banach Fixed Point Theory to prove the Existence and Uniqueness of the solutions to an Initial Value Problem for a system of n-first order differential equations. Then we will use the Leray-Schauder Fixed Point Theory to prove the existence of solutions for a nonlinear Sturm-Liouville Boundary Value Problem. Finally, we will demonstrate how the maximum-area problem can be proved handedly by the Calculus of Variation approach, a subset of Functional Analysis.

KEYWORDS: space, operator, functional, fixed point theory, Initial Value Problem, Boundary Value Problem, max area problem

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1. INTRODUCTION

Concepts and methods of functional analysis were developed from the oldest parts of mathematical analysis (advanced calculus): in the calculus of variation, in the theory of differential equations, and in the theory of integral equations. The essence of functional analysis is the extension of the concepts and methods of elementary analysis to more general (abstract) settings, which are applicable to many problems with different backgrounds. Therefore, functional analysis provides some powerful abstract frameworks and theory that can be used, when the terms are interpreted properly, to solve various applied problems in a variety of disciplines efficiently.

Functional analysis is a branch of mathematical analysis that studies vector spaces with a limit structure (such as a norm or inner product), functions/operators defined on these spaces, and operator equations which include as special cases differential equations and integral equations. Therefore, proofs of the existence of solutions of an initial value problem of a differential equation, and that of an integral equation, can be carried out by a functional analysis approach with a properly chosen space and a properly defined operator.

In this thesis, we will focus on some basic concepts and abstract results in functional analysis, and then demonstrate their power and relevance by solving some applied problems under the framework. We will give the definitions of various spaces, such as metric spaces, vector spaces, normed spaces, inner product spaces, Banach spaces, and Hilbert spaces. Examples are provided to illustrate these spaces and their respective properties. Some useful operators are given, such as differential operators, integral operators, contractions, completely continuous operators, which provide the needed background for later applications of the abstract theories.

Sufficient background information is supplied for the various import spaces that rise in different applications. The spaces include $C[a, b], C^n[a, b], l^p, R^n$. In particular, their completeness as metric spaces is proved, respectively.

Specifically, we will use the Banach Fixed Point Theory[1] to prove the Existence and Uniqueness of the solutions to an Initial Value Problem[2] for a system of n-first order differential equations under the assumption that the functions involved in the system are all continuously differentiable. Then we will use the Leray-Schauder Fixed Point Theory[3], which is an extension of the Brouwer's Fixed Point Theory[4], to prove the existence of solutions for a nonlinear Sturm-Liouville Boundary Value Problem[2] when the function sitting on the right-side of the equation contains the unknown function as a variable. Finally, we will demonstrate how the maximum-area problem can be proved handedly by the Calculus of Variation[5] approach, a subset of Functional Analysis.

2. SPACES

2.1 Metric Space

DEFINITION 2.1: A metric space is a pair (X, d), where X is a set and d is a metric on X(or distance function on X), that is, a function defined on $X \times X$ such that for or $x, y, z \in X$, we have [1]:

(M1) d is real-valued, finite and nonnegative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ (Symmetry).

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

EXAMPLE 2.2: Sequence space l^p

Let $p \geq 1$ be a fixed real number. The element in the l^p is a sequence $x = (x_1, x_2, ...)$ of real numbers such that $|x_1|^p + |x_2|^p + ...$ converges; thus

$$\sum_{j=1}^{\infty} |x_j|^p < \infty.$$

The metric is defined by

$$d(x, y) = (\sum_{j=1}^{\infty} |x_j - y_j|^p)^{\frac{1}{p}}.$$

Proof. Clearly, (M1) to (M3) are satisfied. In order to prove (M4), we need to drive 4 steps [1]:

(a) Prove $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where p and q are conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$.

From $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$pq = p + q, \qquad (p-1)(q-1) = 1, \qquad \frac{1}{p-1} = q - 1$$

So

$$u = t^{p-1} \Longleftrightarrow t = u^{q-1}$$

The value of ab equal the area of a region that is bounded above by u=b, below by t-axis, and on the left by u-axis, on the right by t=a. The value of $\int_0^a t^{p-1}dt$ equal the area of a region that is enclosed by $u = t^{p-1}$, t-axis and t=a. The value of $\int_0^b u^{q-1}du$ equal the area of a region that is enclosed by u-axis, u=b and $t = u^{q-1}$, which is same as $u = t^{p-1}$. See the graph below:

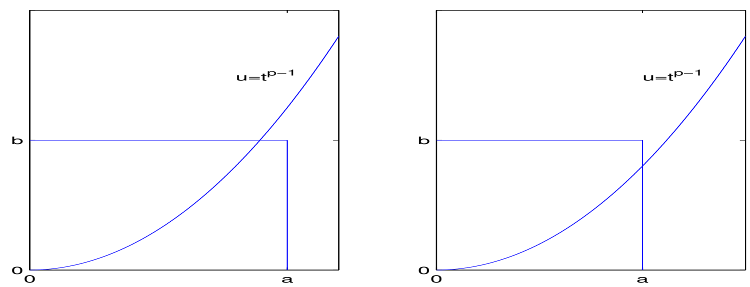

Figure 1: Inequality (2.1)

So we can see that

$$ab \leq \int_0^a t^{p-1}dt + \int_0^b u^{q-1}du = \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.1}$$

(b) Use (2.1) to prove the Holder inequality [1]

$$\sum_{j=1}^{\infty} |x_j y_j| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |y_m|^p)^{\frac{1}{p}}$$

Choose $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in l^p$, such that

$$\sum |x_k|^p = 1, \qquad \sum |y_k|^q = 1.$$

let $a = |x_k|$ and $b = |y_k|$ then from ① we have

$$|x_k y_k| \leq \frac{|x_k|^p}{p} + \frac{|y_k|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1. \tag{2.2}$$

Take any $x' = (x_1', x_2', ...), y' = (y_1', y_2', ...) \in l^p$,if we let

$$x_k = \frac{x_k'}{(\sum |x_k'|^p)^{1/p}}, \qquad y_k = \frac{y_k'}{(\sum |y_k'|^q)^{1/q}}$$

then

$$\sum |x_k|^p = 1, \qquad \sum |y_k|^q = 1.$$

Apply (2.2), then

$$\sum |x_k y_k| = \sum |\frac{x_k'}{(\sum |x_k'|^p)^{1/p}} \frac{y_k'}{(\sum |y_k'|^q)^{1/q}}| = \frac{\sum |x_k y_k|}{(\sum |x_k'|^p)^{1/p} (\sum |y_k'|^q)^{1/q}} \leq 1$$

Thus,

$$\sum |x_k y_k| \leq (\sum |x_k'|^p)^{1/p} (\sum |y_k'|^q)^{1/q}$$

(c) Use Holder inequality to prove Minkowski inequality [1]

$$(\sum_{j=1}^{\infty} |x_j + y_j|^p)^{1/p} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{\infty} |y_m|^q)^{\frac{1}{q}}$$

Let $z_k = x_k + y_k$, then

$$|z_k|^p = |z_k||z_k|^{p-1} = |x_k + y_k||z_k|^{p-1} \leq |x_k||z_k|^{p-1} + |y_k||z_k|^{p-1}$$

Apply the Holder inequality on$|x_k||z_k|^{p-1}$ and $|y_k||z_k|^{p-1}$

$$|x_k||z_k|^{p-1} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |z_k|^{(p-1)q})^{\frac{1}{q}} = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |z_k|^q)^{\frac{1}{q}}$$

Similarly,

$$|y_k||z_k|^{p-1} \leq (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{m=1}^{\infty} |z_k|^q)^{\frac{1}{q}}$$

Add these two inequality together, then we can obtain the Minkowski inequality.

$$\begin{aligned} |z_k|^p &\leq |x_k||z_k|^{p-1} + |y_k||z_k|^{p-1} \\ &\leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |z_k|^q)^{\frac{1}{q}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{\infty} |z_k|^q)^{\frac{1}{q}} \\ &= [(\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}}] (\sum_{k=1}^{\infty} |z_k|^q)^{\frac{1}{q}} \end{aligned}$$

So,

$$(\sum_{k=1}^{\infty} |z_k|^p)^{\frac{1}{p}} = (\sum_{k=1}^{\infty} |z_k|^p)^{1-\frac{1}{q}} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=1}^{\infty} |y_k|^p)^{\frac{1}{p}}$$

$$(\sum_{j=1}^{\infty} |x_j + y_j|^p)^{1/p} \leq (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{\infty} |y_m|^p)^{\frac{1}{p}}$$

(d) Use Minkowski inequality to prove

$$d(x, y) \leq d(x, z) + d(z, y).$$

$$\begin{aligned} d(x, y) &= (\sum |x_k - y_k|^p)^{1/p} = (\sum |x_k - z_k + z_k - y_k|^p)^{1/p} \\ &\leq [(\sum |x_k - z_k| + \sum |z_k - y_k|)^p]^{1/p} \\ &\leq (\sum |x_k - z_k|^p)^{\frac{1}{p}} + (\sum |y_k - z_k|^p)^{\frac{1}{p}} = d(x, z) + d(z, y). \end{aligned}$$

□

EXAMPLE 2.3: $C[a, b]$ space.

$C[a, b]$ is the set of all real-valued continuous functions defined in the interval [a, b]. C[a,b] is a metric space if we define the metric by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|,$$

$J = [a, b]$.

Proof.

$$\begin{aligned} d(x, y) = \max_{t \in J} |x(t) - y(t)| &= \max_{t \in J} |x(t) - z(t) + z(t) - y(t)| \\ &\leq \max_{t \in J}(|x(t) - z(t)| + |y(t) - z(t)|) \\ &\leq \max_{t \in J} |x(t) - z(t)| + \max_{t \in J} |z(t) - y(t)| \\ &= d(x, z) + d(y, z) \end{aligned}$$

□

2.2 Normed Linear Space

DEFINITION 2.4: Linear space [1]:

A linear space (or vector space) is a nonempty set X with two operations '+', ' $\cdot$ ', called addition and scalar multiplication. Suppose x,y,z $\in$ X, α, β are real numbers, then the following 8 properties holds:

$x + y = y + x$

$x + (y + z) = (x + y) + z$

$x + 0 = x$

$x + (-x) = 0$

$\alpha(\beta x) = (\alpha\beta)x$

$1 * x = x$

$\alpha(x + y) = \alpha x + \alpha y$

$(\alpha + \beta)x = \alpha x + \beta x$

DEFINITION 2.5: Normed space :

A normed space X is a linear space with a norm defined on it. Here a norm on a vector space X is a real-valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties [1]:

(N1) $\|x\| \geq 0$

(N2) $\|x\| = 0 \qquad \Longleftrightarrow \qquad x = 0$

(N3) $\|\alpha x\| = |\alpha|\|x\|$

(N4) $\|x + y\| \leq \|x\| + \|y\|$

here x and y are arbitrary vectors on X and α is any scalar.

EXAMPLE 2.6: Space $C[a, b]$.

Suppose x(t)$\in C[a, b]$. Define the norm on $C[a, b]$ by

$$\|x(t)\| = \max_{t \in J} |x(t)|,$$

where $J = [a, b]$.

Proof. We can define a metric on C[a,b] by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|.$$

From (M4), we have for any x(t), y(t) and z(t) in $C[a, b]$.

$$d(x, y) = \max_{t \in J} |x(t) - y(t)| \leq \max_{t \in J} |x(t) - z(t)| + \max_{t \in J} |z(t) - y(t)| = d(x, z) + d(y, z).$$

Let w(t)=-y(t), and z(t)=0, then

$$\begin{aligned}\|x+y\| &= \max_{t\in J}|x(t)+y(t)| \\ &\le \max_{t\in J}|x(t)| + \max_{t\in J}|-y(t)| \\ &= \max_{t\in J}|x(t)| + \max_{t\in J}|y(t)| \\ &= \|x\| + \|y\|\end{aligned}$$

So $C[a,b]$ is a normed space.

$\square$

2.3 Inner Product Space

DEFINITION 2.7: A complex vector space H is called an inner product space if to each ordered pair of vectors x and y in H is associated a complex number (x,y), called the inner product of x and y , such that the following rules holds [1]:

(I1) $(y,x) = \overline{(x,y)}$. The bar denotes complex conjugate.

(I2) $(x+y,z) = (x,z) + (y,z)$.

(I3) $(\alpha x, y) = \alpha(x,y)$, α is a complex number.

(I4) $(x,x) \ge 0$

(I5) $(x,x) = 0$ only if $x = 0$.

PROPOSITION 2.8: Suppose $x, y \in X$, X is an inner product space with the inner product defined by

$$(x,x) = \|x\|^2,$$

then

$$\|x+y\| \le \|x\| + \|y\|.$$

Namely, $d(x,y) = \|x-y\|$ satisfies the triangle inequality [6].

Proof. If $(x, x) = 0$,

$$\|x + y\| = \|y\|.$$

Suppose $(x, x) \neq 0$, consider $\|ax + y\|$, a is a complex number $\bar{a}$ is its conjugate.

$$0 \leq \|ax + y\|^2 = (ax + y, ax + y) = |a|^2(x, x) + a(x, y) + \bar{a}(y, x) + (y, y).$$

Let $a = -\frac{(y,x)}{(x,x)}$, then the right side of above equation becomes [6]

$$|-\frac{(y, x)}{(x, x)}|^2(x, x) - \frac{(y, x)}{(x, x)}(x, y) - \frac{(x, y)}{(x, x)}(y, x) + (y, y)$$

$$= \frac{|(x, y)|^2}{\|x\|^4}\|x\|^2 - 2\frac{|(x, y)|^2}{\|x\|^2} + \|y\|^2 = -\frac{|(x, y)|^2}{\|x\|^2} + \|y\|^2$$

Since $-\frac{|(x,y)|^2}{\|x\|^2} + \|y\|^2 \geq 0$,

$$|(x, y)|^2 \leq \|x\|^2\|y\|^2$$

Taking square root,

$$|(x, y)| \leq \|x\|\|y\|$$

So,

$$\|x + y\|^2 = (x, x) + (x, y) + (y, x) + (y, y) \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

$$\|x + y\| \leq \|x\| + \|y\|$$

$\square$

EXAMPLE 2.9: Euclidean space R^n.

R^n is the set of all ordered n-tuples of real numbers, written

$$x = (x_1, ..., x_n), \qquad y = (y_1, ..., y_n)$$

etc.

Define the inner product by

$$(x, y) = x_1 y_1 + ... + x_n y_n.$$

Since x_i and y_i are all real numbers, (I1) holds. (I2) to (I5) can be proved by simple calculation.

So R^n is an inner product space.

Define a norm on R^n by

$$\|x - y\| = (x - y, x - y)^{\frac{1}{2}},$$

then (N1) to (N4) can be shown by (I1) to (I5).

So R^n is a normed space.

Define a metric on R^n by

$$d(x, y) = \|x - y\| = (x - y, x - y)^{\frac{1}{2}}.$$

By property, substitute x by $x - z$, and y by $z - y$,

$$\|x - z + z - y\| \leq \|x - z\| + \|z - y\|.$$

This proves

$$d(x, y) \leq d(x, z) + d(y, z).$$

so R^n is a metric space, with $d(x, y) = \|x - y\|$.

2.4 Completeness

DEFINITION 2.10: Complete metric space.

A sequence (x_n) in a metric space $X = (X, d)$ is said to be Cauchy if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$d(x_m, x_n) < \epsilon$$

for every $m, n > N$.

The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

EXAMPLE 2.11: Completeness of R.

Proof. Suppose (x_k) is a Cauchy sequence, $x_k \in R$, then for any $\epsilon > 0$, there exists $N > 0$ such that for any $m, n > N$, we have $|x_n - x_m| < \epsilon$.

By triangle inequality,

$$|x_n| - |x_m| \leq |x_n - x_m| < \epsilon.$$

So $|x_n| < |x_m| + \epsilon$ for all $n, m > N$. Let $M = \max\{x_1, x_2, ..., x_N, |x_m| + \epsilon\}$, then for all $n \in \mathbf{N}$, we have$|x_n| \leq M$, so $\{x_n\}$ is bounded.

Suppose $\{x_n\}$ is increasing, since $\{x_n\}$ is bounded above, it has a least upper bound, say b (b$\in R$). By the definition of least upper bound, we have for any $\epsilon > 0$, there exists $x_p \in \{x_n\}$, such that

$$b - \epsilon < x_p < b < b + \epsilon.$$

Also, x_n is increasing, so for any $n > p$

$$b - \epsilon < x_p < x_n < b < b + \epsilon$$

or

$$|x_n - b| < \epsilon$$

By the definition of limit,

$$\lim_{n\to\infty} x_n = b.$$

Similarly, if $\{x_n\}$ is decreasing, we can prove that $\{x_n\}$ is convergent to its greatest lower bound.

If $\{x_n\}$ is not monotone, then we have two cases. Let $[-M, M] = J_0$.

Case 1:

There are only finitely many points of $\{x_n\}$ in the interval J_0. It means that there exists a constant c, such that

$$x_{n_k} = c$$

for some subsequence of $\{x_n\}$. It is clear that

$$x_{n_k} \to c \qquad as \qquad n_k \to \infty.$$

Case 2:

There are infinitely many points of $\{x_n\}$ in the interval J_0. Then we can choose a monotone subsequence of $\{x_n\}$ by the following steps:

Define $a_0 = \inf_{x_n \in J_0} x_n$ and $b_0 = \sup_{x_n \in J_0} x_n$. Divide the interval J_0 into 2 equal intervals : J_1 and J_1', then one of there 2 interval must have infinitely many points of $\{x_n\}$, suppose it is J_1. Then define

$$a_1 = \inf_{x_n \in J_1} x_n, \qquad b_1 = \sup_{x_n \in J_1} x_n.$$

Repeat this process for infinitely many times, then we get two sequence:

$$a_n = \inf_{x_n \in J_n} x_n, \qquad b_n = \sup_{x_n \in J_n} x_n.$$

Clearly, a_n is increasing, and b_n is decreasing. From the proof above, both a_n and b_n converge. It is clear that at least one of the two sequence contains infinite number of distinct numbers, say a_n. Then there exists a subsequence a_{n_k} which is strictly increasing. Suppose

$$a_{n_k} \longrightarrow a^*$$

as $k \longrightarrow \infty$.

Then

$$a_n \longrightarrow a^*$$

and

$$x_n \longrightarrow a^*$$

as $n \longrightarrow \infty$. □

EXAMPLE 2.12: Completeness of R^n .

Proof. Suppose $x = (x_1, ..., x_n)$ and $y = (y_1, ...y_n) \in R^n$ and R^n

$$d(x, y) = (\sum_{j=1}^{n} (x_j - y_j)^2)^{1/2}.$$

Consider a Cauchy sequence $(x_m) = (x_1^m, x_2^m, ..., x_n^m)$, $x_m \in R^n$. Since x_m is Cauchy, for any $\epsilon > 0$, there exists $N > 0$, such that if $m, n > N$, then

$$d(x_n, x_m) = (\sum_{j=1}^{n} (x_j^m - x_j^n)^2)^{1/2} < \epsilon.$$

Squaring,

$$\sum_{j=1}^{n}(x_j^m - x_j^n)^2 < \epsilon^2.$$

So for any fixed j (j=1,...n), if $m, n > N$, then

$$|x_j^m - x_j^n| < \epsilon.$$

Thus, $\{x^m\}_{m=1}^{\infty}$ is Cauchy for each $1 \leq j \leq n$. From Example 2.11, R is complete, so there exists $x_j \in R$ such that

$$x_j^m \to x_j, \qquad as \quad m \to \infty \qquad\qquad \text{(j is fixed)}.$$

Define $x = (x_1, x_2, ..., x_n)$, then $x \in R^n$ and for any $\epsilon > 0$ and fixed j (j=1,2,...n), there exists N_j such that if $m > N_j$, then

$$|x_j^m - x_j| < \frac{\epsilon}{\sqrt{n}}.$$

Choose $N = \max\{N_1, N_2, ...N_n\}$, then when $m > N$ we have

$$|x_j^m - x_j|^2 < \frac{\epsilon^2}{n}$$

$$\sum_{j=1}^{n}(x_j^m - x_j)^2 < \epsilon^2$$

$$d(x_m, x) = [\sum_{j=1}^{n}(x_j^m - x_j)^2)]^{1/2} < \epsilon$$

So,

$$x_m \to x \qquad as \qquad m \to \infty.$$

R^n is complete. □

EXAMPLE 2.13: Completeness of $C[a, b]$.

Proof. Let (x_m) be a Cauchy sequence in C[a,b], then for any $\epsilon > 0$, there exists N, such that if $m, n > N$, then

$$d(x_m, x_n) = \max_{t \in J} |x_m(t) - x_n(t)| < \epsilon.$$

Hence,

$$|x_m(t) - x_n(t)| \leq \max_{t \in J} |x_m(t) - x_n(t)| < \epsilon$$

and $\{x_n(t)\}$ is a Cauchy sequence of numbers for each $t \in [a, b]$.

Thus,

$$\lim_{n \to \infty} x_n(t) = x(t) \tag{2.3}$$

for some number $x(t)$, for each $t \in [a, b]$.

In (2.1), take limit. As $n \longrightarrow \infty$, we obtain

$$|x_n(t) - x(t)| < \epsilon,$$

for any t, for any $n \geq N$.

Since x_N is continuous, for any $\epsilon > 0$, for any $t_0 \in [a, b]$, there exists δ, such that if $t \in [a, b]$ with $|t_0 - t| < \delta$,

$$|x_N(t_0) - x_N(t)| < \frac{\epsilon}{3}.$$

By (2.1), we have

$$|x(t) - x_n(t)| < \frac{\epsilon}{3} \qquad \text{for } \forall t \text{ and } n \geq N.$$

Thus if $|t - t_0| < \delta$ we have

$$|x(t_0) - x(t)| = |x(t_0) - x_N(t_0) + x_N(t_0) - x_N(t) + x_N(t) - x(t)|$$

$$\leq |x(t_0) - x_N(t_0)| + |x_N(t_0) - x_N(t)| + |x_N(t) - x(t)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Therefore, $x(t)$ is continuous and

$$\lim_{n\to\infty} x_n = x \qquad \text{in} \quad C[a,b].$$

So $C[a,b]$ is complete. □

DEFINITION 2.14: Banach spaces.

A Banach spaces is a complete normed space.

EXAMPLE 2.15: R^n

Proof. R^n is complete by Example 2.12, and is a normed space by Example 2.9. So R^n is a Banach space. □

EXAMPLE 2.16: $C[a,b]$

Proof. $C[a,b]$ is complete by Example 2.13, and it is a normed space by Example 2.6.

So C[a,b] is a Banach space. □

DEFINITION 2.17: Hilbert Spaces.

A Hilbert space is a complete inner product space.

EXAMPLE 2.18: R^n

Proof. By Example 2.9, R^n is a inner product space. By Example 2.12, R^n is complete.

So R^n is a Hilbert space. □

3. OPERATORS

DEFINITION 3.1: Suppose X, and Y are two metric spaces or vector spaces, the mapping from X to Y is called an operator.

3.1 Contraction

DEFINITION 3.2: Let $X = (X, d)$ be a metric space. A mapping T: $X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

EXAMPLE 3.3: If $f : [a, b] \to R$ is differentiable and $|f'(x)| \leq \alpha < 1$ for any $x \in [a, b]$, then f is a contraction.

Proof. By Mean Value Theroy, there exists a point $c \in [a, b]$, such that for any $a, b \in [a, b]$

$$\frac{f(a) - f(b)}{a - b} = f'(c).$$

Since $|f'(x)| \leq \alpha < 1$,

$$|f(a) - f(b)| = |a - b||f'(c)| \leq \alpha|a - b|,$$

where $0 < \alpha < 1$. □

EXAMPLE 3.4: Let $X = \{x \in R | x \geq 1\}$, and let the mapping T from X to X be defined by $Tx = \frac{x}{2} + x^{-1}$. Show that T is a contraction.

Proof. Suppose $x, y \geq 1, x, y \in R$. Define $d(x, y) = |x - y|$.

$$d(Tx, Ty) = |Tx - Ty| = |\frac{x - y}{2} - \frac{x - y}{xy}| = |x - y||\frac{1}{2} - \frac{1}{xy}| = d(x, y)|\frac{1}{2} - \frac{1}{xy}|.$$

Since $x, y \geq 1$ so $xy \geq 1$, $0 < \frac{1}{xy} \leq 1$, $|\frac{1}{2} - \frac{1}{xy}| \leq \frac{1}{2}$, $d(Tx, Ty) \leq \frac{1}{2}d(x, y)$.

Let $\frac{1}{2} \leq \alpha \leq 1$ then

$$d(Tx, Ty) \leq \frac{1}{2}d(x, y) \leq \alpha d(x, y)$$

$\square$

3.2 Continuous Operator

DEFINITION 3.5: Let $X = (X, d_1)$ and $Y = (Y, d_2)$ be two metric spaces. A mapping T: $X \rightarrow Y$ is called continuous at x_0, if for any $\epsilon > 0$, there exists δ, such that if $d_1(x_0, x) < \delta$, then $d_2(T(x_0), T(x)) < \epsilon$.

In particular, a contraction operator is continuous.

LEMMA 3.6: If f is a continuous function on a compact set, then f is uniformly continuous.

EXAMPLE 3.7: Suppose $f : X \longrightarrow R$ is a continuous function where

$$X = \{(x_1, x_2) : a \leq x_1 \leq b, c \leq x_2 \leq d\}.$$

Since X is a compact set, and f is continuous, hence f is uniformly continuous.

EXAMPLE 3.8: Prove that Green's function is continuous.

$$G(x, s) = \begin{cases} -y_1(s)y_2(x)/p(x)W(y_1, y_2)(x), & 0 \leq s \leq x, \\ -y_1(x)y_2(s)/p(x)W(y_1, y_2)(x), & x \leq s \leq 1. \end{cases}$$

The function y_1 and y_2 are continuous on the interval $[a, b]$ and $p(x)W(y_1, y_2)(x)$ is a constant, so $G : [a, b] \times [a, b] \rightarrow R$ is continuous. Since $[a, b] \times [a, b]$ is compact, so G is uniformly continuous.

3.3 Completely Continuous Operator

DEFINITION 3.9: Let X,Y be two metric spaces. The operator T is called completely continuous if it is continuous and T maps any bounded subset of X into a relatively compact subset of Y.

DEFINITION 3.10: A relatively compact subset B of Y is a subset whose closure is compact, namely, any sequence in B has a convergent subsequence in Y.

3.4 Ascoli's Theorem [1]

THEOREM 3.11: A bounded equicontinuous sequence (x_n) in $C[a, b]$ has a subsequence which converges.

REMARK 3.12: (a) Bounded: there is some $L > 0$, such that $|x_n(t)| \leq L$ for all t$\in [a, b]$, all n.

(b) Equicontinuous: for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|x_n(t) - x_n(u)| < \epsilon,$$

for all $n \geq 1$ whenever $|t - u| < \delta$ and $t, u \in [a, b]$.

3.5 Lipschitz Condition

DEFINITION 3.13: If $f(t, y)$ is a continuous function defined on the rectangle $D = \{(t, y) : a \leq t \leq b, \quad c \leq y \leq d\}$, and $\frac{\partial f}{\partial y}$ is continuous, then there exists $K > 0$ such that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2|$$

for any (t, y_1) and (t, y_2) in D.

EXAMPLE 3.14: Suppose that the function $f(t, x_1, x_2, ...x_n)$ is continuous and $\frac{\partial f}{\partial x_i}$

is continuous for each $i = 1, 2, ..., n$. Show that there exists a $K > 0$ such that

$$|f(t, x_1, x_2, ...x_n) - f(t, y_1, y_2, ...y_n)| \leq K(|x_1 - y_1| + |x_2 - y_2| + ... + |x_n - y_n|)$$

Proof. Consider

$$|f(t, y_1, y_2, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n) - f(t, y_1, y_2, ..., y_{i-1}, y_i, x_{i+1}, ...x_n)|$$

All the variables are same but the i^{th} variable. So by mean value theorem, we have some $K_i > 0$ such that

$$|f(t, y_1, y_2, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n) - f(t, y_1, y_2, ..., y_{i-1}, y_i, x_{i+1}, ...x_n)| \leq K_i|x_i - y_i|$$

for each $i = 1, 2, ..., n$. Consider

$$\sum_{i=1}^{n}[f(t, y_1, y_2, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n) - f(t, y_1, y_2, ..., y_{i-1}, y_i, x_{i+1}, ...x_n)]$$

$$= f(t, x_1, x_2, ..., x_n) - f(t, y_1, x_2, ..., x_n) + f(t, y_1, x_2, ..., x_n) - f(t, y_1, y_2, x_3, ..., x_n)$$

$$+f(t, y_1, y_2, x_3, ..., x_n) - f(t, y_1, y_2, y_3, x_4, ..., x_n) + ... + f(t, y_1, ..., y_{n-1}, x_n) - f(t, y_1, ..., y_n)$$

$$= f(t, x_1, ..., x_n) - f(t, y_1, ..., y_n)$$

Choose $K = \max\{K_1, K_2, ..., K_n\}$, then

$|f(t, x_1, x_2, ...x_n) - f(t, y_1, y_2, ...y_n)|$

$= |\sum_{i=1}^{n} f(t, y_1, y_2, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n) - f(t, y_1, y_2, ..., y_{i-1}, y_i, x_{i+1}, ...x_n)|$

$\leq \sum_{i=1}^{n} |f(t, y_1, y_2, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n) - f(t, y_1, y_2, ..., y_{i-1}, y_i, x_{i+1}, ...x_n)|$

$\leq K_1|x_1 - y_1| + K_2|x_2 - y_2| + ... + K_n|x_n - y_n|$

$\leq K(|x_1 - y_1| + |x_2 - y_2| + ... + |x_n - y_n|)$

$= K \sum_{i=1}^{n} |x_i - y_i|$ $\square$

4. FUNCTIONAL

DEFINITION 4.1: Let X be a vector space. A mapping from X to the real line R is called a functional.

DEFINITION 4.2: We say that the functional $J[y]$ has a weak extremum for $y = \bar{y} \in X$ if there there exists $\epsilon > 0$ such that $J[y] - J[\bar{y}]$ has the same sign for all y in the domain of definition of the functional which satisfies the condition $\|y - \bar{y}\| < \varepsilon_1$, where $\| \quad \|_1$ denotes the norm in the space $C^1(a, b)$ ($C^1(a, b)$ consists of all functions defined on (a,b) which are continuous and have continuous first derivative) [5].

THEOREM 4.3: A necessary condition for the differential functional $J[y]$ to have an extremum [5] for $y = \hat{y}$ is that its variation vanish for $y = \hat{y}$, *i.e.* that

$$\delta J[y] = 0$$

for $y = \hat{y}$ and all admissible h [5].

Proof. Suppose $J[y]$ has a minimum for $y = \hat{y}$. By the definition of variation,

$$\Delta J[h] = \delta J[h] + \epsilon\|h\| \qquad \epsilon\|h\| \to 0 \quad \text{as} \quad \|h\| \to 0$$

where $\delta J[h]$ is a linear functional which differs from $\Delta J[h]$ by an infinitesimal of order higher than 1 relative to $\|h\|$, denoted by $\epsilon\|h\|$. Thus, for sufficiently small $\|h\|$, $\Delta J[h]$and $\delta J[h]$ have same sign. Suppose $\delta J[h_0] \neq 0$ for some h_0, then for any $\alpha > 0$, we have

$$\Delta J[-\alpha h] = -\delta J[\alpha h].$$

But since $J[y]$ has a minimum for $y = \hat{y}$,

$$\Delta J[h] = J[\hat{y} + h] - J[\hat{y}] \geq 0$$

a contradiction. $\square$

4.1 Euler's Equation

THEOREM 4.4: Let $F(x, y, z)$ be a function with continuous first and second derivatives with respect to all its arguments. Suppose y satisfies the boundary conditions

$$y(a) = A, \quad y(b) = B$$

find the function y for which the functional

$$J[y] = \int_a^b F(x, y, y')dx$$

has a weak extremum. Give y an increment h(x), then:

$$h(a) = h(b) = 0.$$

Since $y + h$ should satisfy the boundary conditions

$$y(a) + h(a) = A, \quad y(b) + h(b) = B.$$

The corresponding increment of the functional is:

$$\begin{aligned}\Delta J &= J[y+h] - J[y] \\ &= \int_a^b F(x, y+h, y'+h')dx - \int_a^b F(x,y,y')dx \\ &= \int_a^b [F(x, y+h, y'+h') - F(x,y,y')]dx\end{aligned}$$

The Taylor series of $F(x, y+h, y'+h')$ expanded about (x_0, y_0, y_0') is:

$$F(x_0, y_0, y_0') + [(y - y_0)F_y(x_0, y_0, y_0') + (y' - y_0')F_{y'}(x_0, y_0, y_0')] + \circ(h, h')$$

$$= F(x_0, y_0, y_0') + [hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0')] + \circ(h, h')$$

$\circ(h, h')$ is a function of order higher than 1 relative to h and h'.

So the increment of J at (x_0, y_0, y_0') is

$$\begin{aligned}\Delta J|_{(x_0,y_0,y_0')} &= \int_a^b [F(x_0, y_0, y_0') + hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0') + \circ(h, h') \\ &\quad - F(x_0, y_0, y_0')]dx \\ &= \int_a^b [hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0') + \circ(h, h')]dx \\ &= \int_a^b [hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0')]dx + \int_a^b \circ(h, h')dx\end{aligned}$$

h is the only variable.

Set $\triangle J[h] = \triangle J|_{x_0,y_0,y_0'}$, $\phi[h] = \int_a^b [hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0')]dx$, $\epsilon[h] = \int_a^b \circ(h, h')dx$, then

$$\triangle J[h] = \phi[h] + \epsilon[h]$$

where $\phi[h]$ is a linear functional of h, and $\epsilon[h] \to 0$ as $h \to 0$.

By the definition of the differential of a functional, $\phi[h]$ is the differential of $J[y]$ at point(x_0, y_0, y_0').

If $J[y]$ have an extremum at (x_0, y_0, y_0'), then

$$\phi[h] = \int_a^b [hF_y(x_0, y_0, y_0') + h'F_{y'}(x_0, y_0, y_0')]dx = 0.$$

Assume $\phi[h] = 0$

Set $A = \int_a^x F_y(s)ds$, then $A' = F_y(x)$.

Integrate by parts

$$\begin{aligned}\int_a^b F_y h dx &= \int_a^b A'hdx = Ah|_{x=a}^b - \int_a^b Ah'dx \\ &= A[h(b) - h(a)] - \int_a^b Ah'dx = -\int_a^b Ah'dx\end{aligned}$$

$$\phi[h] = \int_a^b [-Ah' + F_{y'}h']dx = \int_a^b [-A + F_{y'}]h'dx = 0 \tag{4.4}$$

Set $\alpha = -A + F_{y'}$, then there is a constant c such that $\alpha = c$.

Proof. Let c defined by $\int_a^b (\alpha - c)dx = 0$. Since h is arbitrary, choose $h = \int_a^x (\alpha - c)ds$, then $h(a) = h(b) = 0$

$$\begin{aligned}\int_a^b [\alpha - c]h' &= \int_a^b (\alpha h' - ch')dx = \int_a^b \alpha h'dx - c\int_a^b h'dx \\ &= \int_a^b \alpha h'dx - c[h(a) - h(b)] = \int_a^b \alpha h'dx.\end{aligned}$$

Since $\alpha = -A + F_{y'}$, and (4.4)

$$\int_a^b [\alpha - c]h' = 0.$$

Since $h = \int_a^x (\alpha - c)ds$, $h' = \alpha - c$,

$$\int_a^b (\alpha - c)h'dx = \int_a^b (\alpha - c)^2 dx = 0. \tag{4.5}$$

By (4.5) and $(\alpha - c)^2 \geq 0$, $\alpha = c$

So $\alpha' = 0$.

$$\alpha' = (-A + F_{y'})' = -A' + \frac{d}{dx}F_{y'} = -F_y + \frac{d}{dx}F_{y'} = 0$$

Euler's Equation:

$$F_y - \frac{d}{dx}F_{y'} = 0$$

$\square$

4.2 The Isoperimetric Problem

$$J[y] = \int_a^b F(s, y, y')dx \qquad y(a) = A \quad y(b) = B$$

$$K[y] = \int_a^b G(x, y, y')dx = l$$

if y(x) is an extremal of J[y], but not an extremal of K[y], then

$$F_y - \frac{d}{dx}F_{y'} + \lambda(G_y - \frac{d}{dx}G_{y'}) = 0.$$

Proof. Give y(x) an increment

$$\delta_1 y(x) + \delta_2 y(x)$$

$\delta_1 y(x)$ is nonzero only in a neighborhood of x_1.

$\delta_2 y(x)$ is nonzero only in a neighborhood of x_2.

Using variational derivative, we can write:

$$\Delta J = \{\frac{\delta F}{\delta y}|_{x=x_1} + \epsilon_1\}\Delta\sigma_1 + \{\frac{\delta F}{\delta y}|_{x=x_2} + \epsilon_2\}\Delta\sigma_2 \tag{4.6}$$

where $\Delta\sigma_1 = \int_a^b \delta_1 y(x)dx$ and $\Delta\sigma_2 = \int_a^b \delta_2 y(x)dx$,

ϵ_1, $\epsilon_2 \longrightarrow 0$, as $\Delta\sigma_1$, $\Delta\sigma_2 \longrightarrow 0$.

Let$y^* = y + \delta_1 y(x) + \delta_2 y(x)$, since $K[y] = l$, $\Delta k = K[y^*] - K[y] = 0$

$$0 = \{\frac{\delta G}{\delta y}|_{x=x_1} + \epsilon_1'\}\Delta\sigma_1 + \{\frac{\delta G}{\delta y}|_{x=x_2} + \epsilon_2'\}\Delta\sigma_2 \tag{4.7}$$

Suppose $\{\frac{\delta G}{\delta y}|_{x=x_2} + \epsilon_2'\} \neq 0$, since y is not an extremal of K, x_2 exists.

From (4.7), we can get

$$\Delta\sigma_2 = -\frac{\{\frac{\delta G}{\delta y}|_{x=x_1} + \epsilon_1'\}}{\{\frac{\delta G}{\delta y}|_{x=x_2} + \epsilon_2'\}}\Delta\sigma_1 = -(\frac{\frac{\delta G}{\delta y}|_{x=x_1}}{\frac{\delta G}{\delta y}|_{x=x_2}} + \epsilon')\Delta\sigma_1 \tag{4.8}$$

where $\epsilon' \longrightarrow 0$, as $\Delta\sigma_1 \longrightarrow 0$.

Plug (4.8) into (4.6)

$$\Delta J = \{\frac{\delta F}{\delta y}|_{x=x_1} + \epsilon_1\}\Delta\sigma_1 - (\frac{\frac{\delta G}{\delta y}|_{x=x_1}}{\frac{\delta G}{\delta y}|_{x=x_2}} + \epsilon')\{\frac{\delta F}{\delta y}|_{x=x_2} + \epsilon_2\}\Delta\sigma_1.$$

Let $\lambda = -\frac{\frac{\delta F}{\delta y}|_{x=x_2}}{\frac{\delta G}{\delta y}|_{x=x_2}}$, then

$$\Delta J = \{\frac{\delta F}{\delta y}|_{x=x_1} + \epsilon_1\}\Delta\sigma_1 + \lambda\{\frac{\delta G}{\delta y}|_{x=x_1} + \epsilon'\}\Delta\sigma_1$$

$$= \{\frac{\delta F}{\delta y}|_{x=x_1} + \lambda(\frac{\delta G}{\delta y}|_{x=x_1})\}\Delta\sigma_1 + \epsilon \triangle \sigma_1$$

□

5. APPLICATIONS

5.1 Initial Value Problem

DEFINITION 5.1: Fixed point.

A fixed point of a mapping T : $X \to X$ is an $x \in X$ such that

$$Tx = x.$$

EXAMPLE 5.2: The mapping $x \longmapsto x^2$ has two fixed points 0 and 1.

A rotation of the plane has only one fixed point, the center of rotation.

THEOREM 5.3: Banach Fixed Point Theorem (Contraction Theorem).

Given an non-empty complete metric space X=(X,d), then any contraction $T : X \to X$, has exact one fixed point.

Proof. Existence:

Choose any $x_0 \in X$,and define a sequence x_n by

$$x_0, x_1 = T(x_0), x_2 = T(x_1), ..., x_n = T(x_{n-1}), ...$$

For any $m \in N$, by the definition of contraction we have

$$d(x_{m+1}, x_m) = d(Tx_{m+1}, Tx_m) \leq \alpha d(x_m, x_{m-1})$$

where $\alpha < 1$.

Similarly,

$$d(x_m, x_{m-1}) = d(Tx_m, Tx_{m-1}) \leq \alpha d(x_{m-1}, x_{m-2})$$

Repeat this process m times, then we get

$$d(x_{m+1}, x_m) \leq \alpha^m d(x_1, x_0)$$

By the triangle inequality, and the above inequality, we have for $n > m$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2})... + d(x_{n-1}, x_n) \\ &\leq (\alpha^m + \alpha^{m+1} + \alpha^{-1}) d(x_0, x_1) \\ &= \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha} d(x_0, x_1) \end{aligned}$$

Since $0 < \alpha < 1$, $0 < 1 - \alpha^{n-m} < 1$, so

$$d(x_m, x_n) \leq \frac{\alpha^m}{1 - \alpha} d(x_0, x_1) \tag{5.9}$$

On the right side of (5.9), $d(x_0, x_1)$ is fixed and $0 < 1 - \alpha < 1$, so let m sufficiently large, for any $\epsilon > 0$, we can make (5.9) less than ϵ. This proves $\{x_n\}$ is Cauchy.

Since X is complete, $\{x_n\}$ has a limit in X, say $x_n \to x$, as $n \to \infty$ and $x \in X$.

Consider $d(x, x)$

$$d(x, Tx) \leq d(x, x_m) + d(x_m, Tx) \leq d(x, x_m) + \alpha d(x_{m-1}, x),$$

by the definition of contraction and $x_m = T(x_{m-1})$.

Since $x_m \to x$ as $m \to \infty$, for any $\epsilon > 0$, there exists $N \in Z$, such that if $m - 1 > N$, then $d(x, x_m) < \frac{\epsilon}{2}$ and $d(x_{m-11}, x) < \frac{\epsilon}{2\alpha}$. This implies that

$$d(x, Tx) \leq \frac{\epsilon}{2} + \alpha \frac{\epsilon}{2\alpha} = \epsilon.$$

So we conclude that $d(x, Tx) \leq 0$.

But $d(x, Tx) \geq 0$, so $d(x, Tx) = 0$, which implies $x = Tx$.

Uniqueness:

Suppose there are 2 fixed points of T :

$$Tx = x, \qquad Tx' = x'$$

then

$$d(x, x') = d(Tx, Tx') \leq \alpha d(x, x')$$

where $\alpha < 1$.

Thus, $(1 - \alpha)d(x, x') \leq 0$. Hence, $d(x, x') = 0$, $x = x'$. □

THEOREM 5.4: Picard's Existence and Uniqueness Theorem.

Let f be continuous on a rectangle

$$R = \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}$$

and thus bounded on R, say

$$|f(t, x)| \leq c \qquad \text{for all } (t, x) \in R$$

c is a real number. Suppose that f satisfies a Lipschitz condition on R with respect to its second argument, that is, there is aa constant k such that for (t,x),(t,v) $\in$ R

$$|f(t, x) - f(t, v)| \leq k|x - v|.$$

Then the initial value problem :

$$x' = f(t, x)$$

$$x(t_0) = x_0$$

has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$, where

$$\beta < min\{a, \frac{b}{c}, \frac{1}{k}\}.$$

Proof. Let $C(J)$ consists of all real-valued continuous functions on the closed interval $J = [t_0 - \beta, t_0 + \beta]$. Define the metric by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|.$$

Then $(C(J), d)$ is a complete metric space.

Let $C_0(J)$ be the subspace of $C(J)$, and $C_0(J)$ satisfies

$$|x(t) - x_0| \leq c\beta$$

where $x(t) \in C(J)$.

We can see that for any $x(t) \in C_0(J)$, the domain and range of the function x(t) are closed, so $C_0(J)$ is closed in $C(J)$. Since $C(J)$ is complete, $C_0(J)$ is complete.

Integrate both side of $x'(t) = f(t, x)$ from t_0 to t, then

$$x(t) - x(t_0) = \int_{t_0}^{t} f(s, x(s))ds$$

Applying initial condition $x(t_0) = x_0$, we see that the initial value problem can be written as

$$x = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Define a mapping T from $C_0(J)$ to $C(J)$:

$$Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \qquad x(t) \in C_0(J).$$

If $x(t) \in C_0(J)$, then $s \in J$, and $|x(s) - x_0| \leq c\beta < b$, so $(s, x(s)) \in R$.

To see $Tx(t) \in C_0(J)$, we need $|Tx(t) + x_0| \leq c\beta$. In fact,

$$|Tx(t) + x_0| = |\int_{t_0}^{t} f(s, x(s))| \leq |t - t_0|c$$

because $|f(t, x(t))| \leq c$.

Now show T is a contraction.

For any $x(t), v(t) \in C_0(J)$, we have

$$\begin{aligned} |Tx(t) - Tv(t)| &= |\int_{t_0}^{t} [f(s, x(s)) - f(s, v(s))]ds \\ &\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, v(s))|ds \\ &\leq \int_{t_0}^{t} k|x(s) - v(s)|ds \\ &\leq |t - t_0|k \max_{s \in J} |x(s) - v(s)| \\ &\leq \beta k d(x, v) \end{aligned}$$

Since $\beta k d(x, v)$ dose not depend on t, we can take the maximum on $|Tx(t) - Tv(t)|$,

$$d(Tx, Tv) \leq \alpha d(x, v)$$

where $\alpha = \beta k < 1$. Thus T is a contraction.

By Banach Fixed Point Theorem, there exists a unique $x(t) \in C_0(J)$ such that

$$x(t) = Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

Differentiate both side, then

$$x'(t) = f(t, x(t)).$$

And the initial condition is also satisfied because

$$x(t_0) = x_0 + \int_{t_0}^{t_0} f(s, x(s))ds = x(t_0) + 0 = x(t_0).$$

$\square$

System of Linear Equations.

Consider the system of n first order differential equations:

$$\begin{cases} x_1' = f_1(t, x_1, x_2, ..., x_n) \\ x_2' = f_2(t, x_1, x_2, ..., x_n) \\ \vdots \\ x_{n-1}' = f_{n-1}(t, x_1, x_2, ..., x_n) \\ x_n' = f_n(t, x_1, x_2, ..., x_n) \end{cases}$$

Suppose the initial condition is:

$$x_1(t_0) = a_1, \quad x_2(t_0) = a_2, \quad ..., \quad x_n(t_0) = a_n$$

then from the previous proof, we know that for any fixed k (k=1,2,...n), the equation

$$x_k' = f_k(t, x_1, ..., x_n)$$

is equivalent to

$$x_k = a_k + \int_{t_0}^{t} f_k(s, x_1(s), ..., x_n(s))ds$$

so we can rewrite the system as

$$\begin{cases} x_1 = a_1 + \int_{t_0}^{t} f_1(s, x_1(s), ..., x_n(s))ds \\ x_2 = a_2 + \int_{t_0}^{t} f_2(s, x_1(s), ..., x_n(s))ds \\ \vdots \\ x_n = a_n + \int_{t_0}^{t} f_n(s, x_1(s), ..., x_n(s))ds \end{cases}$$

subject to the initial condition.

Or $x = F(x)$, where

$$x = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

and

$$F(x) = \begin{Bmatrix} a_1 + \int_{t_0}^{t} f_1(s, x_1(s), ..., x_n(s))ds \\ a_2 + \int_{t_0}^{t} f_2(s, x_1(s), ..., x_n(s))ds \\ \vdots \\ a_n + \int_{t_0}^{t} f_n(s, x_1(s), ..., x_n(s))ds \end{Bmatrix}.$$

Prove the Existence and Uniqueness Theorem for the system by applying Banach Fixed Point Theorem to show that F(x) has a fixed point.

Let $\Omega = \{x = (x_1, ..., x_n) | x_k \in C[a, b], k = 1, 2, ..., n\}$ and for $x = (x_i)$, $y = (y_i) \in \Omega$, define

$$d(x, y) = [\sum_{i=1}^{n} (\max_{a \le t \le b} |x_i(t) - y_i(t)|)^2]^{\frac{1}{2}}.$$

Claim that (Ω, d) is a complete metric space.

Proof. (M1) to (M3) is obvious, we now prove (M4).

By triangle inequality, if we fix i (i=1,2,...,n), then for any $t \in [a.b]$, we have

$$|x_i(t) - y_i(t)| = |x_i(t) - z_i(t) + z_i(t) - y_i(t)| \leq |x_i(t) - z_i(t)| + |z_i(t) - y_i(t)|$$

so

$$\begin{aligned}\max_{a\leq t\leq b} |x_i(t) - y_i(t)| &\leq \max_{a\leq t\leq b}(|x_i(t) - z_i(t)| + |z_i(t) - y_i(t)|) \\ &\leq \max_{a\leq t\leq b} |x_i(t) - z_i(t)| + \max_{a\leq t\leq b} |z_i(t) - y_i(t)|\end{aligned}$$

$$[\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - y_i(t)|)^2]^{\frac{1}{2}} \leq [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - z_i(t)| + \max_{a\leq t\leq b} |z_i(t) - y_i(t)|)^2]^{\frac{1}{2}}$$

Applying Minkowski inequality on the right side of above inequality for elements in R^n,

$$\begin{aligned}&[\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - z_i(t)| + \max_{a\leq t\leq b} |z_i(t) - y_i(t)|)^2]^{\frac{1}{2}} \\ &\quad \leq [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - z_i(t)|)^2]^{\frac{1}{2}} + [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |z_i(t) - y_i(t)|)^2]^{\frac{1}{2}}\end{aligned}$$

So

$$\begin{aligned}d(x,y) &= [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - y_i(t)|)^2]^{\frac{1}{2}} \\ &\leq [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |x_i(t) - z_i(t)|)^2]^{\frac{1}{2}} + [\sum_{i=1}^{n}(\max_{a\leq t\leq b} |z_i(t) - y_i(t)|)^2]^{\frac{1}{2}} \\ &= d(x,z) + d(y,z).\end{aligned}$$

Thus, (Ω, d) is a metric space.

To prove completeness of (Ω, d), choose any Cauchy sequence $\{x^m\}$ in Ω. Since $\{x^m\}$, is Cauchy, for any $\epsilon > 0$, there exists $N \in \mathbf{N}$, such that if

$m, k \geq N$, then

$$d(x^m, x^k) = [\sum_{i=1}^{n}(\max_{a \leq t \leq b} |x_i^m(t) - x_i^k(t)|)^2]^{\frac{1}{2}} < \epsilon$$

so

$$\sum_{i=1}^{n}(\max_{a \leq t \leq b} |x_i^m(t) - x_i^k(t)|)^2 < \epsilon^2$$

$$\max_{a \leq t \leq b} |x_i^m(t) - x_i^k(t)| < \epsilon$$

$$|x_i^m(t) - x_i^k(t)| < \epsilon$$

So if i is fixed, then $x_i^m(t)$ is a Cauchy sequence of continuous functions. From example 2.13, we know that C[a,b] is complete. So there exists $x_i(t) \in C[a,b]$ such that

$$\max_{a \leq t \leq b}\{|x_i^m(t) - x_i(t)|\} \qquad \text{as} \quad m \to \infty$$

for any i=1,2,...,n.

Set $x = \{x_1(t), x_2(t), ..., x_n(t)\}$, $t \in [a,b]$, then $x \in \Omega$ since $x_i(t) \in C[a,b]$. Fix i, $i = 1, 2, ..., n$, and for any $\epsilon > 0$, there exists N_i such that if $m > N_i$, then

$$|x_i^m(t) - x_i(t)| < \frac{\epsilon}{\sqrt{n}}.$$

Choose $N = \max\{N_1, N_2, ..., N_n\}$, if $m > N$ then

$$\max_{a \leq t \leq b} |x_i^m(t) - x_i(t)| < \frac{\epsilon}{\sqrt{n}}$$

$$(\max_{a \leq t \leq b} |x_i^m(t) - x_i(t)|)^2 < \frac{\epsilon^2}{n}$$

$$\sum_{i=1}^{n}(\max_{a \leq t \leq b} |x_i^m(t) - x_i(t)|)^2 < \epsilon^2$$

$$d(x_m, x) = [\sum_{i=1}^{n} (\max_{a \leq t \leq b} |x_i^m(t) - x_i(t)|)^2]^{\frac{1}{2}} < \epsilon$$

This shows that $x_m \to x$, as $m \to \infty$.

So (Ω, d) is complete.

Now we need to set up a contraction mapping $F : \Omega \to \Omega$ is, namely there exists $0 < \alpha < 1$ such that for all $x = (x_i)$, $y = (y_i) \in \Omega$

$$d(Fx, Fy) \leq \alpha d(x, y)$$

The metric between Fx and Fy is:

$$d(Fx, Fy) = [\sum_{i=1}^{n} (\max_{a \leq t \leq b} | \int_{t_0}^{t} [f_i(s, x(s)) - f_i(s, y(s))] ds|)^2]^{\frac{1}{2}}.$$

Consider

$$| \int_{t_0}^{t} [f_i(s, x(s)) - f_i(s, y(s))] ds|.$$

It is easy to see

$$| \int_{t_0}^{t} [f_i(s, x(s)) - f_i(s, y(s))] ds| \leq |t - t_0| \max_{a \leq s \leq b} |f_i(s, x(s)) - f_i(s, y(s))|,$$

so

$$\max_{a \leq s \leq b} | \int_{t_0}^{t} [f_i(s, x(s)) - f_i(s, y(s))] ds| \leq |t - t_0| \max_{a \leq s \leq b} |f_i(s, x(s)) - f_i(s, y(s))|.$$

Then squaring both side, take the sum from i=1 to n, and take the square root we can get

$$d(Fx, Fy) \leq |t - t_0| d(f_i(s, x), f_i(s, y)).$$

Since f_i is continuous and $\frac{\partial f}{\partial x_i}$ is continuous for each $i = 1, 2, ..., n$, by Example 3.14 f_i satisfies Lipschitz condition: there exists $K > 0$ such that

$$d(f_i(s, x(s)), f_i(s, y(s))) \leq K \sum_{i=1}^{n} |x_i - y_i| \leq Kd(x, y)$$

for all $x = (x_i)$ and $y = (y_i)$ in Ω.

So

$$d(Fx, Fy) \leq |t - t_0| d(f_i(s, x), f_i(s, y)) \leq K|t - t_0| d(x, y)$$

If we let $t \in (t_0 - \frac{1}{K}, t_0 + \frac{1}{K})$, then $|t - t_0| < \frac{1}{K}$, and $K|t - t_0| < 1$.

Let $\alpha = K|t - t_0|$, then

$$d(Fx, Fy) \leq \alpha d(x, y)$$

and $0 < \alpha < 1$.

Thus, F is a contract mapping.

By Banach Fixed Point Theorem, F(x) has a unique fixed point.

□

5.2 Some Classical Fixed Point Theorems

THEOREM 5.5: Brouwer Fixed Point Theorem.

Any continuous map f of a closed ball $B^n \subseteq R^n$ to itself has a fixed point.

THEOREM 5.6: Leray-Schauder Fixed Point Theorem.

If D is a non-empty,convex , bounded and closed subset of Banach space B and $T : D \to D$ a compact and continuous map, then T has a fixed point in D.

DEFINITION 5.7: Convex:

A subspace A of a vector space X is convex if

$$\lambda x + (1-\lambda)y \in A,$$

for all $x, y \in A$, and all $\lambda \in [0,1]$.

EXAMPLE 5.8: Consider the Banach space $C[a,b]$, with a norm defined by

$$\|y(x)\| = \max_{x\in[a,b]} |y(x)|.$$

Suppose B_η is the ball of $C[a,b]$ defined by

$$B_\eta = \{y(x) \in C[a,b] : \|y\| \le \eta\}.$$

Then B_η is convex.

Proof. For any $y, z \in B_\eta$, and all $\lambda \in [0,1]$, $\lambda z + (1-\lambda)y$ is continuous and

$$\begin{aligned}
\|\lambda z + (1-\lambda)y\| &\le \|\lambda z\| + \|(1-\lambda)y\| \\
&= |\lambda|\|z\| + |1-\lambda|\|y\| \\
&= \lambda\|z\| + (1-\lambda)\|y\| \\
&\le \lambda\eta + (1-\lambda)\eta = \eta
\end{aligned}$$

□

THEOREM 5.9: Boundary Value Problem.

Assume that $f : [a,b] \times R \to R$ is continuous and satisfies

$$|f(x,y)| \le u + v|y|^r,$$

where u, v, r are real numbers and $0 < r < 1$. Then the Sturm-Liouville problem

$$L[y] = -[p(x)y']' + q(x)y = f(x, y(x))$$

with the boundary condition

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \beta_1 y(b) + \beta_2 y'(b) = 0$$

where $|\alpha_1| + |\alpha_2| \neq 0$, and $|\beta_1| + |\beta_2| \neq 0$, has a solution expressed via Green's function

$$y = \int_a^b G(x, s) f(s, y(s)) ds$$

where

$$G(x, s) = \begin{cases} -y_1(s)y_2(x)/p(x)W(y_1, y_2)(x), & a \leq s \leq x, \\ -y_1(x)y_2(s)/p(x)W(y_1, y_2)(x), & x \leq s \leq b. \end{cases}$$

y_1 and y_2 are two independent solutions of $L[y] = 0$, and y_1, y_2 satisfying the first and the second boundary condition respectively.

Proof. First, consider the Green's function. There are two properties of the Green's function: $p(x)W(y_1, y_2)(x)$ is a constant and $G(x, s)$ is uniformly continuous. Prove as below.

$$(pW)' = p'(y_1 y_2' - y_1 y_2') + p(y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1 y_2'')$$

$$= y_1[py_2']' - y_2[py_1']'$$

$$= y_1 q y_2 - y_2 q y_1 = 0.$$

So $p(x)W(y_1, y_2)(x)$ is a constant. Suppose $p(x)W(y_1, y_2)(x) = c$.

We already proved that Green's function is continuous and uniformly continuous in example 3.8.

Consider the Banach space $C[a,b]$, with a norm defined by

$$\|y(x)\| = \max_{x\in[a,b]} |y(x)|.$$

Suppose B_η is the ball of $C[a,b]$ defined by

$$B_\eta = \{y(x) \in C[a,b] : \|y\| \leq \eta\}.$$

It is clear that B_η is non-empty, bounded, closed and convex.

Define a mapping from B_η to $C[a,b]$:

$$y \longrightarrow Ty = \int_a^b G(x,s)f(s,y(s))ds.$$

Since y_1,y_2 is bounded on $[a,b]$, $G(x,s)$ is bounded on $[a,b]$, and there is a real number M, such that

$$|G(x,s)| \leq M.$$

Since $|f(x,y)| \leq u + v|y|^r$,

$$\|Ty\| = \max_{x\in[a,b]} |\int_a^b G(x,s)f(s,y(s))ds| \leq M * \max_{x\in[a,b]} \int_a^b |f(s,y(s))ds|$$

$$\leq M * \int_a^b |u + v|y(s)|^r|ds \leq M * [u(b-a) + v\|y\|^r(b-a)].$$

So,

$$\frac{\|Ty\|}{\|y\|} \leq \frac{M * [u(b-a) + v\|y\|^r(b-a)]}{\|y\|} = \frac{Mu(b-a)}{\|y\|} + v(b-a)\|y\|^{r-1}.$$

Since $0 < r < 1$,

$$\frac{\|Ty\|}{\|y\|} \longrightarrow 0 \qquad \text{as} \quad \|y\| \longrightarrow \infty.$$

So,

$$\|Ty\| < \|y\|,$$

there exists some large $\eta > 0$, such that

$$\|Ty\| < \eta \qquad \text{for all} \quad y \in B_\eta.$$

Namely, $T : B_\eta \to B_\eta$.

Then, we prove T is continuous.

Define a metric on $C[a, b]$ by

$$d(y, z) = \max_{x \in [a,b]} |y(x) - z(x)|$$

Consider $f : [a, b] \times [-\eta, \eta] \to R$, since f is continuous and $[a, b] \times [-\eta, \eta]$ is a compact set, f is uniformly continuous on $[a, b] \times [-\eta, \eta]$.

Since $f(x, y)$ is uniformly continuous on a closed rectangle, for any $\epsilon > 0$ there exists a δ, such that for any fixed $z \in B_\eta$, if $d(y, z) < \epsilon$, then

$$d(f(x, y), f(x, z)) < \frac{\epsilon}{M|b - a|}.$$

And

$$\begin{aligned} d(Ty, Tz) = \max_{x \in [a,b]} | \int_a^b G(x, s)[f(s, y) - f(s, z)]ds| \leq \\ |b - a| \max_{x \in [a,b]} |G(x, s)| * \max_{x \in [a,b]} |[f(x, y) - f(x, z)]| \\ \leq |b - a|M * \frac{\epsilon}{M|b - a|} = \epsilon \end{aligned}$$

So T is continuous on $[a, b] \times [-\eta, \eta]$.

If T is completely continuous, then T has a fixed point on$[a, b]$ by Leray-

Schauder Fixed Point Theorem.

We need to show that the mapping T maps any bounded subset of $C[a,b]$, suppose is B_η, into a relatively compact subset of $C[a,b]$.

In order to prove this, consider Ascoli's Theorem. If we can prove that the sequence $\{T_n\}$ is uniformly bounded and equicontinuous, then by Ascoli's Theorem, there exists a subsequence $\{T_{n_k}\}$ that converge uniformly.

We have already proved that Ty is bounded in a closed ball B_η, so the first condition was satisfied.

Since $f(x,y)$ is bounded on a closed rectangle $[a,b] \times [-\eta,\eta]$, there exists a P, such that for all $(x,y) \in [a,b] \times [-\eta,\eta]$,

$$|f(x,y)| \leq P.$$

Since $G(x,s)$ is uniformly continuous, for any $\epsilon > 0$, there exists δ such that if $|x_1 - x_2| < \delta$, then

$$|G(x_1,s) - G(x_2,s)| \leq \frac{\epsilon}{P|b-a|}.$$

Then

$$\begin{aligned}
|T_n(x_1) - T_n(x_2)| &= |\int_a^b [G(x_1,s) - G(x_2,s)] f(s,y(s)) ds| \\
&\leq |b-a| \max_{x\in[a,b]} |[G(x_1,s) - G(x_2,s)]||f(s,y)| \\
&\leq |b-a| \frac{\epsilon}{P|b-a|} * P = \epsilon.
\end{aligned}$$

Since n is arbitrary,

$$|T_n(x_1) - T_n(x_1)| < \epsilon$$

whenever $|x_1 - x_2| < \delta$ for all n.

Therefore, T is equicontinuous.

By Ascoli's Theorem, there exists a subsequence $\{T_{n_k}\}$ that converge uniformly. It means that T maps B_η, into a relatively compact set B_η. So T is completely continuous.

By Leray-Schauder Fixed Point Theorem, T has a fixed point, suppose is y:

$$y = Ty = \int_a^b G(x,s)f(s,y(s))ds$$

Show that y is a solution of the Sturm-Liouville problem by showing that y satisfying

$$L[y] = -[p(x)y']' + q(x)y = f(x,y(x))$$

and the boundary condition

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0, \qquad \beta_1 y(1) + \beta_2 y'(1) = 0.$$

Since

$$y(x) = \int_a^b G(x,s)f(s,y(s))ds$$

$$= -\int_a^x \frac{y_1(s)y_2(x)f(s,y(s))}{p(x)W(y_1,y_2)(x)}ds - \int_x^b \frac{y_1(x)y_2(s)f(s,y_s)}{p(x)W(y_1,y_2)(x)}ds$$

$p(x)W(y_1,y_2)(x) = c$, is a constant.

Substituted $p(x)W(y_1,y_2)(x)$ by c, and multiply both side by c, then

$$cy(x) = -\int_a^x y_1(s)y_2(x)f(s,y(s))ds - \int_x^b y_1(x)y_2(s)f(s,y(s))ds$$

Taking derivative with respect to x,

$$cy'(x) = -\int_a^x y_1(s)y_2'(x)f(s,y(s))ds - \int_x^b y_1'(x)y_2(s)f(s,y(s))ds$$

by Leibniz's rule.

Multiplying $-p(x)$, and then take the derivative again,

$$\begin{aligned}-c[p(x)y'(x)]' &= (py_2')'\int_a^x y_1(s)f(s,y(s))ds + py_2'y_1f + (py_1')'\int_x^b y_2(s)f(s,y(s))ds - py_1'y_2f \\ &= (py_2')'\int_a^x y_1(s)f(s,y(s))ds + (py_1')'\int_x^b y_2(s)f(s,y(s))ds + py_2'y_1f - py_1'y_2f\end{aligned}$$

Since

$$py_2'y_1f - py_1'y_2f = p(x)W(y_1,y_2)(x)f = cf$$

$$\begin{aligned}-[p(x)y'(x)]' + q(x)y(x) &= \frac{1}{c}(py_2')'\int_a^x y_1(s)f(s,y(s))ds - \frac{1}{c}qy_2\int_a^x y_1(s)f(s,y(s)ds) \\ &\quad + \frac{1}{c}(py_1')'\int_x^b y_2(s)f(s,y(s))ds \\ &\quad - \frac{1}{c}qy_1\int_x^b y_2(s)f(s,y(s))ds + f(x,y(x)) \\ &= \frac{1}{c}[(py_2')' - qy_2]\int_a^x y_1(s)f(s,y(s))ds \\ &\quad + \frac{1}{c}[(py_1')' - qy_1]\int_x^b y_2(s)f(s,y(s))ds + f(x,y(x))\end{aligned}$$

Since y_1, y_2 are the solution of $L[y]=0$, the above equation equal $f(x,y(x))$.

$$y(x) = \frac{1}{c}[-\int_a^x y_1(s)y_2(x)f(s,y(s))ds - \int_x^b y_1(x)y_2(s)f(s,y(s))ds]$$

$$y(0) = \frac{1}{c}[0 - \int_a^b y_1(0)y_2(s)f(s,y(s))ds] = -\frac{1}{c}\int_a^b y_1(0)y_2(s)f(s,y(s))ds$$

$$y'(x) = \frac{1}{c}[-\int_a^x y_1(s)y_2'(x)f(s,y(s))ds - \int_x^b y_1'(x)y_2(s)f(s,y(s))ds]$$

$$y'(0) = \frac{1}{c}[0 - \int_a^b y_1'(0)y_2(s)f(s,y(s))ds] = -\frac{1}{c}\int_a^b y_1'(0)y_2(s)f(s,y(s))ds$$

Since y_1 satisfying the first boundary condition,

$$\alpha_1 y(a) + \alpha_2 y'(a) = -\frac{1}{c}\int_a^b [\alpha_1 y_1(a) - \alpha_2 y_1'(a)]y_2(s)f(s,y(s))ds = 0$$

Similarly, $y(x)$ satisfying the second boundary condition by

$$\beta_1 y_2(b) + \beta_2 y_2'(b) = 0.$$

□

5.3 Maximum-Area Problem

Among all curves of length l in the xy-plane, find the one which encloses the largest area. Suppose the center of the curve is (0,0), and the x-intercept is (-a,0), (a,0). Then the question is: Find the maximum of the functional $J[y] = \int_{-a}^{a} y dx$, with conditions $y(-a) = y(a) = 0$, and $K[y] = \int_{-a}^{a} \sqrt{1+y'^2}dx = l$.

Using Lagrange multiplier, form the functional

$$J[y] + \lambda K[y] = \int_{-a}^{a} (y + \lambda\sqrt{1+y'^2})dx$$

the corresponding Euler's Equation is

$$1 + \lambda\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0$$

Integrate both sides,

$$x + \lambda\frac{y'}{\sqrt{1+y'^2}} = C_1$$

or

$$(C_1 - x)^2 = \lambda^2 \frac{y'^2}{1 + y'^2}$$

$$\frac{\lambda^2}{(C_1 - x)^2} = \frac{1 + y'^2}{y'^2} = \frac{1}{y'^2} + 1$$

$$\frac{\lambda^2 - (C_1 - x)^2}{(C_1 - x)^2} = \frac{1}{y'^2}$$

$$y'^2 = \frac{(C_1 - x)^2}{\lambda^2 - (C_1 - x)^2}$$

$$y' = \frac{dy}{dx} = \frac{(C_1 - x)}{\sqrt{\lambda^2 - (C_1 - x)^2}}$$

$$dy = \frac{(C_1 - x)}{\sqrt{\lambda^2 - (C_1 - x)^2}} dx$$

Integrete both sides

$$y - C_2 = \int \frac{(C_1 - x)}{\sqrt{\lambda^2 - (C_1 - x)^2}} dx$$

Let $u = \lambda^2 - (C_1 - x)^2$, $du = 2(C_1 - x)dx$

$$2(y - C_2) = \int u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}} = 2\sqrt{\lambda^2 - (C_1 - x)^2}$$

$$(y - C_2)^2 = \lambda^2 - (C_1 - x)^2$$

$$(y - C_2)^2 + (x - C_1)^2 = \lambda^2$$

So the curve is a circle.

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