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The Frenet Frame and Space Curves

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THE FRENET FRAME AND SPACE CURVES

A Master's Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

Essential to the study of space curves in Differential Geometry is the Frenet frame. In this thesis we generate the Frenet equations for the second, third, and fourth dimensions using the Gram-Schmidt process, which allows us to present the form of the Frenet equatins for n-dimensions. We highlight several key properties that arise from the Frenet equations, expound on the class of curves with constant curvature ratios, as well as characterize spherical curves up to the fourth dimension. Methods for generalizing properties and characteristics of curves in varying dimensions should be handled with care, since the structure of curves often differ in progressing dimensions.

KEYWORDS: Frenet frame, Frenet equations, orthogonal, tangent vector, normal vector, binormal vector, curvature, torsion, ccr-curve, spherical curve

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1. INTRODUCTION

Fundamentally, classical Differential Geometry of curves is the study of the local properties of curves. To be specific, the local properties determine the behavior of a curve in the neighborhood of a point. We can think of a curve as a path marking out the trail that an object makes as it travels about in space. The key structure of the curve that we want to consider is its shape. Questions of the following nature might arise: is the curve under consideration straight or bending; is its bending smooth or sharp; and does the curve protrude into higher dimensions?

Essential to the study of space curves are the Frenet equations, which in the three-dimensional case use curvature and torsion to express the derivatives of the three vector fields $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ composing the Frenet frame in terms of the vector fields themselves. We can find an analogue for higher dimensions, in which we simply call each n-dimensional bend in the curve the curvature for each respective dimension under consideration. In the present paper, we will prove the Frenet equations up to the fourth dimension and demonstrate the form for n-dimensions.

There are a handful of perspectives from which a space curve can be studied. Imagine the trail a bicyclist might leave behind him on a muddy road. We can call that path a curve in two-dimensions. More colloquially, a curve can be thought of as the trip that is taken by a moving particle. The most common ways to parameterize such a trip would be either as a function of time or of distance traveled. For our purposes, it is convenient to study curves through the lens of arclength. And since we are only concerned about analyzing the shape of the space curve, we can forgo having to consider the speed of the particle as it completes its path. For simplicity, we can happily force the particle to move along its path at unit-speed.

While initially we might envision a curve similar to the markings we make with a pen on a flat sheet of paper, such a plane is not the only surface where curves reside. Particles can move about and form curves on essentially any surface. In fact, if our bicyclist in our illustration rode a significant distance, the resulting curve from the trail formed in the mud would be more relatable to a spherical curve, since our planet is spherical in shape. Curves on varying surfaces can be characterized and their resulting behaviors studied.

2. THE FRENET EQUATIONS

A curve with linearly independent $(n-1)$ derivatives in any n-dimensional space, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t))$, determines the position of a particle at any moment. We say the curve is regular if the derivative of the curve $\alpha'(t) \neq 0$. This derivative is the tangent vector to the curve at the point $\alpha(t)$ and determines the velocity of the particle traveling along the curve. However, since we are interested in the geometry of the curve rather than the applications to Physics, we want to analyze the curve in terms of distance rather than time (see [1], [2], and [4]). So, we reparameterize the coordinates of the curve $\alpha(t)$ in terms of arclength $s(t)$ by

$$s(t) = \int_{t_0}^{t} \|\alpha'(u)\| du.$$

Note that by the Fundamental Theorem of Calculus it follows that

$$\frac{ds}{dt} = \left\| \frac{d\alpha}{dt} \right\| > 0.$$

Since $\alpha(s)$ is a regular curve and it's derivative is strictly positive, then $s(t)$ has a unique inverse. So, we can always parameterize the curve by arclength. Henceforth, every curve discussed will be paremeterized by arclength, where the values of s are contained in the interval I. To denote continuity up to the kth derivative, we use the notation C^k.

DEFINITION 2.1: Let $\alpha : I \to \mathbb{R}^n$ be a curve parameterized by arclength $s \in I$, such that $\alpha \in C^n(I)$. Define the *unit tangent vector* as

$$\mathbf{T}(s) = \frac{d\alpha}{ds} = \frac{\frac{d\alpha}{dt}}{\frac{ds}{dt}}.$$

The Frenet equations define the Frenet frame and provide information about the local behavior of the curve $\alpha(s)$ in a neighborhood of a point on the curve. It forms the orthonormal basis for a set of vector fields. The Frenet equations describe the derivatives of these orthogonal vector fields as the rate of change of these vector fields in terms of the vector fields themselves. We can build the Frenet frame for n-dimensions.

2.1 Two Dimensions

Let $\alpha \in C^2(I)$ be a curve parameterized by arclength. Then $\alpha' = \mathbf{T}$ is the unit tangent vector, and $\mathbf{T} \cdot \mathbf{T} = 1$. Differentiation gives

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{T} = 0. \tag{2.1}$$

Two vectors are orthogonal to each other when their dot product is zero. Excluding the case that $\frac{d\mathbf{T}}{ds} = \mathbf{0}$, equation (2.1) says the derivative of the tangent vector is orthogonal to the tangent vector itself, so we can uniquely write

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \tag{2.2}$$

where $\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| \geq 0$ and $\mathbf{N}$ is orthogonal to $\mathbf{T}$. The unit vector $\mathbf{N}$ is called the *principal unit normal.* The constant, κ, is called the *curvature.* Curvature measures the failure of a curve to be a straight line, providing a measure of bending of the curve.

THEOREM 2.2: Let $\alpha : I \to \mathbb{R}^2$ be a curve parameterized by arclength, such that $\alpha \in C^2(I)$. The Frenet equations are

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T}.$$

Proof. The first Frenet equation is equation (2.2).

Since $\mathbf{N} \cdot \mathbf{T} = 0$, differentiation gives

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{T} + \frac{d\mathbf{T}}{ds} \cdot \mathbf{N} = 0.$$

Substituting equation (2.2), we obtain

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{T} + \kappa\mathbf{N} \cdot \mathbf{N} = 0.$$

Since $\mathbf{N}$ is a unit vector, $\mathbf{N} \cdot \mathbf{N} = 1$. Thus, the equation above simplifies to

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{T} = -\kappa. \tag{2.3}$$

Since $\mathbf{N} \cdot \mathbf{N} = \mathbf{1}$, differentiation gives

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{N} = \mathbf{0}. \tag{2.4}$$

Then $\frac{d\mathbf{N}}{ds}$ is orthogonal to $\mathbf{N}$, while $\mathbf{N}$ is orthogonal to $\mathbf{T}$. So, we can write $\frac{d\mathbf{N}}{ds}$ as a linear combination of these orthogonal vectors, such that

$$\frac{d\mathbf{N}}{ds} = \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}\right)\mathbf{T} - \left(\frac{\mathbf{dN}}{\mathbf{ds}} \cdot \mathbf{N}\right)\mathbf{N}. \tag{2.5}$$

Substituting equations (2.3) and (2.4), gives us

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}. \tag{2.6}$$

Thus, completing the proof of the Frenet equations for the second dimension. $\square$

2.2 Three Dimensions

The Frenet frame considered at a point on a curve in Euclidean $\mathbb{R}^3$-space is formed by three associated orthogonal unit vector fields, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, which form a basis for these vector fields along the curve. We call the third vector, $\mathbf{B}$ the *binormal vector*, which is orthogonal to both the tangent and normal vectors.

The third dimension introduces a second curvature, called *torsion*, commonly denoted as τ. Torsion measures the failure of a curve to lie in a two-dimensional plane. In other words, torsion is the scalar measure of the extension of a curve protruding from the two-dimensional plane. We will make use of this curvature in our proof of the Frenet equations for three-dimensions.

THEOREM 2.3: Let $\alpha : I \to \mathbb{R}^3$ be a curve parametrized by arclength, such that $\alpha \in C^3(I)$. There is a unique unit vector $\mathbf{B} = \mathbf{B}(s)$ orthogonal to $\mathbf{T}$ and $\mathbf{N}$, such that the Frenet equations are

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Proof. We have that $\mathbf{T}$, $\mathbf{N}$ and κ are defined in precisely the same manner as in two dimensions as given by equation (2.2).

Since $\mathbf{N} \cdot \mathbf{N} = 1$, differentiation gives

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{N} = 0. \tag{2.7}$$

Thus, $\mathbf{N}$ and $\frac{d\mathbf{N}}{ds}$ are orthogonal.

Consider the vector

$$\mathbf{u} = \frac{d\mathbf{N}}{ds} - \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}\right)\mathbf{T} - \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{N}\right)\mathbf{N}.$$

Note $\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}\right)\mathbf{T}$ is the projection of the vector $\frac{d\mathbf{N}}{ds}$ onto $\mathbf{T}$, and $\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}\right)\mathbf{N}$ is the projection of the vector $\frac{d\mathbf{N}}{ds}$ onto $\mathbf{N}$.

We show that $\mathbf{u}$ is orthogonal to $\mathbf{T}$ and $\mathbf{N}$ by verifying that $\mathbf{u}\cdot\mathbf{T}=0$ and $\mathbf{u}\cdot\mathbf{N}=0$.

Consider

$$\mathbf{u}\cdot\mathbf{T}=\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}\right)\mathbf{T}\cdot\mathbf{T}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}\right)\mathbf{N}\cdot\mathbf{T}.$$

From the relationships of the vectors $\mathbf{T}$ and $\mathbf{N}$, we verify the equation simplifies to

$$\mathbf{u}\cdot\mathbf{T}=\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}-\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}=0.$$

Consider

$$\mathbf{u}\cdot\mathbf{N}=\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}\right)\mathbf{T}\cdot\mathbf{N}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}\right)\mathbf{N}\cdot\mathbf{N}.$$

Again from the relationships of the vectors $\mathbf{T}$ and $\mathbf{N}$, we verify the equation simplifies to

$$\mathbf{u}\cdot\mathbf{N}=\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}-\frac{d\mathbf{N}}{ds}\cdot\mathbf{N}=0.$$

Since $\mathbf{u}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$, we define $\mathbf{u}$ as a scalar multiple of $\mathbf{B}$. By equation (2.7),

$$\mathbf{u}=\tau\mathbf{B}=\frac{d\mathbf{N}}{ds}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}\right)\mathbf{T}, \tag{2.8}$$

where $|\tau|=\left\|\frac{d\mathbf{N}}{ds}-\left(\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}\right)\mathbf{T}\right\|$. We require $\mathbf{det}[\mathbf{T}\,\mathbf{N}\,\mathbf{B}]>0$; called positive orientation, which holds if and only if $\mathbf{B}=\mathbf{T}\times\mathbf{N}$.

Since $\mathbf{N}\cdot\mathbf{T}=0$, differentiation gives

$$\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}=-\frac{d\mathbf{T}}{ds}\cdot\mathbf{N}=-\kappa\mathbf{N}\cdot\mathbf{N}.$$

Thus,

$$\frac{d\mathbf{N}}{ds}\cdot\mathbf{T}=-\kappa. \tag{2.9}$$

Hence, substituting equation (2.9) into equation (2.8), gives

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}. \tag{2.10}$$

Since $\mathbf{N} \cdot \mathbf{B} = 0$, differentiation gives

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{B} = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

By equation (2.10),

$$(-\kappa\mathbf{T} + \tau\mathbf{B}) \cdot \mathbf{B} = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}.$$

This implies

$$\tau = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}.$$

Hence

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}. \tag{2.11}$$

Thus, completing the proof of the Frenet equations for the third dimension. □

2.3 Four Dimensions

Let $\alpha \in C^4(I)$ be a unit-speed curve in $\mathbb{R}^4$. To build the Frenet 4-frame, consider four vectors orthogonal to one another, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathbf{D}\}$. The fourth dimension also introduces a third curvature, we will denote σ (see [7]). We will make use of this curvature in our proof of the Frenet equations for four-dimensions.

We have that $\mathbf{T}$, $\mathbf{N}$ and κ are defined in precisely the same manner as in two dimensions as given by equation (2.2). As was shown for three dimensions, the vector orthogonal to $\mathbf{T}$ and $\mathbf{N}$ can be written as

$$\frac{d\mathbf{N}}{ds} - \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}\right)\mathbf{T}.$$

Hence we let $\tau = \left\| \frac{d\mathbf{N}}{ds} - \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}\right) \mathbf{T} \right\| \geq 0$, and we write

$$\tau \mathbf{B} = \frac{d\mathbf{N}}{ds} - \left(\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}\right) \mathbf{T},$$

where $\mathbf{B}$ is a unique unit vector. Note τ induces a second curvature function and defines

$$\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}, \tag{2.12}$$

similar to the three dimensional Frenet equation given by (2.10).

Since, $\mathbf{B} \cdot \mathbf{B} = 1$, differentiation gives

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{B} = 0. \tag{2.13}$$

Thus, $\frac{d\mathbf{B}}{ds}$ and $\mathbf{B}$ are orthogonal. The vector orthogonal to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ is written

$$\frac{d\mathbf{B}}{ds} - \left(\frac{d\mathbf{B}}{ds} \cdot \mathbf{T}\right) \mathbf{T} - \left(\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}\right) \mathbf{N} - \left(\frac{d\mathbf{B}}{ds} \cdot \mathbf{B}\right) \mathbf{B}.$$

Note $(\frac{d\mathbf{B}}{ds} \cdot \mathbf{T})\mathbf{T}$ is the projection of vector $\frac{d\mathbf{B}}{ds}$ onto $\mathbf{T}$, and $(\frac{d\mathbf{B}}{ds} \cdot \mathbf{N})\mathbf{N}$ is the projection of $\frac{d\mathbf{B}}{ds}$ onto $\mathbf{N}$, and $(\frac{d\mathbf{B}}{ds} \cdot \mathbf{B})\mathbf{B}$ is the projection of $\frac{d\mathbf{B}}{ds}$ onto $\mathbf{B}$.

By equation (2.13), we define the vector orthogonal to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ as

$$\sigma \mathbf{D} = \frac{d\mathbf{B}}{ds} - \left(\frac{d\mathbf{B}}{ds} \cdot \mathbf{T}\right) \mathbf{T} - \left(\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}\right) \mathbf{N}, \tag{2.14}$$

where σ can be positive or negative. We require $\mathbf{det}[\mathbf{T}\ \mathbf{N}\ \mathbf{B}\ \mathbf{D}] > 0$; called positive orientation.

Since $\mathbf{B} \cdot \mathbf{T} = 0$ differentiation gives

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = -\mathbf{B} \cdot \frac{d\mathbf{T}}{ds} = -\kappa \mathbf{B} \cdot \mathbf{N}.$$

Thus,

$$\frac{d\mathbf{B}}{ds} \cdot \mathbf{T} = 0. \tag{2.15}$$

Since $\mathbf{B} \cdot \mathbf{N} = 0$, differentiating gives

$$\frac{d\mathbf{B}}{ds} = -\mathbf{B} \cdot \frac{d\mathbf{N}}{ds} = -\mathbf{B} \cdot (-\kappa\mathbf{T} + \tau\mathbf{B}).$$

Thus,

$$\frac{d\mathbf{B}}{ds} = -\tau \tag{2.16}$$

Hence, substituting equations (2.15) and (2.16) into equation (2.14), we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} + \sigma\mathbf{D}. \tag{2.17}$$

Since every vector can be written as a linear combination of orthogonal vectors, we can write

$$\frac{d\mathbf{D}}{ds} = \left(\frac{d\mathbf{D}}{ds} \cdot \mathbf{T}\right)\mathbf{T} + \left(\frac{d\mathbf{D}}{ds} \cdot \mathbf{N}\right)\mathbf{N} + \left(\frac{d\mathbf{D}}{ds} \cdot \mathbf{B}\right)\mathbf{B} + \left(\frac{d\mathbf{D}}{ds} \cdot \mathbf{D}\right)\mathbf{D}. \tag{2.18}$$

Since $\mathbf{D} \cdot \mathbf{D} = 1$, differentiation gives

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{D} = 0. \tag{2.19}$$

Since $\mathbf{D} \cdot \mathbf{T} = 0$, differentiation gives

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{T} = 0. \tag{2.20}$$

Since $\mathbf{D} \cdot \mathbf{N} = 0$, differentiation gives

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{N} + \frac{d\mathbf{N}}{ds} \cdot \mathbf{D} = 0.$$

By equation (2.12),

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{N} + (-\kappa \mathbf{T} + \tau \mathbf{B}) \cdot \mathbf{D} = 0.$$

Hence

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{N} = 0. \tag{2.21}$$

Since $\mathbf{D} \cdot \mathbf{B} = 0$, differentiation gives

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{B} + \frac{d\mathbf{B}}{ds} \cdot \mathbf{D} = 0.$$

By equation (2.17),

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{B} + (-\tau \mathbf{N} + \sigma \mathbf{D}) \cdot \mathbf{D} = 0.$$

Hence

$$\frac{d\mathbf{D}}{ds} \cdot \mathbf{B} = -\sigma. \tag{2.22}$$

Hence, substituting equations (2.19), (2.20), (2.21), and (2.22) back into equation (2.18), we have

$$\frac{d\mathbf{D}}{ds} = -\sigma \mathbf{B}.$$

Thus, constructing the Frenet equations for the fourth dimension.

THEOREM 2.4: Let $\alpha : I \to \mathbb{R}^4$ be a curve parameterized by arclength, such that $\alpha \in C^4(I)$. The Frenet equations are

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} + \sigma \mathbf{D}, \qquad \frac{d\mathbf{D}}{ds} = -\sigma \mathbf{B}.$$

2.4 n-Dimensions

Notice for the third and fourth dimensions, we applied the Gram-Schmidt process to generate the Frenet equations. To generalize the Frenet equations analogously for n-dimensions, we simply iterate the same strategy. For the curve $\alpha : I \to \mathbb{R}^n$, parameterized by arclength, such that $\alpha \in C^n(I)$, the Frenet n-frame is a collection of orthogonal vectors $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$, such that $\mathbf{det}[\mathbf{e}_1 \ldots \mathbf{e}_n] > 0$. Then there are curvature functions $\kappa_1, \ldots, \kappa_{n-1}$, such that $\kappa_1, \ldots, \kappa_{n-2} > 0$. The Frenet equations are given in matrix form by

$$\begin{bmatrix} \mathbf{e}_1{}' \\ \mathbf{e}_2{}' \\ \vdots \\ \vdots \\ \mathbf{e}_{n-1}{}' \\ \mathbf{e_n}{}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \ldots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \ddots & \vdots \\ 0 & -\kappa_2 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \kappa_{n-1} \\ 0 & \ldots & \ldots & 0 & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \vdots \\ \mathbf{e}_{n-1} \\ \mathbf{e_n} \end{bmatrix}.$$

2.5 Resulting Properties

The Frenet frame describes the structure of a curve depending on the dimension of the Euclidean subspace within which it is embedded. The Frenet equations for the third and fourth dimensions insist on the following properties, respectively.

PROPOSITION 1: Given a curve $\beta \in C^3(I)$, the graph of the curve, $\{\beta\}$, is a planar curve if and only if $\tau = 0$.

Proof. Suppose $\{\beta\}$ is a planar curve. Then for every value of s, there exists a constant position vector, $\mathbf{p}$, and a constant normal vector $\mathbf{q}$, such that $(\beta(s) - \mathbf{p}) \cdot \mathbf{q} = 0$. The derivative is $\beta'(s) \cdot \mathbf{q} + (\beta(s) - \mathbf{p}) \cdot \mathbf{q}' = \mathbf{0}$, hence

$$\beta'(s) \cdot \mathbf{q} = 0.$$

Taking the derivative again, we have $\beta'(s) \cdot \mathbf{q}' + \beta''(s) \cdot \mathbf{q} = 0$. By the same reasoning above,

$$\beta''(s) \cdot \mathbf{q} = 0.$$

Thus,

$$\beta'(s) \cdot \mathbf{q} = \beta''(s) \cdot \mathbf{q} = 0 \tag{2.23}$$

Now $\beta'(s) = \mathbf{T}$. So, $\beta''(s) = \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$, by Theorem 2.3. Equation (2.23) translates to

$$\mathbf{T} \cdot \mathbf{q} = \mathbf{N} \cdot \mathbf{q} = 0. \tag{2.24}$$

Since $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$, without loss of generality we may assume $\mathbf{q} = \mathbf{B}$. By Theorem 2.3, the derivative of $\mathbf{B}$ is

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} = \mathbf{0}.$$

Since $\kappa > 0$, by assumption, and $\mathbf{N} \neq 0$, it follows that $\tau = 0$.

For the converse, consider $\beta \in C^3(I)$ to be some curve. Assume $\tau = 0$. By Theorem 2.3,

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N} = \mathbf{0}. \tag{2.25}$$

Thus, $\mathbf{B}$ is a constant vector.

We claim that β lies in the plane orthogonal to $\mathbf{B}$, passing through $\beta(0)$. Define

$$f(s) = (\beta(s) - \beta(0)) \cdot \mathbf{B}.$$

The derivative is

$$\frac{df}{ds} = \beta'(s) \cdot \mathbf{B} + \beta(s) \cdot \frac{d\mathbf{B}}{ds}.$$

By equation (2.25),

$$\frac{df}{ds} = \mathbf{T} \cdot \mathbf{B} = 0.$$

This implies that $f(s)$ has the same constant value for all s. Clearly, $f(0) = (\beta(0) - \beta(0)) \cdot \mathbf{B} = 0$. So, $f(s)$ is identically zero, such that for all s, $(\beta(s) - \beta(0)) \cdot \mathbf{B} = 0$. Thus, the curve $\beta(s)$ is a planar curve, which lies in the plane orthogonal to $\mathbf{B}$. $\square$

PROPOSITION 2: Given a curve $\beta(s) \in C^4$, the graph of the curve $\{\beta\}$, is a three-dimensional curve if and only if $\sigma = 0$.

Proof. Suppose $\{\beta\}$ lies in a three-dimensional subspace. Then there exists a constant position vector, $\mathbf{p}$ and a constant normal vector $\mathbf{q}$, such that $(\beta(s) - \mathbf{p}) \cdot \mathbf{q} = 0$. When we take two derivatives, we have the same result as in equation (2.24),

$$\mathbf{T} \cdot \mathbf{q} = \mathbf{N} \cdot \mathbf{q} = 0.$$

Taking the derivative a third time, we have

$$\mathbf{N} \cdot \mathbf{q}' + \frac{d\mathbf{N}}{ds} \cdot \mathbf{q} = 0.$$

By Theorem 2.4 and the fact that $\mathbf{q}$ is a constant vector,

$$\frac{d\mathbf{N}}{ds} \cdot \mathbf{q} = (-\kappa\mathbf{T} + \tau\mathbf{B}) \cdot \mathbf{q} = -\kappa\mathbf{T} \cdot \mathbf{q} + \tau\mathbf{B} \cdot \mathbf{q} = 0.$$

Since $\mathbf{T} \cdot \mathbf{q} = 0$, the equation above simplifies to

$$\mathbf{B} \cdot \mathbf{q} = 0.$$

Since $\mathbf{D}$ is orthogonal to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, without loss of generality we may assume $\mathbf{q} = \mathbf{D}$. Since $\mathbf{D}$ is a constant, its derivative is zero. By Theorem 2.4, the derivative of $\mathbf{D}$ is

$$\frac{d\mathbf{D}}{ds} = -\sigma\mathbf{B} = \mathbf{0}.$$

Since $\kappa > 0$ and $\tau \neq 0$, it follows that $\sigma = 0$.

For the converse, consider $\beta \in C^4(I)$ to be some curve. Assume $\sigma = 0$. By Theorem 2.4,

$$\frac{d\mathbf{D}}{ds} = -\sigma\mathbf{B} = \mathbf{0}. \tag{2.26}$$

Thus, $\mathbf{D}$ is a constant vector. We show that $\beta(s)$ lies in the hyperplane orthogonal to $\mathbf{D}$, passing through $\beta(0)$. Define

$$f(s) = (\beta(s) - \beta(0)) \cdot \mathbf{D}.$$

The derivative is

$$\frac{df}{ds} = \beta'(s) \cdot \mathbf{D} + \beta(s) \cdot \frac{d\mathbf{D}}{ds}.$$

By equation (2.26),

$$\frac{df}{ds} = \mathbf{T} \cdot \mathbf{D} = 0.$$

This implies that $f(s)$ has the same constant value for all s. Clearly, $f(0) = (\beta(0) - \beta(0)) \cdot \mathbf{D} = 0$. So, $f(s)$ is identically zero, such that for all s, $(\beta(0) - \beta(0)) \cdot \mathbf{D} = 0$. Thus, the curve $\{\beta\}$ lies entirely in the third dimension, which lies in the hyperplane orthogonal to $\mathbf{D}$. $\square$

3. FUNDAMENTAL THEOREM OF CURVES

The Fundamental Theorem of Curves involves determining the existence and uniqueness of a curve whose curvature is given by functions of its arclength. As is suggested by its title, the Fundamental Theorem of Curves is key to the theory of curves in Differential Geometry. It characterizes the curve under consideration with respect to the curve's position in space (see [6]).

We begin with given curvatures. We want to show that we can derive a Frenet frame and a Frenet curve, from those given curvatures. In order to carry out the proof of the Fundamental Theorem of Curve Theory, we need to make use of the Existence and Uniqueness Theorem for a system of Ordinary Differential Equations (see [5]).

EXISTENCE AND UNIQUENESS THEOREM FOR LINEAR SYSTEMS: If the entries of the square matrix $A(t)$ are continuous on an open interval I containing t_0, then the initial value problem $x' = A(t)x$, where $x(t_0) = x_0$, has one and only one solution $x(t)$ on the interval I.

THE FUNDAMENTAL THEOREM OF CURVES: Given functions $\lambda_1(s), \lambda_2(s), \ldots, \lambda_{n-1}(s)$, such that $\lambda_1(s), \lambda_2(s), \ldots, \lambda_{n-2}(s) > 0$, then there exists a curve $\alpha : I \to \mathbb{R}^n$ parameterized by arclength, where $\lambda_i = \kappa_i \in C^{n-1-i}$. Moreover, two oriented curves in Euclidean $\mathbb{R}^n$ space, having the same curvature functions, are congruent under an orientation preserving rigid motion.

Proof. The given curvatures can be arranged in the following matrix form:

$$K = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \dots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \ddots & \vdots \\ 0 & -\kappa_2 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \kappa_{n-1} \\ 0 & \dots & \dots & 0 & -\kappa_{n-1} & 0 \end{bmatrix}$$

Let $F(s) = (\mathbf{e}_1(s) \dots \mathbf{e}_n(s))^{\mathbf{T}}$, where the e_i's are unknown vector-valued functions to be determined. Assume $F(s_0) = I$. For the given matrix-valued function K, the Frenet frame for n-dimensions is equivalent to the matrix equation

$$F' = KF. \tag{3.27}$$

Since this is a system of linear first-order differential equations, by the Existence and Uniqueness Theorem for Linear Systems, there exists a unique solution $F(s)$. Multiplying equation (3.27) through by the transpose matrix, we have

$$(FF^T)' = F'F^T + F(F^T)'.$$

Hence

$$(FF^T)' = F'F^T + F(F')^T. \tag{3.28}$$

Substituting equation (3.27) into equation (3.28),

$$(FF^T)' = KFF^T + F(KF)^T.$$

Invoking the property from Linear Algebra that $(AB)^T = B^T A^T$, we have

$$(FF^T)' = KFF^T + FF^T K^T. \tag{3.29}$$

This can be viewed as a differential equation for the unknown function FF^T.

Since $F(s_0)(F(s_0))^T = \mathbf{I}$, then by the Existence and Uniqueness Theorem for Linear Systems, the constant function $F(s)(F(s))^T = \mathbf{I}$ satisfies equation (3.28) for all s, since K is skew-symmetric. Therefore, by the Existence and Uniqueness Theorem for Linear Systems, $FF^T = \mathbf{I}$, on the entire interval. So, $F(s)$ is an orthogonal matrix. Since $F(s_0) = \mathbf{I}$, then $\mathbf{det}F(s) = 1$, preserving orientation.

The matrix $F(s)$ determines a unique vector-valued function $\mathbf{e_1}(s)$. For given initial conditions $\alpha(s_0) = c$, a unique curve $\alpha(s)$ with $\alpha(s)' = \mathbf{e_1}(s)$ is defined

$$\alpha(s) = c + \int_{s_0}^{s} \mathbf{e_1}(t)dt.$$

Because $\mathbf{e_1}' = \kappa_1 \mathbf{e_2} \neq 0$, the vector $\mathbf{e_2}$ corresponds to the second vector of the Frenet n-frame for the curve $\alpha(s)$. Analogously, for all the other $\mathbf{e_i}$ vectors.

Thus, $F(s)$ represents the Frenet n-frame for $\alpha(s)$ at each point.

The given curvature functions λ_i coincide with the Frenet curvatures of the curve $\alpha(s)$. In particular, $\alpha(s)$ is called a Frenet curve which follows from the fact that $\alpha' = \mathbf{e_1}$, $\alpha'' = \kappa_1 \mathbf{e_2}$, and so on for every $i = 1, \ldots, n-1$.

Now suppose $\beta(s)$ is any other curve with curvatures $\lambda_1(s), \ldots, \lambda_{n-1}(s)$. Let $A = (\mathbf{e_1}(0) \ldots \mathbf{e_n}(0))^T$ be an orthogonal matrix with $\mathbf{det}A = 1$. The curves $A\alpha(s)$ and $\beta(s)$ both satisfy the Frenet equations and equal A when $s = s_0$. By the Existence and Uniqueness Theorem, $\beta(s) = A\alpha(s)$. $\square$

4. CURVES WITH CONSTANT CURVATURE RATIOS

DEFINITION 1: A curve $\alpha : I \to \mathbb{R}^n$ is said to have constant curvature ratios if all the quotients $\frac{\kappa_{i+1}}{\kappa_i}$ are constant.

In fact, generalized helices in $\mathbb{R}^3$ are characterized by the fact that $\frac{\tau}{\kappa}$ is constant. This is a result cited as Lancret's Theorem (see [3]), and it follows that curves with constant curvature ratios (i.e. ccr-curves) are a subset of generalized helices. This is only true, however, for odd-n. Instead of working through a rigorous proof, we will investigate two explicit examples of helices: one in the third dimension and the other in the fourth dimension, in order to demonstrate ccr-curves.

4.1 Circular Helix in $\mathbb{R}^3$

Define a circular helix parameterized by time in the third dimension as $\alpha(t) = (a\cos t, a\sin t, bt)$, where $a > 0$ We want a unit-speed parameterization of this helix in terms of its arclength, because we would rather analyze the curve from the perspective of distance, rather than time. So, using the formula for arclength:

$$s(t) = \int_0^t ||\alpha'(t)||dt.$$

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = ||\alpha'(t)||. \tag{4.30}$$

Since

$$||\alpha'(t)|| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

and $x'(t) = -a\sin t$, $y'(t) = a\cos t$, and $z'(t) = b$, we have

$$||\alpha'(t)|| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2 + b^2}.$$

Let $c = \sqrt{a^2 + b^2}$. Then

$$s = \int_0^t c\,dt = ct.$$

Hence, the unit-speed parametrized helix is defined as

$$\beta(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{bs}{c}\right).$$

Since we are in the third-dimension, our curve is defined by curvature κ and torsion τ, which arise from Theorem 2.3. According to Definition 1, we verify that the ratio of curvatures $\frac{\tau}{\kappa}$ is constant for the circular helix.

First, we find κ. By Theorem 2.3,

$$\mathbf{T}'(s) = \frac{d\mathbf{T}}{ds} = \frac{1}{c^2}\left(-a\cos\left(\frac{s}{c}\right), -a\sin\left(\frac{s}{c}\right), 0\right) = \kappa\mathbf{N}.$$

Since $a > 0$, by definiton

$$\kappa = ||\mathbf{T}'(s)|| = \sqrt{\frac{1}{(c^2)^2}\left(a^2\cos^2\left(\frac{s}{c}\right) + a^2\sin^2\left(\frac{s}{c}\right)\right)}.$$

By Trigonometric identities,

$$\kappa = \sqrt{\frac{a^2}{(c^2)^2}} = \frac{a}{c^2}. \tag{4.31}$$

This implies κ is constant.

By definition,

$$\mathbf{N} = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \frac{\frac{1}{c^2}(-a\cos\left(\frac{s}{c}\right), -a\sin\left(\frac{s}{c}\right), 0)}{\frac{a}{c^2}}.$$

Hence

$$\mathbf{N} = \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right) \tag{4.32}$$

Now we find τ. Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$,

$$\mathbf{B} = \left(\frac{1}{c}\left(-a\sin\left(\frac{s}{c}\right), a\cos\left(\frac{s}{c}\right), b\right)\right) \times \left(-\cos\left(\frac{s}{c}\right), -\sin\left(\frac{s}{c}\right), 0\right).$$

Performing the cross product, combining like terms, and utilizing Trigonometric identities, we have

$$\mathbf{B} = \frac{1}{c}\left(b\sin\left(\frac{s}{c}\right), -b\cos\left(\frac{s}{c}\right), a\right). \tag{4.33}$$

By Theorem 2.3, $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$. Hence

$$\tau = -\mathbf{N} \cdot \frac{d\mathbf{B}}{ds}. \tag{4.34}$$

So, find $\mathbf{B}'$.

$$\mathbf{B}' = \frac{d\mathbf{B}}{ds} = \frac{1}{c^2}\left(b\cos\left(\frac{s}{c}\right), b\sin\left(\frac{s}{c}\right), 0\right). \tag{4.35}$$

Plugging equations (4.32) and (4.35) into equation (4.34), we solve for τ:

$$\tau = \frac{1}{c^2}\left(b\cos\left(\frac{s}{c}\right), b\sin\left(\frac{s}{c}\right), 0\right) \cdot \left(\cos\left(\frac{s}{c}\right), \sin\left(\frac{s}{c}\right), 0\right).$$

So,

$$\tau = \frac{1}{c^2}\left[b\cos^2\left(\frac{s}{c}\right) + b\sin^2\left(\frac{s}{c}\right)\right].$$

By Trigonometric identities, hence

$$\tau = \frac{b}{c^2}. \tag{4.36}$$

This implies that τ is also constant.

Explicitly determining the ratio of τ and κ by equations (4.31) and (4.36), hence

$$\frac{\tau}{\kappa} = \frac{\frac{b}{c^2}}{\frac{a}{c^2}} = \frac{b}{a}.$$

This ratio we know is constant. Thus, the circular helix in $\mathbb{R}^3$ is a ccr-curve.

4.2 Cylindrical Helix in $\mathbb{R}^4$

Now for an even-dimensional example: a cylindrical helix in $\mathbb{R}^4$. Define a cylindrical helix as a curve for which there exists a fixed unit vector $\mathbf{u}$ such that $\mathbf{T} \cdot \mathbf{u}$ is constant along the curve. We will attempt to show that the ratio of curvatures of the cylindrical helix are constant, in order to prove that the cylindrical helix is a ccr-curve. This example will demonstrate how we cannot simply assume the behavior of a curve in higher dimensions based on the characteristics we know of a curve in lower dimensions.

By the definition of dot product, $\mathbf{T} \cdot \mathbf{u} = \|\mathbf{T}\|\|\mathbf{u}\| \cos\theta$. Of course θ is constant, being the angle between $\mathbf{T}$ and $\mathbf{u}$, and since these are unit vectors,

$$\mathbf{T} \cdot \mathbf{u} = \cos\theta \tag{4.37}$$

is constant.

The derivative is

$$\frac{d\mathbf{T}}{ds} \cdot \mathbf{u} + \mathbf{T} \cdot \frac{d\mathbf{u}}{ds} = 0.$$

By Theorem 2.4, and using the fact that the derivative of the unit vector $\mathbf{u}$ is zero,

$$\kappa\mathbf{N} \cdot \mathbf{u} = 0.$$

Since $\kappa > 0$, hence

$$\mathbf{N} \cdot \mathbf{u} = 0. \tag{4.38}$$

Write the vector $\mathbf{u}$ as a linear combination of orthogonal vectors, such that

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{T})\mathbf{T} + (\mathbf{u} \cdot \mathbf{N})\mathbf{N} + (\mathbf{u} \cdot \mathbf{B})\mathbf{B} + (\mathbf{u} \cdot \mathbf{D})\mathbf{D}. \tag{4.39}$$

Denote $\xi = \mathbf{u} \cdot \mathbf{B}$ and $\gamma = \mathbf{u} \cdot \mathbf{D}$. Substituting equations (4.37), (4.38), ξ, and γ into equation (4.39), we have

$$\mathbf{u} = (\cos\theta)\mathbf{T} + \xi\mathbf{B} + \gamma\mathbf{D}. \tag{4.40}$$

Since $\mathbf{u}$ is a unit vector, $||\mathbf{u}|| = 1 = \sqrt{\cos^2\theta + \xi^2 + \gamma^2}$. Squaring both sides, we have $||\mathbf{u}||^2 = 1 = \cos^2\theta + \xi^2 + \gamma^2$. By Trigonometric identities,

$$\sin^2\theta = \xi^2 + \gamma^2. \tag{4.41}$$

Define ϕ, such that $\xi = \sin\theta\cos\phi$ and $\gamma = \sin\theta\sin\phi$. Substituting back into equation (4.41) and using Trigonometric identities gives

$$\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = \sin^2\theta.$$

Thus, equation (4.40) becomes

$$\mathbf{u} = (\cos\theta)\mathbf{T} + (\sin\theta\cos\phi)\mathbf{B} + (\sin\theta\sin\phi)\mathbf{D}.$$

Differentiate $\mathbf{u}$ with the intent to find the curvature ratio.

$$\mathbf{u}' = 0 + (\cos\theta)\frac{d\mathbf{T}}{ds} + (\sin\theta\cos\phi)\frac{d\mathbf{B}}{ds} - \frac{d\phi}{ds}(\sin\theta\sin\phi)\mathbf{B}$$

$$+ (\sin\theta\sin\phi)\frac{d\mathbf{D}}{ds} + \frac{d\phi}{ds}(\sin\theta\cos\phi)\mathbf{D}. \quad (4.42)$$

By the Theorem 2.4,

$$\mathbf{u}' = (\cos\theta)(\kappa\mathbf{N}) + (\sin\theta\cos\phi)(-\tau\mathbf{N} + \sigma D)$$

$$- \frac{d\phi}{ds}(\sin\theta\sin\phi)\mathbf{B} + (\sin\theta\sin\phi)(\sigma\mathbf{B}) + \frac{d\phi}{ds}(\sin\theta\cos\phi)\mathbf{D}. \quad (4.43)$$

Since $\mathbf{u}$ is a constant vector, $\mathbf{u}' = \mathbf{0}$. Combining like terms,

$$\mathbf{0} = (\kappa\cos\theta - \tau\sin\theta\cos\phi)\mathbf{N} - (\sin\theta\sin\phi)\left(\frac{d\phi}{ds} + \sigma\right)\mathbf{B} + (\sin\theta\cos\phi)\left(\frac{d\phi}{ds} + \sigma\right)\mathbf{D}.$$

When the sum of orthogonal vectors is zero, each component of the sum is zero:

$$\kappa\cos\theta - \tau\sin\theta\cos\phi = 0 \quad (4.44)$$

$$\sin\theta\sin\phi\left(\frac{d\phi}{ds} + \sigma\right) = 0 \quad (4.45)$$

$$\sin\theta\cos\phi\left(\frac{d\phi}{ds} + \sigma\right) = 0 \quad (4.46)$$

From equation (4.44), solve for $\frac{\tau}{\kappa}$, such that

$$\frac{\tau}{\kappa} = \frac{\cos\theta}{\sin\theta\cos\phi} = \cot\theta\sec\phi. \quad (4.47)$$

Consider equations (4.45) and (4.46). If $\sin\theta = 0$, then $\theta = 0$ or $\theta = \pi$.

In the case that $\theta = 0$, then $\mathbf{T} \cdot \mathbf{u} = 1$, by equation (4.37). Thus, $\mathbf{T} = \mathbf{u}$.

In the case that $\theta = \pi$, then $\mathbf{T} \cdot \mathbf{u} = -1$, by equation (4.37). Thus, $\mathbf{T} = -\mathbf{u}$.

In either case, $\mathbf{T}$ is a straight line in the direction of the unit vector $\mathbf{u}$. This is not an interesting event, since curvature is not present.

The other alternatives for satisfying equations (4.45) and (4.46) are

$$\sin\phi = 0, \qquad \cos\phi = 0, \qquad \text{or} \qquad \frac{d\phi}{ds} + \sigma = 0.$$

The only interesting case is

$$\frac{d\phi}{ds} = -\sigma.$$

Since, $\sigma \neq 0$, ϕ is not constant.

From equation (4.47), $\cot\theta$ is constant since it is composed of the constants $\sin\theta$ and $\cos\theta$. But $\sec\phi$ is not constant, since ϕ is not constant.

Therefore, $\frac{\tau}{\kappa}$ is not a constant curvature ratio for the four-dimensional cylindrical helix curve. Thus, the cylindrical helix is not a ccr-curve.

5. CHARACTERIZATION OF SPHERICAL CURVES

Consider a sphere with a center, denoted $\mathbf{c}$. The sphere has a radius R. The space curve $\alpha(s)$ traces a path over the surface of the sphere. What are the defining properties of this curve? Let's do some exploration!

5.1 Curves on 2-Spheres

THEOREM 5.1: Let $\alpha : I \to \mathbb{R}^3$ be a unit-speed curve parameterized by arclength, such that $\alpha \in C^3(I)$. Then $\alpha(s)$ lies on a sphere of radius R if and only if

$$R^2 = \frac{1}{\kappa^2} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]^2.$$

Proof. Begin with assuming the curve lies on a sphere in $\mathbb{R}^3$-space. It follows that

$$\|\boldsymbol{\alpha}(s) - \mathbf{c}\| = R,$$

and hence

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = R^2. \tag{5.48}$$

By differentiating we obtain

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot (\boldsymbol{\alpha}(s) - \mathbf{c})' + (\boldsymbol{\alpha}(s) - \mathbf{c})' \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0. \tag{5.49}$$

By definition, $(\boldsymbol{\alpha}(s) - \mathbf{c})' = \mathbf{T}$. Thus, equation (5.49) becomes

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{T} + \mathbf{T} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 2\mathbf{T} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0.$$

Hence

$$\mathbf{T} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0. \tag{5.50}$$

The derivative is $\mathbf{T} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c})' + \mathbf{T}' \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0$.

So,

$$\mathbf{T} \cdot \mathbf{T} + \frac{d\mathbf{T}}{ds} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0.$$

By Theorem 2.3 and since $\mathbf{T} \cdot \mathbf{T} = 1$,

$$1 + \kappa \mathbf{N} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0.$$

Hence

$$\mathbf{N} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = -\frac{1}{\kappa}. \tag{5.51}$$

The derivative is $\mathbf{N} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c})' + \mathbf{N}' \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \frac{d}{ds}\left(-\frac{1}{\kappa}\right)$.

So,

$$\mathbf{N} \cdot \mathbf{T} + \frac{d\mathbf{N}}{ds} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = -\left(\frac{1}{\kappa}\right)'.$$

By Theorem 2.3 and since $\mathbf{N} \cdot \mathbf{T} = 0$,

$$(-\kappa \mathbf{T} + \tau \mathbf{B}) \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = -\left(\frac{1}{\kappa}\right)'.$$

Since $\mathbf{T}$ and $(\boldsymbol{\alpha}(s) - \mathbf{c})$ are orthogonal, $-\kappa \mathbf{T} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = 0$. Hence

$$\mathbf{B} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)' \tag{5.52}$$

Since any vector can be expressed as a linear combination of orthogonal vectors, set

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) = a\mathbf{T} + b\mathbf{N} + c\mathbf{B}.$$

Since $a = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{T}$, by equation (5.50), $a = 0$. Also, $b = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{N}$, and by equation (5.51), $b = -\left(\frac{1}{\kappa}\right)$. Finally, $c = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{B}$, and by equation (5.52), $c = -\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'$.

Thus,

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) = -\frac{1}{\kappa}\mathbf{N} - \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{B} \tag{5.53}$$

Since $\mathbf{N}$ is orthogonal to $\mathbf{B}$, take the magnitude of both sides of equation (5.53) and apply the Pythagorean Theorem.

Hence

$$R^2 = \frac{1}{\kappa^2} + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]^2 \tag{5.54}$$

For the converse, assume equation (5.54) and let

$$\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa}\mathbf{N} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{B}.$$

We want to show that $\boldsymbol{\gamma}(s)$ is a constant vector, implying that it is the center of the sphere. The derivative of the equation above is

$$\boldsymbol{\gamma}'(s) = \boldsymbol{\alpha}'(s) + \left(\frac{1}{\kappa}\mathbf{N}\right)' + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{B}\right)'$$

With expansion, we have

$$\boldsymbol{\gamma}'(s) = \mathbf{T} + \left(\frac{1}{\kappa}\mathbf{N}' + \left(\frac{1}{\kappa}\right)'\mathbf{N}\right) + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{B}' + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\mathbf{B}\right]$$

By Theorem 2.3,

$$\boldsymbol{\gamma}'(s) = \mathbf{T} + \frac{1}{\kappa}(-\kappa\mathbf{T} + \tau\mathbf{B}) + \left(\frac{1}{\kappa}\right)'\mathbf{N} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'(-\tau\mathbf{N}) + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\mathbf{B}.$$

With distribution, we have

$$\boldsymbol{\gamma}'(s) = \mathbf{T} - \mathbf{T} + \frac{\tau}{\kappa}\mathbf{B} + \left(\frac{1}{\kappa}\right)'\mathbf{N} - \left(\frac{1}{\kappa}\right)'\mathbf{N} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\mathbf{B}.$$

Hence

$$\boldsymbol{\gamma}'(s) = \frac{\tau}{\kappa}\mathbf{B} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\mathbf{B}. \tag{5.55}$$

To show that $\boldsymbol{\gamma}'(s) = 0$, we need to show that $\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = -\frac{\tau}{\kappa}$.

Rewriting equation (5.54), we have

$$\left(\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]^2\right)' = \left(R^2 - \frac{1}{\kappa^2}\right)'.$$

Thus,

$$2\kappa'\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)'\right) = 0 + \frac{2\kappa'}{\kappa^3}. \tag{5.56}$$

Note that

$$\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'' + \left(\frac{1}{\tau}\right)'\left(\frac{1}{\kappa}\right)' = \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'. \tag{5.57}$$

The $2\kappa'$ on either side of the equation (5.56) cancels. Substituting equation (5.57), equation (5.56) becomes

$$\frac{1}{\tau}\left(-\frac{1}{\kappa^2}\right)\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' = \frac{1}{\kappa^3}.$$

Hence

$$\left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right] = \frac{1}{\kappa^3}(-\tau\kappa^2) = -\frac{\tau}{\kappa}. \tag{5.58}$$

Substituting equation (5.58) back into equation (5.55), we obtain our desired result that $\boldsymbol{\gamma}'(s) = 0$. Thus, $\boldsymbol{\gamma}(s)$ is a constant vector, say $\boldsymbol{\gamma}(s) = \mathbf{c}$, and the result follows.

$\square$

5.2 Curves on 3-Spheres

THEOREM 5.2: Let $\alpha : I \to \mathbb{R}^4$ be a unit-speed curve parameterized by arclength, such that $\alpha \in C^4(I)$. Then $\alpha(s)$ lies on a sphere of radius R if and only if

$$R^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2 + \left[\frac{1}{\sigma}\left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' + \frac{\tau}{\sigma}\left(\frac{1}{\kappa}\right)\right]^2.$$

Proof. Begin with assuming the sphere lies in Euclidean $\mathbb{R}^4$-space. We want to characterize the spherical curve $\alpha(s)$ in terms of four-dimensional curvatures κ, τ, and σ.

We start from equation (5.52). The derivative of (5.52) is

$$\mathbf{B} \cdot (\boldsymbol{\alpha}(s)) - \mathbf{c})' + \frac{d\mathbf{B}}{ds} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \left[-\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'.$$

By Theorem 2.4 and the fact that $\mathbf{B} \cdot \mathbf{T} = 0$,

$$-\tau\mathbf{N} \cdot \mathbf{T} + \sigma\mathbf{D} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \left[-\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'.$$

By equation (5.51),

$$-\tau\left(-\frac{1}{\kappa}\right) + \sigma\mathbf{D} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \left[-\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]'.$$

Hence

$$\mathbf{D} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \frac{1}{\sigma}\left[-\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' - \frac{1}{\sigma}\left(\frac{\tau}{\kappa}\right). \tag{5.59}$$

For the sake of simplicity, denote $\frac{1}{\kappa} = \rho$.

So, equation (5.59) becomes

$$\mathbf{D} \cdot (\boldsymbol{\alpha}(s) - \mathbf{c}) = \frac{1}{\sigma}\left[-\frac{\rho'}{\tau}\right]' - \frac{\tau\rho}{\sigma}. \tag{5.60}$$

Since every vector can be written as a linear combination of orthogonal vectors, write $(\boldsymbol{\alpha}(s) - \mathbf{c}) = a\mathbf{T} + b\mathbf{N} + c\mathbf{B} + d\mathbf{D}$.

Since $a = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{T} = 0$, by equation (5.50), $a = 0$. Since, $b = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{N}$, by equation (5.51), $b = -\rho$. Also, $c = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{B}$, and by equation (5.52), $c = -\frac{\rho'}{\tau}$. Finally, $d = (\boldsymbol{\alpha}(s) - \mathbf{c}) \cdot \mathbf{D}$, and by equation (5.60), $d = \frac{1}{\sigma}\left[-\frac{\rho'}{\tau}\right]' - \frac{\tau\rho}{\sigma}$.

Thus,

$$(\boldsymbol{\alpha}(s) - \mathbf{c}) = -\rho\mathbf{N} - \frac{\rho'}{\tau}\mathbf{B} + \left(\frac{1}{\sigma}\left[\frac{-\rho'}{\tau}\right]' - \frac{\tau\rho}{\sigma}\right)\mathbf{D}. \tag{5.61}$$

Use the orthogonality of $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ to obtain

$$R^2 = \rho^2 + \left(\frac{\rho'}{\tau}\right)^2 + \left(\frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' + \frac{\tau\rho}{\sigma}\right)^2 \tag{5.62}$$

For the converse, assume equation (5.62) and let

$$\boldsymbol{\gamma}(s) = \boldsymbol{\alpha}(s) + \rho\mathbf{N} + \frac{\rho'}{\tau}\mathbf{B} + \left(\frac{1}{\sigma}\left[\frac{\rho'}{\tau}\right]' + \frac{\tau\rho}{\sigma}\right)\mathbf{D}.$$

We want to show that $\boldsymbol{\gamma}(s)$ is a constant vector, implying it is the center of the 3-sphere. The derivative of the equation above is

$$\boldsymbol{\gamma}'(s) = \boldsymbol{\alpha}'(s) + (\rho\mathbf{N})' + \left(\frac{\rho'}{\tau}\mathbf{B}\right)' + \left(\left[\frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' + \frac{\tau\rho}{\sigma}\right]\mathbf{D}\right)'.$$

With expansion, we have

$$\begin{aligned}\boldsymbol{\gamma}'(s) = \mathbf{T} + (\rho\mathbf{N}') + &\left(\frac{\rho'}{\tau}\mathbf{B}' + \left(\frac{\rho'}{\tau}\right)'\mathbf{B}\right) \\ &+ \left(\left[\frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' + \frac{\tau\rho}{\sigma}\right]\mathbf{D}' + \left[\frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' + \frac{\tau\rho}{\sigma}\right]'\mathbf{D}\right).\end{aligned} \tag{5.63}$$

By Theorem 2.4, the derivative simplifies to

$$\boldsymbol{\gamma}'(s) = \frac{\sigma\rho'}{\tau} + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)'. \tag{5.64}$$

Now consider equation (5.62). The derivative is

$$0 = \rho\rho' + \frac{\rho'}{\tau}\left(\frac{\rho'}{\tau}\right)' + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)\left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)'.$$

Invoke a little trick to encourage simplification by strategically adding and subtracting $\frac{\sigma\rho'}{\tau}$, such that

$$0 = \rho\rho' + \frac{\rho'}{\tau}\left(\frac{\rho'}{\tau}\right)' + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)\left(-\frac{\sigma\rho'}{\tau} + \frac{\sigma\rho'}{\tau} + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)'\right).$$

By distribution,

$$\begin{aligned} 0 = \rho\rho' + \frac{\rho'}{\tau}\left(\frac{\rho'}{\tau}\right)' - \frac{\sigma\rho'}{\tau}\left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right) \\ + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)\left(\frac{\sigma\rho'}{\tau} + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)'\right). \end{aligned} \tag{5.65}$$

This simplifies to

$$0 = \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)\left(\frac{\sigma\rho'}{\tau} + \left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)'\right). \tag{5.66}$$

For the case that $\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' \neq 0$, then $\left(\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)'\right)' = 0$. By equation (5.64), this implies that $\boldsymbol{\gamma}'(s) = 0$. Thus, $\boldsymbol{\gamma}(s)$ is a constant vector, say $\boldsymbol{\gamma}(s) = \mathbf{c}$, and the result follows.

On the other hand, for the case that $\frac{\tau\rho}{\sigma} + \frac{1}{\sigma}\left(\frac{\rho'}{\tau}\right)' = 0$, then we are back in the three-dimensional case with the identity from equation (5.58):

$$\left(\frac{\rho'}{\tau}\right)' = -\tau\rho.$$

Substituting this identity into equation (5.63) would take us back to the three-dimensional equation (5.55), in which case we have already shown that $\boldsymbol{\gamma}(s) = \mathbf{c}$ is a constant vector. $\square$

6. CONCLUSION

The relevance of the Frenet frame in studying the characteristics and properties of curves in space strongly apparent. While curves in Euclidean $\mathbb{R}^2$-space and $\mathbb{R}^3$-space are intuitively visual, we generalized curves in higher-dimensions utilizing the Frenet equations to understand space curves whose properties are not quite as obvious. Indeed, even the Fundamental Theorem of Curves builds upon the Frenet equations. And as we demonstrated, they are vital in analyzing curves with or without constant curvature ratios and in characterizing spherical curves.

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