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Groups Satisfying the Converse to Lagrange's Theorem

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GROUPS SATISFYING THE CONVERSE TO LAGRANGE'S THEOREM

A Master's Thesis

Presented to

The Graduate College of

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In partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

Lagrange's theorem, which is taught early on in group theory courses, states that the order of a subgroup must divide the order of the group which contains it. In this thesis, we consider the converse to this statement. A group satisfying the converse to Lagrange's theorem is called a CLT group. We begin with results that help show that a group is CLT, and explore basic CLT groups with examples. We then give the conditions to guarantee either CLT is satisfied or a non-CLT group exists for more advanced cases. Additionally, we show that CLT groups are properly contained between supersolvable and solvable groups.

KEYWORDS: lagrange, group, subgroup, order, index

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I dedicate this thesis to my parents.

TABLE OF CONTENTS

1. INTRODUCTION

One way of thinking of groups is as collections of symmetries. The dihedral group D_{2n} contains the symmetries of a regular n-gon, including rotations and reflections. If we restrict to the rotations of a regular n-gon, we have the cyclic group C_n. The symmetric group S_n consists of all the re-orderings of n items. Every group contains a trivial action. For each action in the group, there is an opposite action that, when combined with the original, yields the trivial action. A more rigorous definition is a nonempty set along with a closed, associative binary operation that contains an identity and inverses for each element.

Group theory has roots in studying permutations. Consider a function $\phi(x_1, x_2 \ldots, x_n)$. For example, let $\phi(x_1, x_2, x_3) = x_1 + x_2 + x_3^3$. We can shuffle the variables to obtain a permutation of ϕ, such as $\phi(x_2, x_1, x_3)$ which permuted x_1 and x_2. There are 3!=6 possible ways to re-order x_1, x_2, x_3. However, it is easy to see that the only distinct permutations of ϕ are $x_1+x_2+x_3^3$, $x_1+x_3+x_2^3$, and $x_2+x_3+x_1^3$. In 1770, Lagrange showed by example that for a function ϕ which takes n variables, if d is the number of distinct permutations of ϕ, then d divides $n!$. This development was in a time before group theory was a recognized branch of mathematics. The modern theorem of Lagrange deals with the orders of groups and subgroups, and can be used to prove his original claim. In this paper, we investigate the converse to this statement.

In Section 2, we cover preliminary group theory results, and introduce Lagrange's Theorem as well as its converse. We define CLT (converse to Lagrange's theorem) groups, and show that not every finite group satisfies CLT. Additionally, we cover partial statements of CLT, given by Sylow's and Cauchy's Theorems. Sections 3, 4, and 6 show that CLT is satisfied for certain classes of groups, with basic results covered in Section 3, and more advanced ones left for Sections 4 and 6. In Section 5, we show that CLT groups are contained between supersolvable and solvable groups.

2. PRELIMINARIES AND BASIC RESULTS

Lagrange's Theorem, which is taught early on in introductory group theory courses, is stated as follows.

Theorem 2.1. Let G be a finite group. Then the order of every subgroup H of G divides the order of G.

Proof. Let $N \leq G$ and G act on G/N by left multiplication. Then

$$\operatorname{Orbit}(N) = \{gN \mid g \in G\} = G/N,$$

$$\operatorname{Stab}(N) = \{g \in G \mid gN = N\} = N.$$

And so we have

$$\begin{aligned} |G| &= |\operatorname{Stab}(N)||\operatorname{Orbit}(N)| \\ &= |N||G/N| \\ &= |N|[G:N]. \end{aligned}$$

So the order of N divides the order of G. □

The proof, in fact, reveals an even stronger result than the theorem. Not only does the order of a subgroup divide the order of the whole, but for a group G with $H \leq G$, we know that $|G| = |H|[G : H]$. A corollary to Lagrange's theorem is that, for a finite group G and $x \in G$, the order of x divides the order of G. If d is the order of x, then the cyclic subgroup $\langle x\rangle = \{1,\ x\ x^2, \dots,\ x^{d-1}\}$ has order d. By the theorem, d divides the order of G.

Definition 2.2. Let G be a group. Then the *automorphism group of* G, denoted $\operatorname{Aut}(G)$, is the group of isomorphisms from G to itself.

Definition 2.3. Let G be a group with subgroup H. If $\phi(H) = H$ for every ϕ in $\mathrm{Aut}(G)$, then H is *characteristic* in G.

Let G be some group with $H \leq G$ and, for each $g \in G$, define $\phi_g \in \mathrm{Aut}(G)$ to be conjugation by g. If $\phi_g(H) = gHg^{-1} = H$ for each ϕ_g, then H is *normal* in G, for which we use the notation $H \trianglelefteq G$. It is easy to show that if H is characteristic in G, then it is normal. This is because conjugation is an automorphism. We also have the following.

Proposition 2.4. Let G be a group such that H is normal in G and K is characteristic in H. Then K is normal in G.

Proof. We have $H \trianglelefteq G$, and so, for $g \in G$, $\phi_g : H \to H$ given by $\phi_g(h) = ghg^{-1}$ is an automorphism of H. Since K is characteristic in H, $\phi_g(K) = K$ for each $g \in G$, and so $K \trianglelefteq G$. □

Definition 2.5. For a group G and $N \trianglelefteq G$, the *quotient group* of N in G is the set of cosets of N in G, and $aNbN = abN$. This is denoted G/N, and $|G/N| = [G : N]$.

Theorem 2.6. (Cauchy's Theorem) Let G be a finite group. If p is prime and p divides the order of G, then there exists an element of G which has order p.

Cauchy's Theorem is useful because for each prime dividing the order of a group, we are guaranteed a subgroup of that order. If G is a group and p is some prime dividing the order of the group, then we are guaranteed that there is an element $x \in G$ that has order p. Then the cyclic group generated by x is a subgroup of G with order p.

Definition 2.7. Let G be a group and $H \leq G$. We define the following subgroups:

1. The *center* of H, denoted $Z(H)$, is the set of elements $x \in H$ that commute with every element in H.
2. The *centralizer* of H in G, denoted $C_G(H)$, is the set of elements $x \in G$ that commute with every element of H.
3. The *normalizer* of H in G, denoted $N_G(H)$, is the set of elements in $x \in G$ such that $xHx^{-1} = H$.

It is easy to show that $Z(H) \leq C_G(H) \leq N_G(H)$: Let $x \in Z(H)$. Then $xy = yx$ for all $y \in H$. But by definition, $x \in H \leq G$. So x is an element of G which commutes with the elements in H, and so $x \in C_G(H)$.

Now, suppose $x \in C_G(H)$ and let $y \in H$. Then $xyx^{-1} = yxx^{-1} = y \in H$, demonstrating that $x \in N_G(H)$.

Definition 2.8. Let G be a finite group and p prime. If p^m is the highest power of p which divides the order of G, then a subgroup of G of order p^m is called a *Sylow p-subgroup*. The set of all Sylow p-subgroups for G is denoted $\mathrm{Syl}_p(G)$.

Theorem 2.9. (Sylow's Theorem)

1. Let G be a finite group and p some prime. If p divides the order of G, then G has a Sylow p-subgroup.
2. Let G be a finite group and p some prime. Then all Sylow p-subgroups are conjugate to each other.
3. Let p be a prime factor of a finite group G such that n is the highest power of p dividing $|G|$. Then $|G| = p^n m$ for some m. Let n_p be the number of Sylow p-subgroups in G. Then n_p divides m and p divides $n_p - 1$.

Proposition 2.10. Let G be a finite group and p some prime such that $P \in \mathrm{Syl}_p(G)$. Then P is normal in G if and only if $n_p = 1$.

Proof. If P is the only Sylow p-subgroup of a group G, then gPg^{-1} is another subgroup of the same order, so $gPg^{-1} = P$ for all $g \in G$. So $P \trianglelefteq G$. Conversely, if P is normal in G, then by part 2 of Sylow's Theorem, every Sylow p-subgroup is a conjugate of P, and so must be P itself. And so it is unique and $n_p = 1$. $\square$

Let the order of a group G have prime factorization $p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$. Cauchy's Theorem provides us with subgroups of orders $p_1, p_1, \ldots, p_n$ while part one of Sylow's Theorem shows the existence of subgroups of orders $p_1^{a_1}, p_2^{a_2}, \ldots p_n^{a_n}$. We will now consider an example.

Theorem 2.11. (Third Isomorphism Theorem) Suppose G is a group with normal subgroups H and N such that N is a subgroup of H. Then N is normal in H and

$$(G/N)/(H/N) \cong G/H.$$

Theorem 2.12. (Lattice Isomorphism Theorem) Let G be a group and $N \trianglelefteq G$. Then there exists a bijection between the set of subgroups of G containing N and set of subgroups of G/N. Also, $K/N \trianglelefteq G/N$ if and only if $K \trianglelefteq G$.

The proofs of the previous two results can be found at [5, pg 98] and [5, pg 99] respectively.

Definition 2.13. A *subnormal series* of a group G is a sequence of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

such that each G_i is normal in G_{i+1}. If each G_i is also normal in G ,then this chain is called a *normal series*.

Definition 2.14. A group G is *solvable* if it has a normal series whose quotient groups G_{i+1}/G_i are all abelian.

Remark 2.15. An equivalent definition of a solvable group is a group G with subnormal series $1 = G_0 \trianglelefteq \cdots \trianglelefteq G_n = G$ such that each G_{i+1}/G_i is cyclic.

Definition 2.16. A group G is *supersolvable* if it has a normal series whose quotient groups G_{i+1}/G_i are all cyclic.

Note that it is easy to show that a supersolvable group is solvable, since cyclic groups are abelian.

Definition 2.17. Suppose G and H are groups such that $\phi : G \to \mathrm{Aut}(H)$ is a homomorphism. Then the *semi-direct product* of H and G with respect to ϕ, denoted $H \rtimes_\phi G$ is the cartesian product $H \times G$ with the rule $(h, g)(h', g') = (h(\phi(g)[h']), gg')$

Whenever ϕ is the trivial homomorphism, then this is precisely the direct product of H and G. And so semi-direct products are a richer generalization of the direct product. Suppose there exist subgroups H, N of a group G where $N \trianglelefteq G$, $H \cap N = \{e\}$, and $G = NH$, the inner semi-direct product defined for some ϕ. Then $G \cong N \rtimes_\phi H$. If we also have that $H \trianglelefteq G$, then $G \cong N \times H$.

The converse to Lagrange's theorem is that for a finite group G, if d divides G, then there exists a subgroup $H \leq G$ of order d. This is not true in general, as we will demonstrate in the following example.

Example 2.18. Consider the alternating group A_4, which has order 12. If the converse to Lagrange's theorem were true, then there would exist a subgroup $H \leq A_4$ with order 6. Assume such an H exists. Then H has index 2. Thus the two cosets are H and $G - H$. If $g \in H$, then $gH = H = Hg$. If $g \notin H$, then $gH = G - H = Hg$, and so $H \trianglelefteq G$. Hence $|A_4/H| = [A_4 : H] = 2$ and so for any $s \in G$, we have $s^2H = (sH)^2 = H$. Let s be some 3-cycle. Then $s = (s^2)^2$, so $s \in H$. But there are 8 3-cycles in A_4, contradicting the fact that $|H| = 6$.

Definition 2.19. A finite group which satisfies the converse to Lagrange's theorem is called a *CLT group*. Note that these groups are often called Lagrangian.

In order to show that a finite group G is a CLT group, we let d be an arbitrary divisor of $|G|$, and show that there exists a subgroup H of G with order d.

Proposition 2.20. The symmetric group S_4 is a CLT group.

Proof. S_4 has order $4! = 24$, which has prime factorization $2^3 \cdot 3$. The divisors of this are

1, 2, 3, 4, 6, 8, 12, 24. Subgroups of order 1 and 24 clearly exist, as they are the identity and whole of S_4 respectively. Cauchy's Theorem provides the existence of subgroups with orders 2 and 3, while Sylow's Theorem gives us a subgroups of order $2^3 = 8$. This leaves us to find subgroups of orders 4, 6 and 12. The Klein four-group, which consists of the identity and double transpositions, is a subgroup of order 4. The alternating group A_4 is a subgroup of order 12. Finally, the symmetric group S_3 is embedded in S_4 and has order 6. We have now found a subgroup for each divisor of $|S_4|$, and so S_4 is indeed a CLT group.

□

Lemma 2.21. For $x, y_i \in \mathbb{R}$, if $x \leq \sum_{i=1}^{n} y_i$, then there exist $x_1, \ldots, x_n \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} x_i$ and $x_i \leq y_i$ for each i.

Proof. There exists $l \leq n$ such that $\sum_{i=1}^{l-1} y_i \leq x \leq \sum_{i=1}^{l} y_i$. Now, for each i, define

$$x_j = \begin{cases} y_j & j < l \\ x - \sum_{i=1}^{l-1} y_i & j = l \\ 0 & j > l \end{cases}$$

Then clearly $x_i \leq y_i$ for $i \neq l$, and $x_l = x - \sum_{i=1}^{l-1} y_i \leq \sum_{i=1}^{l} y_i - \sum_{i=1}^{l-1} y_i = y_l$. Furthermore, $x = \sum_{i=1}^{n} x_i$. □

Proposition 2.22. If $a \mid \prod_{i=1}^{k} n_i$, then there exist $a_1, a_2, \ldots, a_k$ such that $a = \prod_{i=1}^{k} a_i$ and $a_i \mid n_i$ for each i.

Proof. Let $p_1^{\alpha_{i,1}} \cdots p_r^{\alpha_{i,r}}$ be the prime factorization of each n_i. Then the prime factorization of $n_1 n_2 \cdots n_k$ is $p_1^{\sum_{i=1}^{k} \alpha_{i,1}} \cdots p_r^{\sum_{i=1}^{k} \alpha_{i,r}}$. Let $a = p_1^{\beta_1} \cdots p_r^{\beta_r}$. Then we have $0 \leq \beta_i \leq \sum_{i=1}^{k} \alpha_{i,r}$ for

$i = 1, \ldots, k$. By Lemma 2.21, we have $\beta_{i,j}$ such that $\beta_j = \sum_{i=1}^{k} \beta_{i,j}$ and $\beta_{i,j} \leq \alpha_{i,j}$. Now for each i, take $a_i = p_1^{\beta_{i_1}} \cdots p_r^{\beta_{i,r}}$. Then $a = a_1 \cdots a_k$ and each $a_i \mid n_i$. □

Definition 2.23. A number N is said to be *square-free* if its prime factorization contains only one occurrence of each prime.

Proposition 2.24. An alternate definition for a square-free integer N is if whenever $d^2|N$, then $d^2 = 1$.

Proof. We will prove that the two definitions of square-free are equivalent.

($\Rightarrow$) Let $N = p_1 p_2 \ldots p_k$, and suppose that $d^2 \mid N$ but $d^2 \neq 1$. Then $d \neq 1$ and $d \mid N$, so d shares prime divisors with N. Without loss of generality, let $d = p_1$. Then $d^2 = p_1^2$, so $p_1^2 \mid N$, contradicting our assumption. So we must have $d^2 = 1$, showing $d^2 \mid N \Rightarrow d^2 = 1$.

($\Leftarrow$) We will show the contrapositive is true. Let $N = p_1^{a_1} \ldots p_k^{a_k}$, where at least one $a_i > 1$. Without loss of generality, assume $a_1 > 1$. Then $p_1^2 \mid N$. But $p_1 \neq 1$. Hence

$d^2|N \not\Rightarrow d^2 = 1$. So we must have $a_1 = a_2 = \ldots = a_k = 1$. Thus the definitions are equivalent. □

3. ELEMENTARY CLT GROUPS AND PROPERTIES

For this and the remaining sections, groups are assumed to be finite.

Proposition 3.1. Cyclic Groups are CLT.

Proof. Let $G = \langle x \rangle$ be a cyclic group of order n, and let $d|n$. Then $n = md$ for some m. And so $H = \langle x^m \rangle$ is a subgroup of G such that $|H| = d$. Hence G is a CLT group. □

Proposition 3.2. Abelian groups are CLT.

Proof. Let G be abelian such that $|G| = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, where each p_i is prime. Assume $d \mid |G_i|$. If $d = 1$, the result is trivial, so suppose $d > 1$. Then $p_i \mid d$ for some i. By Cauchy's Theorem, for each p_i, there exists $H \trianglelefteq G$ with $|H| = p_i$. Note that $|G/H| < |G|$ since H is nontrivial. Also $d/p_i \mid |G/H|$. By induction, there exists $N/H \leq G/H$ of order d/p_i. By the Lattice Isomorphism Theorem, this corresponds to $N \trianglelefteq G$ where $|N| = d$. So G is CLT. □

Lemma 3.3. If $G = p^n$ for some prime p and $n > 0$, then G has a nontrivial center.

Proof. Let $|G| = p^n$. If $G = Z(G)$, then we are done. So suppose $G \neq Z(G)$. From the class equation, $|G| = |Z(G)| + \sum_{i=1}^{k} [G : C_G(g_i)]$. Since each $g_i \notin Z(G)$, $p|[G : C_G(g_i)]$. Otherwise, $G/C_G(g_i)$ would be trivial. It is also clear that $p \mid |G|$. Hence p divides the order of $Z(G)$, and so $Z(G)$ is nontrivial. □

Proposition 3.4. Let $|G| = p^n$ for some prime p and $n > 0$. If $p^\alpha \mid |G|$, then there exists $H \leq G$ such that $|H| = p^\alpha$. Furthermore, H may be taken to be normal.

Proof. By Lemma 3.3, $|Z(G)| \neq 1$. That is, $|Z(G)| = p^\beta$, $\beta \neq 0$. From this, we have $|G/Z(G)| < |G|$. If $\alpha \leq \beta$, then $p^\alpha \mid p^\beta$. Note that $Z(G)$ is abelian and so CLT by Proposition 3.2. Hence there exists $H \leq Z(G) \leq G$ such that $|H| = p^\alpha$. Furthermore, $H \trianglelefteq G$, since H is contained in $Z(G)$. Now suppose $\alpha > \beta$. Since the center is nontrivial, $|G/Z(G)| < |G|$. By induction, there exists a normal subgroup $H/Z(G)$ of $G/Z(G)$ such that $|H/Z(G)| = p^{\alpha-\beta}$. By the Lattice Isomorphism Theorem, $H \trianglelefteq G$, and $|H| = p^\alpha$. By [5, pg 188], H may be chosen to be normal. □

Corollary 3.5. Every p-group is CLT.

Proof. Let G be a p-group, then $|G| = p^a$ for some prime p and some a. Let d divide the order of G. Then $d = p^b$, where $b \leq a$. Then by Proposition 3.4, there exists a subgroup of G of order d, and so G is CLT. □

Corollary 3.6. If G is a group with $p^\alpha \mid |G|$ for some prime p , then there exists a subgroup H of G such that $|H| = p^\alpha$.

Proof. Let β be the highest power of p dividing $|G|$. Then there exists $P \in \mathrm{Syl}_p(G)$, which is a p-group of order p^β. Since $\beta \geq \alpha$, $p^\alpha \mid p^\beta$. It follows from Corollary 3.5 that there exists a subgroup of G with order p^α. □

This result generalizes Sylow's Theorem in that it guarantees p-subgroups for every power of each prime dividing the order of a group G, while Sylow's Theorem only gave us the existence of subgroups for the maximum powers of each prime dividing $|G|$.

Theorem 3.7. A product of CLT groups is itself CLT.

Proof. We will prove this by inducting on the number of groups. Let $|G_1| = n_1$ and $|G_2| = n_2$. Then $|G_1 \times G_2| = n_1 n_2$. Let $a \mid n_1 n_2$. By Proposition 2.22, there exist a_1, a_2 such that $a = a_1a_2$, $a_1 \mid n_1$, and $a_2 \mid n_2$. Since G_1 and G_2 are CLT, there exist $H_1 \leq G_1$ and $H_2 \leq G_2$ such that $|H_1| = a_1$ and $|H_2| = a_2$. Then $|H_1 \times H_2| = a_1a_2 = a$ and $H_1 \times H_2 \leq G_1 \times G_2$. Now, assume $G_1, \dots G_n, G_{n+1}$ are CLT groups. By induction, $\prod\limits_{k=1}^{n} G_k$ is CLT. Then we have $\prod\limits_{k=1}^{n+1} G_k = (\prod\limits_{k=1}^{n} G_k) \times G_{n+1}$. This is the direct product of two CLT groups, and so is CLT. $\square$

We've shown CLT groups are closed under direct products. This is not the case, however, for subgroups or quotient of CLT groups.

Example 3.8. Proposition 2.20 showed that S_4 is a CLT group. However, A_4, which is a subgroup of S_4 of order 12, has no subgroup of order 6, and so is is not CLT. This gives us an example of a non-CLT subgroup of a CLT group.

Example 3.9. Consider $A_4 \times C_2$. This has order $2^3 \cdot 3$. Finding subgroups of orders 1 and 24 is trivial. Corollary 3.6 gives us subgroups of order 2, 3, 4, and 8. Now, A_4 is a subgroup of order 12, and $A_3 \times C_2$ is a subgroup of order 6. And so $A_4 \times C_2$ is CLT. However, $(A_4 \times C_2)/C_2 \cong A_4$ is not CLT, giving us an example of a CLT group with a CLT subgroup, whose quotient is not CLT. This is also an example of a product of two groups that is CLT, but only one of the factors is CLT.

3.1 Groups of Square-Free Order

Proposition 3.10. If $|G| = pq$ for some primes p and q, then G is CLT.

Proof. Subgroups of orders 1 and pq clearly exist. Cauchy's Theorem gives us subgroups of orders p and q. $\square$

Here, we have an example of a group with square-free order. That is, $|G|$ is a square-free number, which we defined in the preliminary section as a number whose prime factors have multiplicity one. In this case, the prime factorization contained only two primes. As we will show, this result can be extended to any group with square-free order.

Lemma 3.11. Let G be a group and $N \trianglelefteq G$. If G/N and N are both solvable, then G is solvable.

Proof. Let $N \trianglelefteq G$ such that both N and G/N are solvable. Since N and G/N are each solvable, there exist subnormal series

$$1 = N_k \trianglelefteq \cdots \trianglelefteq N_1 = N$$

$$1 = G_j/N \trianglelefteq \cdots \trianglelefteq G_1/N = G/N$$

whose factor groups are each cyclic. By the Lattice Isomorphism Theorem and since each $G_{i+1}/N \trianglelefteq G_i/N$, we obtain subnormal series $N = G_j \trianglelefteq \cdots \trianglelefteq G_1 = G$. By the Third Isomorphism Theorem, each $G_i/G_{i+1} \cong (G_i/N)/G_{i+1}/N)$ is cyclic. And so we have our desired subnormal series

$$1 = N_1 \trianglelefteq \cdots \trianglelefteq N_1 = N = G_j \trianglelefteq \cdots \trianglelefteq G_1 = G$$

with cyclic quotients. Hence G is solvable. $\square$

Definition 3.12. Let G be a finite group where p is a prime dividing the order of G, and $P \in \mathrm{Syl}_p(G)$. Then a normal *p-complement*, Q, of P is a normal subgroup of G such that $P \cap Q = 1$ and $PQ = G$.

Lemma 3.13. [4, pg 327] Let P be a Sylow p-subgroup of a group G. If P is a subgroup of $Z(N_G(P))$, then P has a normal p-complement in G.

Lemma 3.14. Let p be the smallest prime dividing $|G|$. If $P \in \mathrm{Syl}_p(G)$ and P is cyclic, then there exists a normal complement of P in G.

Proof. Let $x \in N_G(P)$. Then $xPx^{-1} = P$. Let $\phi : N_G(P) \to \mathrm{Aut}(P)$ be given by $\phi(x) = f_x$, where $f_x(g) = xgx^{-1}$. Now, P is cyclic, so abelian, and hence $P \leq \ker \phi$. We have the induced homomorphism

$$\theta : N_G(P)/P \to \mathrm{Aut}(P).$$

Since $|P| = p^k$ is the highest power of p dividing $|G|$, $p \nmid |N_G(P)/P|$. Any prime dividing the order of $N_G(P)/P$ must be greater than p. Now since P is cyclic of order p^k, $|\mathrm{Aut}(P)| = (p-1)p^{k-1}$. Any prime factor of $|\mathrm{Aut}(P)|$ must therefore be less than or equal to p, and so $\gcd(|N_G(P)/P|, |\mathrm{Aut}(P)|) = 1$, and thus θ is the trivial mapping. Let $g \in P$ and take $\theta(xP)[g] = xgx^{-1} = g$. It follows that $xg = gx$, and so $g \in Z(N_G(P))$. So $P \leq Z(N_G(P))$. From Lemma 3.13, it follows that there exists a normal p-complement of P in G. □

Theorem 3.15. If G has square-free order, then G is solvable.

Proof. Let G be a group with square-free order. Let $|G| = p_1p_2 \ldots p_r$. We will induct on r.

If $r = 1$, then $|G| = p$, and so G is cyclic, hence solvable. Assume the result is true for $r = k$. Now suppose that $r = k + 1$. Then $|G| = p_1p_2 \ldots p_{k+1}$, where $p_1 < p_2 < \ldots p_{k+1}$. By Lemma 3.14, there exists $N \trianglelefteq G$ such that $|G/N| = p_1$. But $|N| = p_2 \ldots p_{k+1}$. By induction, N is solvable. Also, G/N is cyclic, and thus solvable. By Lemma 3.11, G is solvable. □

Lemma 3.16. Suppose N is square-free and $mn = N$. Then $\gcd(m, n) = 1$.

Proof. Suppose $\gcd(m, n) = d$. Then $m = m'd$ and $n = n'd$. So $N = mn = m'n'd^2$. That is, $d^2 \mid N$, and so $d^2 = 1$. By Proposition 2.24, $1 = d = \gcd(m, n)$. $\square$

Definition 3.17. Let G be a finite group. If H is a subgroup of G whose order is relatively prime to its index in G, then H is called a *Hall subgroup*.

Sylow p-subgroups are Hall subgroups whose order has only one prime factor, while p-complements are Hall subgroups whose index has only one prime. The next result from Hall generalizes the existence of these subgroups in solvable groups.

Theorem 3.18. [6, Theorem 9.3.1] If G is solvable and $|G| = mn$, where m and n are relatively prime, then there exists a Hall subgroup of G of order m.

Theorem 3.19. A group G of square-free order is a CLT group.

Proof. Let G be a group of square-free order. By Theorem 3.15, G is solvable. Suppose $m \mid |G|$. Then $|G| = mn$ for some n. By Lemma 3.16, m and n are relatively prime. It follows by Theorem 3.18 that G has a Hall subgroup of order m. Hence G is a CLT group.

$\square$

3.2 Nilpotent Groups

Definition 3.20. A central series of a group G is a series of subgroups

$$1 = G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

such that each G_{i+1}/G_i is contained in the center of G/G_i.

In a similar fashion to our definitions of supersolvable and solvable groups, we have a new classification.

Definition 3.21. A group G is called *nilpotent* if it has a finite central series.

All abelian groups are nilpotent. A nonabelian example of a nilpotent group is the quaternion group, which has central series $\{1\} \trianglelefteq \{1, -1\} \trianglelefteq Q_8$.

Lemma 3.22. [6, Corollary 10.3.1] Suppose G is nilpotent and H is a proper subgroup of G. Then H is a proper subgroup of $N_G(H)$.

This is the normalizer property, and it generalizes on a property of abelian groups: that every subgroup of an abelian group is normal.

Theorem 3.23. [5, pg 191] A group G is nilpotent if and only if it is the direct product of its Sylow p-subgroups.

Proof. ($\Rightarrow$) Suppose G is nilpotent. First, we will show that every Sylow p-subgroup of G is normal. Let $P \in \mathrm{Syl}_p(G)$ for some $p \mid |G|$. Since $P \trianglelefteq N_G(P)$, it is unique in $N_G(P)$ of order p^k. Since automorphisms preserve orders, we get that P is characteristic in $N_G(P)$. But $N_G(P) \trianglelefteq N_G(N_G(P))$. So by Proposition 2.4, $P \trianglelefteq N_G(N_G(P))$. And so $N_G(N_G(P)) \leq N_G(P)$, showing $N_G(N_G(P)) = N_G(P)$. By Lemma 3.22, this means $N_G(P)$ cannot be a proper subgroup of G, and so $N_G(P) = G$. Thus $P \trianglelefteq G$. But if every Sylow p-subgroup of G is normal in G, then G is the direct product of them.

($\Leftarrow$) Now, suppose $G \cong P_1 \times \cdots \times P_n$ is the direct product of its Sylow p-subgroups. Note that $Z(P_1 \times \cdots \times P_n) \cong Z(P_1) \times \cdots \times Z(P_n)$. Each P_i is a p-group, so $Z(P_i) \neq 1$ by Lemma 3.3. If $G \neq 1$, then $|G/Z(G)| < |G|$. By induction, $G/Z(G)$ is nilpotent, giving us central series

$$Z(G)/Z(G) = Z_1/Z(G) \trianglelefteq \cdots \trianglelefteq Z_k/Z(G) = G/Z(G).$$

By the Third Isomorphism Theorem and Lattice Isomorphism Theorem, we have central series $1 \trianglelefteq Z(G) \trianglelefteq Z_1 \trianglelefteq \cdots \trianglelefteq Z_k = G$, showing G is nilpotent. $\square$

Note that Sylow p-subgroups are p-groups, and are CLT by Corollary 3.5. Since G is a finite direct product of these, it must be CLT by Theorem 3.7. However, we will show a stronger result.

Theorem 3.24. Let G be nilpotent and d divide the order of G. Then G has a normal subgroup of order d.

Proof. Let G be nilpotent such that $|G| = p_1^{a_1} \cdots p_n^{a_n}$. Then each $P_i \in \mathrm{Syl}_{p_i}(G)$ is unique and thus normal in G. By Theorem 3.23, $G \cong P_1 \times P_2 \times \cdots P_n$, where $P_i \in \mathrm{Syl}_{p_i}(G)$. Let d divide $|G|$. Then $d = p_1^{b_1} \cdots p_n^{b_n}$, where $0 \leq b_i \leq a_i$. By Proposition 3.4, each P_i has a normal subgroup H_i of order $p_i^{b_i}$. Take

$$H = H_1 \times \cdots \times H_n.$$

Then H is a normal subgroup of G with order d. $\square$

Therefore nilpotent groups satisfy a stronger version of CLT in that for each divisor d of the order, there is a normal subgroup of order d.

4. GROUPS OF ORDERS p^2q AND p^2q^2

In the previous section, we showed that a group of order pq, where p and q are distinct primes, are CLT. We then generalized this result to any group having a square-free order. In this section, we will again restrict the prime factorization of the orders of our groups to two primes, except now with the condition that either one or both of the primes are squared.

4.1 Groups of Order p^2q

Definition 4.1. Let A be an $n \times n$ matrix. Then the *characteristic polynomial* of A is given by

$$\rho(\lambda) = \det(\lambda I - A)$$

where I is the $n \times n$ identity matrix.

Theorem 4.2. [6, Theorem 9.3.2] A group of order $p^a q^b$, where p and q are primes, is solvable.

Corollary 4.3. If G is a group of order $p^a q^b$, where p and q are prime, then there exists $N \trianglelefteq G$ with index p or q.

Proof. Since G has order $p^a q^b$, it is solvable by Theorem 4.2. Let N be a maximal normal subgroup of G. Then G/N is simple and solvable, and so cyclic of prime order. Hence $[G : N] = |G/N| = p$ or q. $\square$

Lemma 4.4. Matrices that are conjugates of each other have the same characteristic polynomial.

Proof. Let A and B be $n \times n$ matrices with corresponding characteristic polynomials $\rho_A(\lambda)$ and $\rho_B(\lambda)$. Suppose $A = RBR^{-1}$. That is, A and B are conjugates of each other. Then we have

$$\begin{aligned}
\rho_A(\lambda) &= \det(\lambda I - A) \\
&= \det(\lambda I - RBR^{-1}) \\
&= \det(R\lambda I R^{-1} - RBR^{-1}) \\
&= \det(R(\lambda I - B)R^{-1}) \\
&= \det(R)\det(\lambda I - B)\det(R^{-1}) \\
&= \det(\lambda I - B) \\
&= \rho_B(\lambda)
\end{aligned}$$

□

Theorem 4.5. Let G be a non-abelian group of order p^2q. Then G is a CLT group if and only if $q \nmid p+1$ or $q = 2$.

Proof. Suppose G has order p^2q. By Sylow's Theorem, G has subgroups of order p^2 and q. By Cauchy's Theorem, G has a subgroup of order p. Hence G is CLT if and only if it has a subgroup of order pq. Let $Q \in \mathrm{Syl}_q(G)$. By Corollary 4.3, G has a normal subgroup N with prime index. If $[G : N] = p$, then $|N| = pq$, and so G is a CLT group. If $[G : N] = q$, then $|N| = p^2$, and so $N \cong C_{p^2}$ or $N \cong C_p \times C_p$. Let $N \cong C_{p^2}$. Now, C_p is characteristic in G, so $C_pQ \leq G$ and $|C_pQ| = pq$, showing that G is CLT. Now, suppose $C_p \times C_p \cong N \trianglelefteq G$.

We have $G \cong (C_p \times C_p) \rtimes_\phi C_q$ with $\phi : C_q \to \mathrm{Aut}(C_p \times C_p) \cong \mathrm{GL}_2(\mathbb{F}_p)$, and so

$$\begin{aligned} |\,\mathrm{Aut}(C_p \times C_p)| &= |\,\mathrm{GL}_2(\mathbb{F}_p)| \\ &= (p^2-1)(p^2-p) \\ &= p(p-1)^2(p+1) \end{aligned}$$

If ϕ is trivial, then $G \cong C_p \times C_p \times C_q$, which contains $C_p \times C_p \times 1$ as a subgroup of order pq. So we may assume ϕ is nontrivial. Let x generate C_q and ϕ map x to $A \in \mathrm{GL}_2(\mathbb{F}_p)$. Then A is a non-identity element of order q. By Lagrange's theorem, $q \mid p(p-1)^2(p+1)$. Clearly $q \nmid p$, so either $q \mid p+1$ or $q \mid p-1$.

Case 1: Suppose A is not be diagonalizable. Then the Jordan Canonical Form of A is $\left(\begin{smallmatrix} \alpha & 1 \\ 0 & \alpha \end{smallmatrix}\right)$. That is,

$$A = R \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} R^{-1}.$$

And in general,

$$A^q = R \begin{pmatrix} \alpha^q & q\alpha^{q-1} \\ 0 & \alpha^q \end{pmatrix} R^{-1} = I.$$

So we have $0 = q\alpha^{q-1} \in \mathbb{F}_p$. Hence $p \mid q$, which contradicts the fact that p and q are distinct primes. So it is necessary that A be diagonalizable.

Case 2: Now, suppose A is diagonalizable. Then we have

$$A = R \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R^{-1}.$$

That is,

$$A^q = R \begin{pmatrix} \alpha^q & 0 \\ 0 & \beta^q \end{pmatrix} R^{-1}.$$

Case 2a: If $q \mid p-1$, then $\alpha^{p-1} = 1$. So $\alpha \in (\mathbb{F}_p)^\times$. There exists $\vec{u} \in \mathbb{F}_p^2$ such that $A\vec{u} = \alpha\vec{u} \in \langle \vec{u} \rangle$. That is, $x\vec{u}x^{-1} \in \langle \vec{u} \rangle \cong C_p$. Hence $\langle x, \vec{u} \rangle \cong C_p \rtimes C_q \leq G$ has order pq. And so G is a CLT group.

Case 2b: Suppose $q \mid p+1$. Without loss of generality, we may assume that $q \nmid p-1$, otherwise 2a applies. That is, $q \neq 2$. By Cauchy's Theorem, there exists $A \in \mathrm{GL}_2(\mathbb{F}_p)$ of order q. We will now show that at least one of the eigenvalues of A is contained in $\mathbb{F}_{p^2} - \mathbb{F}_p$. By way of contradiction, assume neither are. That is, $\alpha, \beta \in \mathbb{F}_p$. Since A is invertible, $\alpha, \beta \neq 0$. So both are units. That is, they are contained in $\mathbb{F}_p^\times$, which has order $p-1$. By Lagrange's theorem, $\alpha^{p-1} = \beta^{p-1} = 1$. But note that

$$I = A^q = R \begin{pmatrix} \alpha^q & 0 \\ 0 & \beta^q \end{pmatrix}.$$

It follows that $\alpha^q = \beta^q = 1$, but q is the lowest power doing so. Hence $q \mid p-1$, contradicting our assumption. So either α or β is contained in $\mathbb{F}_{p^2} - \mathbb{F}_p$. Without loss of generality, assume it is α. We have

$$
\begin{aligned}
\rho_A(\lambda) &= \det \begin{pmatrix} \lambda - \alpha & 0 \\ 0 & \lambda - \beta \end{pmatrix} \\
&= (\lambda - \alpha)(\lambda - \beta) \\
&= \lambda^2 - (\alpha + \beta)\lambda + \alpha\beta
\end{aligned}
$$

Since $x^2 - (\alpha + \beta)x + \alpha\beta \in \mathbb{F}_p[x]$, it is true that $\alpha + \beta \in \mathbb{F}_p$. So we must have $\beta \notin \mathbb{F}_p$. Otherwise, $\alpha = \alpha + \beta - \beta \in \mathbb{F}_p$, contradicting our assumption. Now, let $G = (C_p \times C_p) \rtimes_\phi C_q$ (where c generates C_q) under the action $c\vec{v}c^{-1} = A\vec{v}$. Suppose H is a subgroup of order pq. Let $\vec{v} \in C_p \times C_p$ have order p. We need to find a vector of order q. Let $\vec{w} \in C_p \times C_p$. Then $(\vec{w}c)^q = (\vec{w} + A\vec{w} + \cdots + A^{q-1}\vec{w})$. So $(A - I)(\vec{w}c)^q = (A^q - I)\vec{w} = 0$. But note that

$$
\begin{aligned}
\det(A - I) &= \det(R \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} R^{-1} - I) \\
&= \det \begin{pmatrix} \alpha - 1 & 0 \\ 0 & \beta - 1 \end{pmatrix} \\
&= (\alpha - 1)(\beta - 1) \\
&\neq 0
\end{aligned}
$$

So we must have $(\vec{w}c)^q = 0$, showing $\vec{w}c$ is an element of order q. Thus

$$H = \langle \vec{v}, \vec{w}c \rangle$$

, and so $\vec{w}c\vec{v}(\vec{w}c)^{-1} \in H$. But

$$\begin{aligned} \vec{w}c\vec{v}(\vec{w}c)^{-1} &= \vec{w}c\vec{v}c^{-1}\vec{w} \\ &= \vec{w} + c\vec{v}c^{-1} + w^{-1} \\ &= c\vec{v}c^{-1} \\ &= A\vec{v} \neq \lambda\vec{v}, \end{aligned}$$

where $\lambda \in \mathbb{F}_p$. (We showed the eigenvalues of A must be in $\mathbb{F}_{p^2} - \mathbb{F}_p$.) Thus $A\vec{v}$ and $\vec{v}$ are linearly independent, so H has order p^2q, contradicting the assumed order of H. Therefore there cannot be a subgroup of order pq. Hence it is necessary that $q \nmid p+1$ or, if it does, then $q = 2$. $\square$

Example 4.6. Suppose $|G| = 7^2 \cdot 3$. 3 divides 7-1 and, since $3 \neq 2$, 3 does not divide 7+1. It suffices to show there is a subgroup of order $3 \cdot 7 = 21$. Now, let $Q \in \mathrm{Syl}_3(G)$. By 4.3, there exists $N \trianglelefteq G$ such that $[G : N] = 3$ or 7. If $[G : N] = 7$, then $N = 21$. Suppose $[G : N] = 3$. Then $|N| = 7^2$. So $N \cong C_{49}$ or $N \cong C_7 \times C_7$. If $N \cong C_{49}$, then we have C_{49}. C_7 is characteristic in N, so $C_7Q \leq G$ has order 21. Let $N \cong C_7 \times C_7$. Then $G \cong (C_7 \times C_7) \rtimes_\phi C_3$. Suppose ϕ is non-trivial, and let x generate C_3. Then $\phi(x) = A \in GL_2(\mathbb{F}_7)$, where $A \neq I$. We have $A = R\left(\begin{smallmatrix} \alpha & 0 \\ \beta & 0 \end{smallmatrix}\right)R^{-1}$. But $A^3 = I$. Since $3 \mid 7-1$, $A^{7-1} = I$. So $\alpha^{7-1} = 1$, hence $\alpha \in (\mathbb{F}_7)^\times$. So there exists $\vec{u} \in \mathbb{F}_{\not p}{}^2$ such that $A\vec{u} = \alpha\vec{u} \in \langle\vec{u}\rangle \cong C_7$. And so $\langle x, \vec{u}\rangle \cong C_3 \times C_7 \leq G$ has order 21.

Example 4.7. Let C_3 be generated by x and suppose

$$\phi : C_3 \to \mathrm{Aut}(C_5 \times C_5) \cong \mathrm{GL}_2(\mathbb{F}_5)$$

maps x to some $M \in \mathrm{GL}_2(\mathbb{F}_5)$. We have $C_5^2 \rtimes_\phi C_3$ has order $5^2 \cdot 3 = 75$. Then $M^3 = \phi(x^n) = I$, so M has order 3. Take

$$M = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Note that $M^2 + M + I = 0$ and $M^4 + M^2 + I = 0$. Let $v \in C_5 \times C_5$ so that (v, x^i) is in G. Then i is either 0, 1, or 2. If $i = 0$, then $(v, x^i) = (v, 1)$ has order 1 or 5. Now, note that

$$\begin{aligned}
(v, x^i)^3 &= (v, x^i)^2 \cdot (v, x^i) \\
&= (v + M^i v, x^2 i) \cdot (x, v^i) \\
&= (v + M^i v + M^{2i} v, x^{3i}) \\
&= ((I + M^i + M^{2I})v, 1).
\end{aligned}$$

If $i = 1$ or $i = 2$, then this is $(0, 1)$, the identity. And so (v, x^i) has order 3. Hence each element in G has order of either 1, 3, or 5. Suppose there is a subgroup of G with order 15. By Cauchy's Theorem, there is an element of order 15, a contradiction. There cannot be such a subgroup, and so G is not CLT.

4.2 Groups of Order p^2q^2

We have now considered groups of order pq and p^2q. A natural step would be to consider groups of order p^2q^2.

Theorem 4.8. [2, pg 2] Let G be a non-abelian group of order p^2q^2, where p, q are primes with $q < p$. Then G is non-CLT if and only if q is odd and q divides $p+1$.

Proof. ($\Leftarrow$) We have that $q < p$. By Sylow's Theorem, $n_p \equiv 1 \mod p$ and $n_p \mid q^2$. So $n_p = 1$ or q or q^2. We must have the $n_p = 1$, and so $P \in \mathrm{Syl}_p(G)$ is normal. Now, suppose G contains a subgroup H of order pq^2. There exists $Q \in Syl_q(G)$ that is a subgroup of H. Since q divides $p+1$, q does not divide $p+1$, since it is odd. Now by Sylow's Theorem, $n_q = 1$ or p. But $n_q \equiv 1 \mod q$. Since $q \nmid p-1$, we must have that $n_q = 1$, and so $Q \trianglelefteq H$. And so the normalizer of Q in G is either H or all of G. Hence we must either $[G : N_G(Q)] = 1$ or $[G : N_G(Q)] = p$. It follows that the number of conjugates $n_q(G)$ of Q is either 1 or p. If $n_q(G) = p$, then $p \equiv 1 \mod q$, contradicting q being odd. And so $n_q(G) = 1$, so $Q \trianglelefteq G$. Since P, Q are normal in G such that $P \cap Q = 1$, $G \cong Q \times P$ is abelian, contradicting our assumption. Hence there cannot be a subgroup of G of order pq^2, showing G is non-CLT.

($\Rightarrow$) Suppose G is non-CLT. As shown in the first part of the proof, G has a normal Sylow p-subgroup P. If Q is some group of order q, then the product of Q and P is a subgroup of order p^2q. If G contained a subgroup of order pq^2, then the intersection of it with a subgroup of order p^2q would yield a subgroup of order pq, making G CLT. And so there cannot be such a subgroup. Therefore there cannot be a normal subgroup in G of order p, as its product with a Sylow q-subgroup would be one. P is non-cyclic and so G has $p+1$ subgroups of order p. Now, every subgroup of order p is normal in P. If H is a subgroup of order p, then $N_G(H) \geq N_P(H) = P$ (since $H \trianglelefteq P$). Consequently, $|N_G(H)| \geq p^2$, and so $1 < [G : N_G(H)] \leq q^2$. Now, q divides the number of conjugates of H which is equal to q or q^2. And so $q \mid p+1$. $\square$

5. FURTHER CLASSIFICATION

Recall from the first section the definitions of supersolvable and solvable groups, and that all supersolvable groups are solvable. As we will show in this section, CLT groups are properly contained between supersolvable and solvable groups.

5.1 CLT Groups are Solvable

Lemma 5.1. [6, Theorem 9.3.3] If a finite group G contains a p-complement for every prime p dividing its order, then G is solvable.

Suppose n_i is the maximum power for each prime p_i such that $p_i^{n_i}$ divides the order of a group G. Then if G has a subgroup with index $p_i^{n_i}$ for each p_i, then G is solvable. Now, recall the definition of a Hall subgroup: a subgroup whose order is relatively prime to its index.

Lemma 5.2. [3, Lemma 1] Let $|G| = n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $n_i = n/p_i^{a_i}$ for each i. Then G is solvable if and only if G has a subgroup of order n_i for each i.

Proof. ($\Rightarrow$) Assume G is solvable such that $G = n = p_1^{a_1} \cdots p_n^{a_n}$. Take $n_i = n/p_i^{a_i}$. Then $|G| = n_i p_i^{a_i}$ and $\gcd(n_i, p_i^{a_i}) = 1$. By 3.18, there exists a Hall subgroup of G of order n_i.

($\Leftarrow$) Let $|G| = n = p_1^{a_1} \ldots p_n^{a_n}$ and suppose for each prime dividing the order of G, there exists a subgroup H_i of G of order $n/p_i^{a_i}$. Then H_i is a p-complement for each p_i dividing G. By Lemma 5.1, G is solvable. $\square$

With this result, we are now ready to show the first containment.

Theorem 5.3. [3, Theorem 1] All CLT groups are solvable.

Proof. Let G be CLT. If $|G| = 1$, then this is trivial. So assume $|G| = n$ has prime factorization $p_1^{a_1} \dots p_k^{a_k}$. Define $n_i = n/p_i^{a_i}$ for $i = 1, \dots, k$. Each n_i divides n and so G has a subgroup of order n_i for each $i = 1, \dots, k$. By Lemma 5.2, G is solvable. □

Example 5.4. The converse to this is not true. For example, A_4 is solvable but note CLT. In general, if H is a group of odd order, then $A_4 \times H$ is solvable but is not CLT. Let $|H| = h$, where h is odd. Note that A_4 is solvable. By the Feit-Thompson Theorem, H is solvable. Hence $A_4 \times H$ is solvable and $|A_4 \times H| = 12h$. We will now show that $A_4 \times H$ does not have a subgroup of order $6h$. Suppose that such a group, K, exists. Then $[A_4 \times H : K] = 2$, and so K is normal. Also, $(A_4 \times H)/K$ has order 2, and hence is abelian. Let $g, h \in A_4 \times H$. Then $ghK = hgK$, and so $g^{-1}h^{-1}ghK = K$. Hence $[g, h] \in K$, showing $(A_4 \times H)' = A_4' \times H' \leq K$. By Lagrange's theorem, $|A_4'|$ divides $|K|$. But $A_4' = \{(1), (12)(34), (13)(24), (14)(23)\}$, so $4 \mid 6h$, which is a contradiction since h is odd. Therefore, $A_4 \times H$ cannot have a subgroup of order $6h$, and so is not a CLT group.

5.2 Supersolvable Groups Satisfy CLT

Definition 5.5. A *chief series* of a group G is a normal series

$$1 = G_r \trianglelefteq \cdots \trianglelefteq G_1 = G$$

such that for any i, there does not exist a proper subgroup H between G_i and G_{i+1} where $H \trianglelefteq G$.

Lemma 5.6. [6, Corollary 10.5.2] Suppose G is finite and supersolvable of order $p_1 p_2 \dots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$ are primes. Then G has a chief series

$$1 = A_r \trianglelefteq A_r - 1 \trianglelefteq \cdots \trianglelefteq A_0 = G$$

where each A_{i-1}/A_i has order p_i.

Lemma 5.7. [3, Lemma 2] Let $|G| = n = p_1^{a_1} \dots p_n^{a_n}$ such that $p_1 < p_2 < \cdots < p_n$. If G is supersolvable, then there exist normal subgroups of G with orders 1, p_n, $p_n^2, \dots,$ $p_n^{a_n}$.

Proof. Let G be finite supersolvable. Let $|G| = p_1 p_2 \cdots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$. By Lemma 5.6, G has chief series $A_0 \geq A_1 \geq \cdots \geq A_r = 1$, where A_{i-1}/A_i has order p_i. Let $|G| = p_1^{a_1} \dots p_t^{a_t}$, where $p_1 < \cdots < p_t$. Then we have

$$|A_r| = 1 = p_t^0$$

$$|A_{r-1}/A_r| = p_t, \text{ and so } |A_{r-1}| = p_t$$

$$|A_{r-2}/A_{r-1}| = p_t, \text{ and so } |A_{r-2}| = p_t^2$$

$$\vdots$$

$$|A_{r-a_t}/A_{r-(a_t-1)}| = p_t, \text{ and so } |A_{r-a_t}| = p_t^{a_t}$$

Since this is a chief series, each A_i is normal in G. □

In Theorem 5.3, we showed that all CLT groups are solvable, and that this is a proper containment. Now, we are ready for our next containment- that all supersolvable groups are CLT.

Theorem 5.8. [3, Theorem 2] All supersolvable groups are CLT.

Proof. Assume G is supersolvable. We will induct on the number of primes which divide $|G|$. If $|G| = 1$, then this is trivial. If $|G| = p^a$, then G is a p-group and is a CLT group, by Corollary 3.5. Assume G is CLT if $|G| = p_1^{a_1} \dots p_n^{a_n}$ for some n. Let $|G| = p_1^{a_1} \dots p_n^{a_n} p_{n+1}^{a_{n+1}}$, where $p_1 < p_2 < \dots < p_{n+1}$. As usual, suppose d divides the order of G. Then $d = p_1^{b_1} \dots p_{n+1}^{a_{n+1}} = r p_{n+1}^{a_{n+1}}$, where $0 \leq b_i \leq a_i$ and $r = p_1^{b_1} \dots p_n^{b_{n+1}}$. G is supersolvable, and hence solvable. Thus we can find a subgroup $H \leq G$ such that $|H| = |G|/p_{n+1}^{a_{n+1}} = p_1^{a_1} \dots p_n^{a_n}$.

Since G is supersolvable, H is supersolvable. By the induction hypothesis, H is a CLT group since it has n prime divisors. Since $r \mid |H|$, there is a subgroup $R \leq H$ of order r. So R is a subgroup of G with order r. Since G is supersolvable, we may apply Lemma 5.7 to find $P \trianglelefteq G$ with $|P| = p_{n+1}^{b_{n+1}}$. Since P is normal, $RP \leq G$. Also, since the orders of R and P are relatively prime, $|RP| = \dfrac{|R||P|}{|R \cap P|} = |R||P| = rp_{n+1}^{b_{n+1}} = d$. This shows that G contains a subgroup of order d, and it follows that G is a CLT group. $\square$

Example 5.9. The converse is not true. Let G be a CLT group. The group $A_4 \times C_2$ is a CLT group, but not supersolvable. More generally, let G be some CLT group. Then, by Theorem 3.7, $(A_4 \times C_2) \times G$ is also CLT. Note that $A_4 \cong A_4 \times \{e\}$ is not CLT. By Theorem 5.8, A_4 is not supersolvable. It follows that $(A_4 \times C_2) \times G$ is not supersolvable.

Indeed, this sequence of group class containment extends to the following classes covered in Sections 1 and 3:

$$\text{Cyclic} \subset \text{Abelian} \subset \text{Nilpotent} \subset \text{Supersolvable} \subset \text{CLT} \subset \text{Solvable}$$

6. GROUPS OF ORDER $p^r q$

In the third section, we found the conditions for a group of order p^2q to be CLT.

Definition 6.1. Let p be some prime. A group G is *strictly p-closed* if G has a unique Sylow p-subgroup P such that G/P has exponent dividing $p-1$.

Definition 6.2. An abelian group G is *elementary abelian* if every nontrivial element has order p, where p is prime.

Note that all elementary abelian groups are p-groups. The following lemmas are given by Pinnock in [8].

Lemma 6.3. let G be finite and solvable. Then a minimal normal subgroup of G is an elementary abelian subgroup for some prime p.

Lemma 6.4. Suppose N is a minimal normal subgroup of a group G such that N is elementary abelian for some prime p. Then $|N| = p$ if and only if $G/C_G(N)$ is abelian of exponent dividing $p-1$.

Definition 6.5. Let $N \trianglelefteq G$. If N has a normal series whose terms are normal in G and factors are cyclic, then N is said to be *G-supersolvable*.

Lemma 6.6. If $N \trianglelefteq G$ is G-supersolvable and G/N is supersolvable, then G itself is supersolvable. In particular, cyclic-by-supersolvable groups are supersolvable.

Theorem 6.7. [8, Proposition 4.4] If G is strictly p-closed for some prime p, then G is supersolvable.

Proof. We will induct on the order of G. Assume G is strictly p-closed and $S \in \mathrm{Syl}_p(G)$. Consider $|S| = 1$. Then G/S is abelian of exponent dividing $p-1$. But $G \cong G/S$ is abelian, and so G is supersolvable. So consider the case where $|S| \neq 1$. Let Z be the center of S. Since S is a p-group, by Lemma 3.3 Z is nontrivial. Also, since Z is characteristic in S and S is normal in G, we have that Z is normal in G. Thus Z contains a minimal normal subgroup N of G. Since N is contained in the center of S, S centralizes N. By Lemma 6.3, N is elementary abelian for some prime p. Note that by assumption, G/S is abelian of exponent dividing $p-1$. And so, by the Third Isomorphism Theorem, $G/C_G(N) \cong (G/S)/(C_G(N)/S)$ is abelian of exponent dividing $p-1$. By Lemma 6.4, N is cyclic of order p. Since N is nontrivial, $|G/N| < |G|$. Since S is a p-group, it has order p^n, and so $|S/N| = p^{n-1}$. Since S is normal in G, S/N is normal in G/N. Hence S/N is the unique Sylow p-subgroup of G/N. By the Third Isomorphism Theorem, $(G/N)/(S/N) \cong G/S$ is abelian of exponent dividing $p-1$ by the assumption. Hence G/N is strictly p-closed. So by induction, G/N is supersolvable. Since N is cyclic, G/N is cyclic-by-supersolvable. Hence G is supersolvable. $\square$

We will now obtain a result about CLT groups of order $p^r q$.

Theorem 6.8. Let G be a group of order $p^r q$, where p and q are primes such that q divides $p-1$. Then G is a CLT group.

Proof. Let $P \in \mathrm{Syl}_p(G)$. By Sylow's Theorem, $n_p \equiv 1 \mod p$. But n_p divides q, which divides $p-1$. Hence we must have $n_p = 1$, and so $P \trianglelefteq G$. Now, $|G/P| = q$ is cyclic, and so abelian. Also, the order of each element in G/P divides q, which divides $p-1$. Thus G/P has exponent dividing $p-1$. By Theorem 6.7, G is supersolvable. By Theorem 5.8, G is a CLT group. $\square$

Example 6.9. In particular, all groups of order $2p^r$ are supersolvable, and so CLT.

Note that this is not a generalization of Theorem 4.5. For example, a group of order $3^2 \cdot 5$ is a CLT group by that result. However, the order clearly does not meet the conditions for Theorem 6.8, and so it does not apply here.

7. MORE RESULTS

Previously, we showed that all groups of order pq are CLT. We then gave conditions to guarantee that a group of order p^2q is CLT and for a group of order p^2q^2 to not be CLT. Another natural step from p^2q, rather than squaring both primes, would be to consider groups of order p^3q. Marius Tărnăceanu gives the following result.

Theorem 7.1. [9, Theorem 1.1] Let p and q be primes. Then there exists a non-CLT group of order p^3q if and only if q divides $p+1$ or q divides p^2+p+1.

In Theorem 4.8, the conditions we gave to guarantee a non-abelian group of order p^2q^2 (where $q < p$) to not be CLT were that q be odd and q divide $p+1$. However, there are equivalent conditions for this result.

Theorem 7.2. [2, pg 2] Let G be a non-abelian group of order p^2q^2, where $q < p$. Then G is not CLT if and only if one of the following holds:

1. $q = 2$, $p \equiv 3 \mod 4$, $G \cong (C_p \times C_p) \rtimes_\phi C_4$, and ϕ is one-to-one.
2. $G \cong K_4 \rtimes_\phi C_9$ or $K_4 \rtimes_\phi (C_3 \times C_3)$, where K_4 is the Klein-four group.

We can generalize these order to groups of orders p^aq^b, where p and q are primes. Such groups are solvable by Theorem 4.2. John Nganou extends this in [7, pg 3]. He defines a number n to be a CLT number if every group of order n is CLT. An example he gives unlike what we have covered in this thesis, are numbers of the form p^mq^2, where p, q are primes and $q^2 \mid p-1$.

The result of Theorem 3.24 showed that if G is a nilpotent group and d divides $|G|$, then there exists a normal subgroup of G of order d. And so nilpotent groups are strongly CLT. One may investigate if these are the only groups which satisfy this, or if it can be extended to a larger class.

Finally, we end with an interesting result which gives conditions that guarantee a group to be CLT, regardless of order.

Theorem 7.3. [1, Theorem 6] Let G be a finite group such that its commutator subgroup G' is isomorphic to A_4. Then G is CLT.

Example 7.4. Consider the group S_4. $S_4/A_4 \cong C_2$, and so $S_4' \leq A_4$. Now, take $(a\ b\ c) \in S_4$, where a, b, c are distinct. Then

$$[(a\ b) : (a\ c)] = (a\ b)(a\ c)(a\ b)(a\ c) = (a\ b\ c)$$

is contained in S_4'. Since A_4 is generated by the 3-cycles of S_4, we have $A_4 \leq S_4'$. And so the commutator subgroup of S_4 is A_4. As we showed earlier, S_4 is a CLT group.

8. CONCLUSION

In this thesis, we explored finite groups satisfying the converse to Lagrange's theorem. We investigated groups by their orders, namely groups with orders that are square-free or consisting of two prime factors. We also showed that there is a proper containment of supersolvable groups in CLT groups, and of CLT groups in solvable groups. Finally, we outlined other results which, amongst others, are worthy of further investigation.

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