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Transitioning from the Abstract to the Concrete: Reasoning Algebraically

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TRANSITIONING FROM THE CONCRETE TO THE ABSTRACT:

REASONING ALGEBRAICALLY

A Master's Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics Education

By

Andrea Lynn Martin

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ABSTRACT

Why are students not making a smooth transition from arithmetic to algebra? The purpose of this study was to understand the nature of students' algebraic reasoning through tasks involving generalizing. After students' algebraic reasoning had been analyzed, the challenges they encountered while reasoning were analyzed. The data was collected through semi-structured interviews with six eighth grade students and analyzed by watching recorded interviews while tracking algebraic reasoning. Through data analysis of students' algebraic reasoning, three themes emerged: 1) it was possible for students to reach stage two (informal abstraction) and have an abstract understanding of the mathematical pattern even if they were not transitioning to stage three (formal abstraction), 2) students relied heavily on visualizations of the tasks as reasoning tools to reach stage two (informal abstraction), and 3) using the context of the task to understand the mathematical patterns proved to be the most powerful way to reach stage two (informal abstraction). When analyzing challenges students faced reasoning algebraically one theme emerged: students often needed guidance transitioning from stage to stage of the generalization process. The findings of this study will provide teachers with evidence of the importance of algebraic reasoning tools and strategies to better equip students with algebraic tools.

KEYWORDS: algebraic challenges, generalization, algebraic reasoning, informal algebra, formal algebra

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My motto this past year has been, Proverbs 3:4-5: Trust in the Lord with all your heart and lean not on your own understanding; in all your ways acknowledge Him and He will direct your path. Heavenly Father, thank you for giving me a passion for teaching math and direction during the writing of this thesis. May I honor You in everything I do.

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CHAPTER I: OVERVIEW OF THE STUDY

Every veteran algebra teacher can recall conversations with friends, family and sometimes even complete strangers concerning their respective struggles with algebra. I would often make the comment, “Yes, it is true. Students don’t transition well from arithmetic to algebra.” Recently, when reflecting on this go-to comment of mine, I wondered, what does that even mean? I have taught students over the years whom I know are intelligent, creative problem-solvers, and yet they too struggled to transition from arithmetic to algebra. I recognized their potential and knew there had to be something I could do differently to help these frustrated students. Maybe getting to the bottom of the differences between arithmetic and algebra would be the best place to start.

According to Merriam-Webster’s American Dictionary (2005), arithmetic is a part of mathematics that includes computing nonnegative numbers. Arithmetic deals with concepts you can discuss, see, and recognize. However, algebra is a branch of mathematics that dives into abstractions of the underlying structure of mathematics. This often includes using symbols to represent generalizations of mathematical patterns that are discussed, seen and recognized. Algebraic reasoning is being able to analyze quantitative relationships, notice structure, and generalize (Dougherty et al., 2015). Using Dougherty’s definition of algebraic reasoning, it seems the transition from analyzing and noticing structure of nonnegative numbers should set one up for a smooth transition to analyzing and noticing the structure of mathematical patterns to make abstractions. So why are students struggling to make a smooth transition?

Rationale for the Study

An example from my algebra class when I was a student will set the stage for the topic of this thesis which was the nature of students' abilities and challenges to reason algebraically. I remember learning about arithmetic sequences and being taught how to find the *nth* term of the sequence. My algebra teacher presented the formula $a_n = a_1 + d(n-1)$ while she shared what each variable in the formula represented. The *d* represented the common difference and *n* represented the term of the sequence you would be asked to calculate. Then she talked through a few examples with the class before assigning homework over this topic. The homework looked like this: Using this sequence: 8, 10, 12, 14, 16….Find the 25th term in the sequence. Being the good algebra student I thought I was, I found a common difference of 2 by subtracting the first two terms in the sequence. Then I substituted the numbers, 2 for the *d* and 25 for the *n,* into the formula without asking any questions about how or why this formula worked. Steps to my solution looked exactly like this: $a_{25} = 8 + 2(25-1)$ which would be $8 + 2(24) = 8 + 48 = 56$. And there you have it, the 25th term in the sequence is 56. Well done. But what had I done? This problem had no meaning to me and therefore no algebraic reasoning was taking place like analyzing, noticing structure, or generalizing. Algebra students should be encouraged to find and understand the structure of mathematics rather than blindly following procedures without questioning the rules of the game (Mason, 1996). Maybe reflections like these will lead me to unpack the reason intelligent, creative problem-solvers are struggling with algebra.

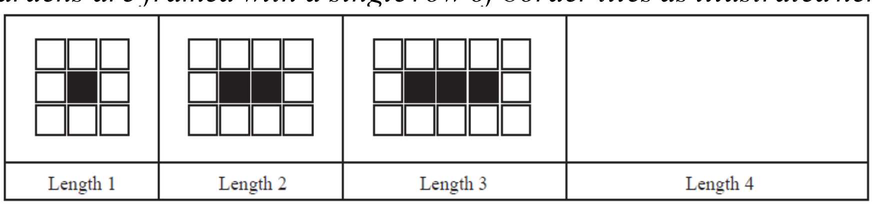
This reflection of how vastly different the definition of algebraic reasoning was compared to how algebra was taught in the 1980's led me to an article by Shelley Kriegler (2007) titled, *Just What is Algebraic Thinking?* This article discussed the importance of pairing informal algebraic reasoning tools such as analyzing mathematical patterns and problem-solving strategies with

formal algebraic ideas which include generalizing patterns with symbols. Her article shared examples of questions that would evoke algebraic thinking. Then months later I found three more articles written by John Lannin (2003, 2004, 2006) suggesting the use of tasks similar to Kriegler's (2007). The tasks that led students to rich and deep algebraic reasoning were called generalization tasks by Kriegler and Lannin.

This next task was taken from the article *Just What is Algebraic Thinking?* by Kriegler (2007) and was one of the three tasks students were asked to solve in this study (see Figure 1). Using generalization tasks such as the one below provided evidence of the nature of students' abilities and challenges for algebraic reasoning.

In this task students were asked to analyze the growth of the garden. Prompt *c* asks students to find the number of border tiles surrounding a garden of length 25. The first frame has eight border tiles, the second frame has ten border tiles, and so

Gardens are framed with a single row of border tiles as illustrated here.



a) *In the space provided, draw the garden of length 4.*
b) *What patterns do you notice about how the gardens grow?*
c) *Find the number of border tiles of the garden of length 25.*
d) *How would you find the number of border tiles of a garden of any length?*
e) *Use this same idea to find the length of a garden surrounded by 70 border tiles.*

Figure 1. The Garden Task with Five Prompts.

on. In other words, the border tile pattern is 8, 10, 12, 14, 16….similar to the sequence I was asked to find the 25^{th} term of by my algebra teacher. However, notice this sequence has a context. This sequence has pictures to represent each stage of the garden's growth. Students were

asked about pattern recognition before being asked the number of border tiles for a larger garden. This task encouraged algebraic thinking like analyzing, noticing structure, and generalizing.

This research study involved observing six eighth grade algebra students solving generalization tasks like the garden task to observe the nature of their abilities and challenges to reason algebraically. The findings of the analyzed data led to strategies teachers can use to help students link arithmetic to algebra by using students' natural reasoning processes and lead them to generalize mathematical patterns. Although algebra consists of many topics, there is one common theme: the importance of making use of student intuition to lead students to generalize the underlying structure of mathematics which uncovers the power of the abstract.

Purpose of the Study

The purpose of this study was to observe, analyze and understand the nature of students' algebraic reasoning through generalization tasks. After students' algebraic reasoning had been analyzed, the challenges they encountered while reasoning were analyzed. The findings of this study proved to be valuable to provide the importance of using tasks, either generalization tasks or other similar tasks that encourage algebraic reasoning. These tasks provided students with practice accessing algebraic reasoning tools they can use to transition from the concrete to the abstract and evidence of when and why students encountered challenges as they generalized the mathematical patterns. Providing students with encouragement that they already possess valuable reasoning tools and can practice and learn new strategies to aid along the way will give students confidence to be successful in algebra and to excel in more advanced mathematics courses (Kriegler, 2007).

Research Questions

The following research questions guided this study:

1. What was the nature of students' algebraic reasoning through tasks involving generalizing?
2. What were the challenges students encountered as they reasoned algebraically through tasks involving generalizing?

Research Design

This study was conducted as an action research design using qualitative data collected from student interviews. Action research is "a process of self-study; thus a teacher engaged in action research may, for example, study ways to increase student learning in his or her class, focusing on his or her intentions, methods, and desired outcomes as part of the investigation." (Hendricks, 2013, p. 3). Hattie and Yates (2014) emphasize the importance of learning through the eyes of your students while they are learning through your eyes. How can you know what goes through the mind of a student unless you ask?

This study included three semi-structured interviews of six eighth grade students from May to August of 2019. During the 45-minute sessions the students were asked to think aloud while solving without direct guidance from the researcher. During the problem-solving process, if a student became unsure how to proceed, then the researcher provided aid so the student could continue through the generalization process. However, the researcher made note of when guidance was needed to expose the challenges hindering each student to reason independently. After the 18 interviews were conducted and recorded, the interviews were watched while tracking the stages of generalization with an observational protocol (Appendix A). The interviews were also transcribed so they could be read several times looking for overarching themes in the nature of students' algebraic reasoning and challenges that kept them from solving the tasks independently.

Significance of the Study

Algebra teachers do not want students who are just good at memorizing rules and following procedures and believe blindly following directions is the key to being proficient in algebra (Kieran, 1992). They want students who analyze, generalize and notice structure. They want those intelligent problem-solvers using their creativity to make mathematical discoveries. Students might believe that mathematics is objective with each question having only one correct answer. However, mathematical reflection is never complete, but rather waiting for a deeper conversation at a later date (Mason, 1996). When this reflection is taking place, the potential for algebraic reasoning is present.

Algebra teachers want their students to learn algebra in a richer, deeper way than they were taught. They are looking for ways to help students, not just pass the class, but rather become deep thinkers by asking and answering the "why" behind mathematics. Teachers are searching for activities that focus on key algebraic ideas like analyzing, noticing structure, and generalizing. Generalization tasks like the garden task will encourage those mathematical discussions.

This study provided valuable feedback for the researcher gained by carefully watching and listening to students' natural algebraic reasoning skills. This data was not only a record of algebraic reasoning skills that students already possess, which should encourage teachers that students are good problem-solvers, but also provided evidence of possible stepping-stones through the stages of generalization needed to arrive at formal algebra. These stepping-stones are algebraic reasoning tools which could help students make the transition from the concrete to the abstract.

Assumptions

Three assumptions were made before conducting the student semi-structured interviews:

1. Students had not seen scaffolded questions that led to generalization.
2. Students asked the researcher to clarify a task, or the next step in a task, when stuck in the problem-solving process. Since challenges students face while generalizing was defined to be the moment the researcher provided input for the student to continue in the reasoning process, students' familiarity with the researcher could have caused them to ask for help even when it was not needed.
3. Students were solving, for the first time, the tasks presented to them during the interview.

Limitations

The following was a list of the limitations of this study:

1. Since the study had only six participants, all from the same school with the same teacher, the results might not be representative of all students' approaches to reasoning algebraically.
2. All six students in this study had just completed Algebra I. Therefore, the data provided may not be representative of students from all levels of algebra.
3. Because the researcher was the teacher of the participants for the previous two years, familiarity could have affected the results if the teacher assumed rather than asked for clarification during the problem-solving process.
4. This study used only three tasks involving generalizing and each task was representing a linear function model. Limiting the tasks to only one type of function might not represent the same analysis had the students been asked to algebraically reason through tasks with different function models.
5. Presenting the questions with scaffolded questions could have altered students' natural strategies for solving the tasks involving generalizing.

Definition of Terms

This was a list of the terms used in the analysis and discussion of the study:

1. Algebraic challenges were defined as stumbling blocks that keep students from completing the given task, at which point, they ask the researcher for clarification or guidance to proceed.
2. Algebraic reasoning was being able to analyze quantitative relationships, notice structure and generalize (Dougherty et al., 2015).
3. Context of the task referred to the story behind each mathematical pattern.
4. Explicit patterns involved finding a strategy to find any term in a mathematical sequence (Lannin, 2004).

5. Formal algebraic procedures pertained to the rules, definitions, properties and formulas typically found in algebra curriculums (Kriegler, 2007).
6. Generalization was discussing, informally abstracting and formally abstracting mathematical patterns, in addition to being able to justify the reasons for the given generalization (Doughtery et al., 2015; Lannin, 2003).
7. Informal mathematical procedures referred to intuitive, logical, reasoning skills (Center for Algebraic Thinking, 2020).
8. Recursive patterns used the previous number in a sequence to find the next term (Lannin, 2004).
9. Scaffolding questions were prompts to lead students independently through the generalization process of any given mathematical task.
10. Visualization of a task was the use of charts, tables, diagrams or pictures, was a valuable tool to search for patterns and justifications of those patterns (Lannin, 2003; Stump, 2011). Visualization in this study did not mean recognition of the mathematical pattern but rather a visual aid to look for patterns.

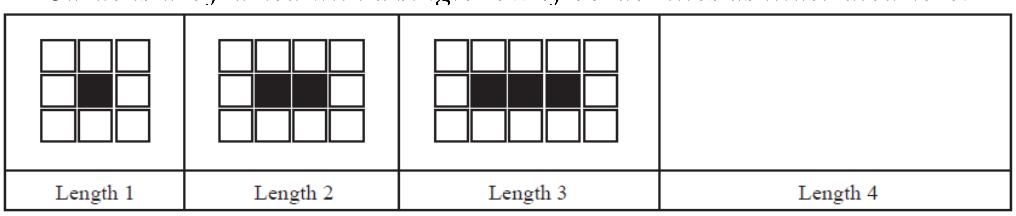
Summary

This qualitative interview study was motived by an effort to encourage algebra students to look for and understand the structure of mathematics rather than blindly following procedures without questioning the rules of the game (Mason, 1996). The purpose of this study was to observe, analyze and understand the nature of students' algebraic reasoning through the generalization process. Therefore six eighth grade algebra students participated in three, 45-minute semi-structured interviews, with each interview consisting of students solving one of three generalization tasks. After students' algebraic reasoning had been analyzed, the challenges they encountered while reasoning were analyzed. The analysis of the data was guided by using an observational protocol to search for when and how students analyzed, noticed structure, and generalized mathematical patterns. The researcher also looked for reasoning tools, both informal and formal, that students accessed to independently reason algebraically. The findings of this study led to evidence of how students generalize mathematical patterns by accessing both informal and formal reasoning tools to help them transition from the concrete to the abstract.

CHAPTER II: REVIEW OF RELATED LITERATURE

Algebraic reasoning includes analyzing, generalizing, and noticing structure in mathematical patterns (Dougherty et al., 2015) and two of the National Council of Teachers of Mathematics (NCTM) eight mathematical practices align with this definition of algebraic reasoning. One of the mathematical practices is to look for and make use of structure (MP 7), and another practice is to look for and express regularity in repeated reasoning (MP 8) (NCTM, 2014). The following prompts (*a* and *b* in Figure 2), related to the garden task from the article *Just What is Algebraic Thinking?* by Kriegler (2007), demonstrate the difference between MP 7 and MP 8. Algebraic reasoning can be encouraged by eliminating the numbers from the task (see Figure 2, prompt *b*) thus inviting students to generalize the pattern.

Gardens are framed with a single row of border tiles as illustrated here.



a) Find the number of border tiles of the garden of length 25.
b) How would you find the number of border tiles for a garden of any length?

Figure 2. The Garden Task with Two Prompts.

Challenging students to generalize mathematical patterns leads them to a transition from the concrete to the abstract. This generalization elicits discussing key algebraic concepts like the common difference, which speaks to the recursive nature of this pattern (Lannin, 2004). The recursive pattern in the garden task is explained by finding the number of border tiles surrounding a garden by adding two more border tiles to the previous garden while the explicit pattern would be the pattern for finding the number of border tiles around a garden of any length (Lannin, 2003). The explicit pattern could be found by multiplying the common difference of

two one less time than the garden length you are calculating and then adding that product to the eight tiles that surround the first garden (Lannin, 2003).

The difference between MP 7 and MP 8 was explained by Mason (1996) when he referred to changing the "particular" to the "general". This way of thinking motivates students to take an opportunity to "look at" and "look through" a problem to find patterns in its mathematical structure. In other words, students start analyzing and noticing structure with concrete numbers until they independently begin to recognize the abstraction or generalization of the mathematical pattern. This chapter will include the meaning and importance of algebraic reasoning, the fostering of algebraic reasoning through generalization tasks, and the challenges students encounter while reasoning algebraically through tasks involving generalizing.

Meaning and Importance of Algebraic Reasoning

According to Blanton and Kaput (2005), algebraic reasoning is the activity of doing, thinking and talking about mathematics from a generalized and relational perspective. A generalized perspective is an awareness that mathematical conjectures or properties of equality hold true for all instances of a similar nature, while a relational perspective includes an understanding that the mathematical properties or definitions can apply to all expressions, both numeric and abstract.

Dougherty et al. (2015) describe an algebraic reasoning skillset as being able to analyze quantitative relationships, generalize, model, justify or prove, predict, problem solve, and notice structure" (Dougherty et al., 2015, p. 274). Because algebra can cover many topics and is very difficult to define, this study will focus on generalizing by analyzing and noticing structure of the mathematical patterns. Through this process, there will also be evidence of problem-solving,

prediction and discussing relationships but these algebraic ideas will weave in and out of students' discussions as they try to generalize mathematical patterns.

Algebraic reasoning is a combination of mathematical thinking with formal algebraic procedures so students can conceptualize mathematical patterns (Kriegler, 2007; Lannin, 2003). The ability to see mathematical structure gives students confidence to focus on the big picture rather than procedural skills (Dougherty et al., 2015). Students with algebraic reasoning skills can initiate the problem-solving process independently and have confidence in their own reasoning skills, which will equip them to go beyond answer-getting and have the abilities to analyze and understand mathematical patterns (Green, 2014).

Algebraic reasoning includes informal algebraic reasoning alongside formal algebraic procedures (Kriegler, 2007). If we want students to understand the algebraic concepts, they must see and recognize the underlying structure which means algebra should be learned from a structural and conceptual approach (Kieran, 1992). Algebra should not be a series of routine computations or comfortingly familiar processes, but a thought-provoking experience, not seen as rules to be memorized but as a way of problem-solving that makes sense (Green, 2014). The use of generalization tasks that elicit algebraic reasoning lead students through the informal and formal reasoning processes, which will also be called the stages of generalization (Mason, 1996).

Fostering Students' Algebraic Reasoning

As seen in the last section, algebraic reasoning can be difficult to define. This study has chosen to adopt Dougherty's definition for algebraic reasoning which includes analyzing, noticing structure, and generalizing mathematical patterns (2015). Therefore, the importance of algebraic reasoning through generalization tasks would include analyzing,

explaining, noticing structure, and generalizing mathematical patterns in either a informal or formal abstraction which is the root of understanding necessary for formal algebra (Blanton & Kaput, 2005; Stump, 2011). Generalizing patterns valuably bridges the gap between the concrete and the abstract (Tabach et al., 2008). Once these connections are made students' algebraic understanding deepens and they learn to appreciate the power of the abstract (Windsor & Norton, 2011). According to Radford (1996) generalizing is not confined to just looking for mathematical patterns but is also one of the key tools to finding scientific knowledge and day-to-day knowledge as well.

Researchers agree algebraic reasoning through generalization includes both algebraic reasoning and algebraic symbolism. For algebraic reasoning to be more than just generalization, it needs to be paired with formal algebra to justify and validate the generalization (Kieran, 1992). Blanton and Kaput (2005) and Lannin (2003) refer to algebraic reasoning as the students' activities of generalizing about the mathematical relationships and establishing those generalizations through a verbal explanation. However, they also stress the importance of eventually expressing these generalizations in formal algebraic ways. For example, using the garden task, students may be capable of recognizing that two additional border tiles are being added to the garden and could verbally explain how to find the number of border tiles surrounding a garden of any length. However, could they translate their verbal generalization to a symbolic form as well? Generalization is defined by the Center for Algebraic Thinking (2020) into stages: seeing, saying, recording, and algebraic representation. These stages are a summary of Mason's (1996) work on generalization:

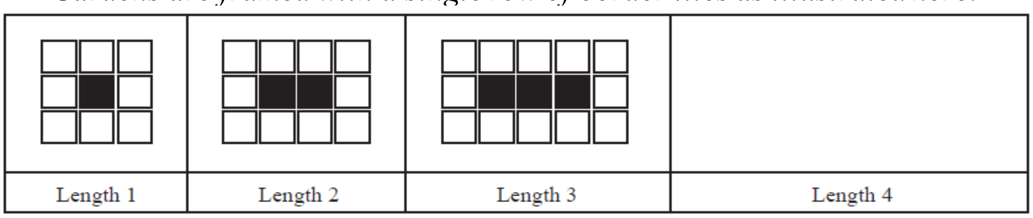
> Taking time to state your rule in simple language helps you to find a formula. By inviting children to express their rules out loud, you create an opportunity for children to exercise their powers of observation and description to an audience, to show tolerance and appreciation of other people's struggles to express

themselves, and to encounter other ways of seeing and thinking which might be useful in the future (pp. 84-85).

Mason's (1996) stages are represented as articulating (saying), getting a sense of, and manipulating (seeing) concrete numbers during the problem-solving process until students arrive at an understanding (getting a sense of) of the mathematical pattern. Once students have a solid understanding of the mathematical pattern, they will have confidence to explain the generalization in their own words (recording). Then students will begin to translate their verbal generalizations into symbolic forms (algebraic representation).

The garden task is an example of one of the generalization tasks presented to students during the interview sessions of this study. Using tasks like the garden task (see Figure 3) to prompt students to analyze mathematical patterns will lead them through the stages of generalization by accessing key algebraic reasoning tools (Lannin, 2006). These reasoning tools will include

Gardens are framed with a single row of border tiles as illustrated here.



a) In the space provided, draw the garden of length 4.
b) What patterns do you notice about how the gardens grow?
c) Find the number of border tiles of the garden of length 25.
d) How would you find the number of border tiles of a garden of any length?
e) Use this same idea to find the length of a garden surrounded by 70 border tiles.

Figure 3. The Garden Task with Five Prompts.

discussing and visualizing mathematical patterns. Discussing while visualizing the tasks helps students recognize important relationships and communicate ideas confidently and independently (Lannin, 2004). Visualization of a task, through the use of charts, tables, diagrams or pictures, is a valuable tool to search for patterns and justifications of those patterns (Lannin, 2003; Stump, 2011). Visualization in this study does not mean recognition of the mathematical pattern but

rather a visual aid to look for patterns. For example, in the garden task, a visualization of the garden might be a student drawing of the garden, different color algebra tiles depicting the garden and its surrounding border tiles, or a table comparing garden tiles to border tiles. According to Lannin (2006), seeing that the number of border tiles is increasing by two in each consecutive garden is not as valuable as providing an explanation of how the specific features of the garden task provide context and understanding to the mathematical pattern. For example, it is valuable to notice two border tiles are added to each frame which is the recursive pattern of this task, but it would be more valuable to notice that, no matter how long the garden is, there will be the same number of border tiles above and below the garden, which gives context to the number two in the recursive pattern. This understanding leads students to justify their answers and explain the mathematical patterns in their own words.

The garden task is from the article *Just What is Algebraic Thinking?* by Kriegler (2007) which focuses on the two components of algebraic thinking: the development of mathematical thinking tools and the study of fundamental algebraic ideas. The garden task will be used as an example to show how students transition through the stages of generalization and make independent discoveries of the recursive and explicit patterns to represent the underlying mathematical structure of the situation given (Tabach et al., 2008). Recursive expressions in the garden task emphasize finding the total number of border tiles surrounding the garden by adding two more tiles to the previous sum. According to Lannin (2004), students naturally see and recognize the recursive pattern when they analyze mathematical patterns. Explicit patterns provide a generalization to find the total number of border tiles regardless of the length of the garden (Lannin, 2004; Mason, 1996). In this day and age, with the use of computers and other forms of technology to calculate patterns, neither the recursive nor the explicit formula is

preferred; however, Lannin (2004) highlights the importance of giving students tasks where recursive patterns are inefficient so they learn to find power in recognizing the explicit pattern of the mathematical situation.

Since Kriegler (2007) has already conducted a study using the garden task, a brief discussion of how students in her study approached the task will be shared. Initially students had no problems seeing the pattern well enough to draw the next image. When asked what patterns do you notice, find the border tiles surrounding a garden of length 25, and how would you find the number of border tiles of a garden of any length, students accessed problem-solving strategies to reach the stages of generalization (seeing, saying, recognizing and algebraic expression), such as making a table, using models and diagrams, and working backwards. For example, some students noticed the first picture is a garden surrounded by eight border tiles and each picture contains two additional tiles. This is an example of what Lannin (2004) would call noticing the common difference of the recursive pattern. Student work also showed that students who used formal algebraic ideas had relied on their informal procedures. They used a visualization of the task to transition from the concrete to the abstract. Kriegler (2007) also mentioned that the conceptual understanding of the mathematical pattern was derived from students' connections between the mathematical pattern and the geometric design of the garden. For example, the students who noticed each picture had two additional border tiles also noticed the first picture started with eight border tiles. Then they concluded that an explicit symbolic generalization representing the number of border tiles for a garden of any length could be found in the expression $8 + 2(n - 1)$. According to Lannin (2004), allowing students to use their own visualization of the pattern leads them to a recognition of the generalization and an opportunity

to compare their generalization to other students and discover different symbolic expressions can be equivalent.

Kriegler (2007) made no mention of how many students were able to write their verbal generalization into a symbolic form, nor did she discuss the challenges students faced when reasoning algebraically. Students do face challenges when reasoning algebraically and common challenges will be addressed in the next section.

Challenges in Students' Algebraic Reasoning

When students are given generalization tasks to encourage algebraic reasoning, finishing the activity provided is not the point but rather situating the student into recognizing the important and not-so-important mathematical patterns, which will be helpful in future mathematics courses (Mason, 1996). Students encounter challenges attempting to find those mathematically helpful patterns. For example, in reference to the garden task, students might reference the ratio of garden tiles to border tiles or the changing sizes of the rectangles. All of these are valid discoveries, but if they are not leading to informal or formal abstractions of the mathematical pattern, then they should not be the focus.

Students naturally discuss and visualize when reasoning algebraically through tasks involving generalizing; however, justifying mathematical patterns is not natural (Lannin, 2006). Students should be encouraged to provide reasons for their explanations of the mathematical patterns for deeper reflection. This mathematical reflection encourages students to construct valid arguments to defend their generalization or critique the generalizations of others. According to Lannin (2006), agreeing or disagreeing with peers' generalizations does not come naturally. Students need practice being able to justify their own answers and why their

classmates may have made errors. Classroom conversations about what the correct answer is and the possibility that several different answers could all be correct will help students transition into formal algebraic procedures and upper level mathematics.

When reasoning algebraically through tasks that involve generalizing, students encounter challenges translating their verbal generalization into a symbolic form. If students have already recognized the mathematical pattern but are not comfortable using symbols to write an algebraic expression such as $8 + 2(n - 1)$, then it is understandable they will not make this transition (Quinlan, 2001). Using this symbolic expression would mean students have to know they are letting n represent the length of the garden and $(n - 1)$ represent a number one less than the garden's length and that $2(n - 1)$ is representing two times a number that is one less than the garden. It is easy to see why students get lost in this process. Their lack of confidence can get them stuck trying only concrete, numerical strategies, like guess and check, rather than looking for an understanding of how the pieces of the given symbolic expression point to the mathematical pattern represented (Lannin, 2004). Classroom discussions of the verbal generalizations cause reflection on mathematical structure and development of confidence (Mason, 1996), but students struggling with formal algebra need help accessing reasoning tools that will help them make the transition from informal to formal abstraction.

Summary

Algebraic reasoning is the activity of analyzing, noticing structure, and generalizing mathematical patterns. Through these mathematical reflections, students will be transitioning through the stages of generalization to form an abstraction of the pattern. The stages of generalization explain the process from manipulating concrete numbers until an understanding of

the pattern can be verbalized and then translated to a symbolic form. Although there are challenges to developing algebraic reasoning through tasks involving generalizing and a risk in taking time to foster a deeper and richer understanding of the underlying mathematical structure, the risk is worth it. Using generalization tasks to encourage algebraic reasoning bridges the gap between arithmetic and algebra. There is no right or wrong way to achieve generalizations but rather power in the ownership of discovery and creativity that each student can call his or her own (Mason, 1996).

CHAPTER III: METHODOLOGY

Educational researchers have reported concern that students do not have a robust understanding of algebraic concepts and underlying mathematical structure (Blanton & Kaput, 2005; Green, 2014; Kieran, 1992). Students are not transitioning from analyzing mathematics they can see and recognize to making verbal and symbolic abstractions of their analyses. The purpose of this study was to observe, analyze and understand the nature of students' algebraic reasoning through generalization tasks. After students' algebraic reasoning had been analyzed, the challenges they encountered while reasoning were analyzed. The findings of this study provided evidence of the importance of using tasks, either generalization tasks or other similar tasks that encouraged algebraic reasoning. This analysis will give educators a glimpse of how to use students' informal algebraic reasoning strategies to solidify a conceptual understanding of mathematical patterns. This chapter will discuss the research design, site of the study, and participants involved in the study. Also included will be an explanation of the ethical considerations taken, followed by methods by which the data was collected and analyzed.

Research Design

This study was conducted to answer two research questions: 1) What was the nature of students' algebraic reasoning through tasks involving generalizing? and 2) What were the challenges students encountered while reasoning algebraically through tasks involving generalizing? This action research study was a qualitative analysis through semi-structured interviews of students using algebraic reasoning while solving generalization tasks. A qualitative approach to research was chosen because "in qualitative inquiry, the intent is not to generalize to

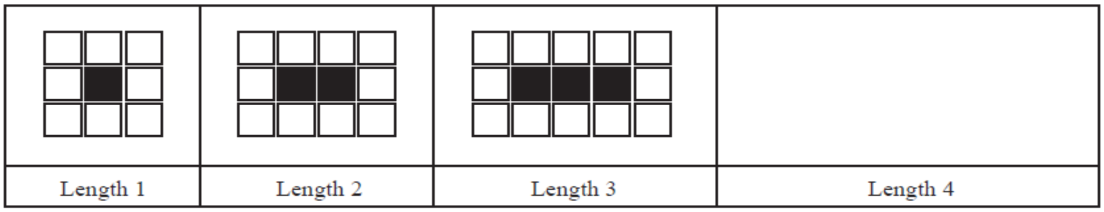
a population, but to develop an in-depth exploration of a central phenomenon" (Creswell, 2008, p. 213). The central phenomenon is students are struggling with algebraic concepts. Therefore, this study's focus was to analyze the nature of students' algebraic reasoning skills to develop teaching and learning strategies that connect independent student thinking to the algebraic ideas. Then, careful notice was made of the challenges' students encountered while generalizing patterns and possible strategies to assist students to a richer understanding of the mathematical concepts.

Six students participated in the study: Ben, Chad, Susan, George, Lance and Jasmine (pseudo names). Originally four students, two of which were Ben and Chad, were asked to participate but two of the original students declined and Susan, George, Lance and Jasmine accepted the invitation to participate along with Chad and Ben. The student interviews were one-on-one and each of the six participants attended three 45-minute interviews, each time solving one of the three generalization tasks.

The generalization tasks (Appendix B) were chosen before the interview specifically to initiate algebraic reasoning such as analyzing, noticing structure, and generalizing. The garden task (see Figure 4), the well task (see Figure 5) and the number sequence task (see Figure 6) were the three tasks chosen for each of the three interviews.

The researcher began each interview encouraging students to discuss what they were thinking during the generalization process. Since the focus of the study was to determine algebraic reasoning tools students naturally used to generalize, there were times the researcher asked probing questions to encourage students to justify their mathematical patterns to determine the depth of their conceptual understandings of the generalizations. For example, if a student stated that there would be 56 border tiles surrounding a garden of length 25, the researcher asked

Generalization Task #1: The Garden Task *(modified from Kriegler, 2007, p. 5)*



Gardens are framed with a single row of border tiles as illustrated here.

a) In the space provided, draw the garden of length 4.
b) What patterns do you notice about how the gardens grow?
c) Find the number of border tiles of the garden of length 25.
d) How would you find the number of border tiles of a garden of any length?
e) Use this same idea to find the length of a garden surrounded by 70 border tiles.

Figure 4. The Garden Task with Five Prompts and a Reference.

Generalization Task #2: The Well Task (from the article *Three Components of Algebraic Thinking: Generalization, Equality, Unknown Quantities*, EdTech Leaders Online)

A frog is at the bottom of a 10-foot well. If he climbs three feet in the first hour and slides back one foot in the second hour and repeats this pattern, how long will it take him to climb out of the well?

a) What pattern or patterns do you notice?
b) How long would it take the frog to exit an 11-foot well?
c) How long would it take the frog to exit a 20-foot well?
d) Could you find the number of hours needed to climb out of a well of any depth?
e) Can you use this idea to find the depth of the well if it takes the frog 15 hours to climb out?

Figure 5. The Well Task.

Generalization Task #3: The Number Sequence Task (from the article *Three Components of Algebraic Thinking: Generalization, Equality, Unknown Quantities*, EdTech Leaders Online)

1, 3, 5, 7, 9, ____, _____, _____

a) Find the next three numbers in the sequence.
b) What pattern do you notice?

Figure 6. The Number Sequence Task.

how he/she determined his/her answer. During discussion of key concepts of the mathematical patterns the researcher did not interject guidance during the problem-solving process but rather

remained available to answer any clarification questions about the tasks. However, when it was apparent the students were challenged to continue with the task, the researcher provided guidance by presenting strategies to help students organize their own ideas to highlight pattern recognitions. For example, if a student said he/she did not know how to explain how to find the number of border tiles for a garden of any length, the researcher suggested he/she look back at previous concrete examples that had been calculated. When guidance was offered and independent algebraic reasoning had ceased, these instances were recorded as algebraic challenges.

Site of the Study

The site of this study is a private K-12 school in Southwest Missouri where the researcher works. The school is comprised of 300 students from ages 5 to 18, and the eighth grade class from which the participants were selected had 23 students between the ages of 13 and 14. Students attending this school must apply and interview before being accepted. The students are either residents of the small town where the school is located or within an hour drive of the campus. All recorded semi-structured interviews were conducted in a classroom of the private school. Before conducting any of the interviews, the school administrator, parents of participants and the participants signed a written consent form accompanied by a letter explaining all the details of the research study (Appendices C & D).

Participants

Since the objective of the study was to observe the nature of students' algebraic reasoning, the six students asked to participate in the study varied in their understanding and confidence

with algebraic concepts. The students were chosen by convenience sampling which is "selecting participants because they are willing and available to be studied" (Creswell, 2008, p. 155). The researcher had already developed a rapport in class with Ben, Chad, Susan, George, Lance and Jasmine during pre-algebra and algebra of the previous two academic years. Because of the familiarity with the students, the researcher would say Ben and Chad enjoyed algebra and discussions related to mathematics while Susan, George, Lance and Jasmine did not share this same view.

Ethical Considerations

Conducting recorded student interviews had the risk of making students feel uncomfortable while solving algebraic problems. Therefore, all methods and considerations were cleared with the Institutional Review Board, IRB-FY2019-244 on October 6, 2018 (Appendix E), Missouri State University advisors, and school administrators. During the interviews, careful attention was taken if students became uncomfortable with the interview or frustrated with a task. Each student knew the interview could be stopped at any time. It was beneficial that students were already familiar discussing mathematics with the researcher which allowed them to feel at ease to ask questions during the problem-solving process. Also, because students often get frustrated explaining the methods and procedures they use to solve the mathematics problems, the researcher deemphasized arriving at the correct answers and emphasized the importance of explaining the algebraic reasoning steps used in the process.

Data Collection Procedures

Each student was asked to participate in three different 45-minute interview sessions solving one of each of the generalization tasks. The eighteen interviews were conducted from May to August of 2019. The researcher was responsible for all phases of this research study which included applying for the IRB approval, selecting the generalization tasks, and transcribing and analyzing the data. Before each interview, a sheet was prepared with one generalization task and its scaffolded questions. Students were reminded at the beginning of each interview that it was important for them talk out loud so there would be a record of each step in their problem-solving process and reasoning for each step as well. The students were given 45 minutes and asked to solve the task to the best of their ability. The interviews were recorded on an iPad with password protection and then transcribed using pseudo names to protect each student's identity. While watching and listening to video-recorded student interviews, the researcher typed transcriptions of the verbal conversations. Although this was time-consuming, it resulted in careful attention to the discussion and a deeper reflection on the student/researcher conversations. Notes were also taken while reading through the typed transcriptions to find themes concerning the students' algebraic reasoning skills and challenges they encountered.

Data Analysis

After conducting and recording eighteen interviews, the researcher watched and transcribed the interviews; however, there had to be a way to organize the analysis of thirteen hours of algebraic conversation. The first time watching the interviews the researcher listened and jotted notes of themes detected in student algebraic reasoning. Since the focus of the analysis was to look for the nature of students' algebraic reasoning while solving generalization tasks, the second

time the interviews were watched with a specific focus. This focus was guided by an observational protocol to track the stages of generalization (Appendix F) and when or if students arrived at each stage in the problem-solving process. The Center for Algebraic Thinking (2020) summarized Mason's (1996) four stages of generalization to be seeing, saying, recording, and algebraic representation; however, since this study's focus was to determine if students were transitioning from the concrete to the abstract, the analysis of the data will refer to three stages of generalization rather than four, and they will be categorized as concrete (stage one), informal abstraction (stage two) and formal abstraction (stage three). Through the observational protocol it was noted if students independently transitioned from one stage to the next or if they encountered challenges and needed guidance. Additionally, notes were made if students verbal comments when students independently visualized and recognized mathematical patterns (Lannin, 2004).

Notes on the observational protocol were highlighted and color-coded when and if algebraic thinking took place. Red and orange were used to represent recursive and explicit patterns respectively. Blue and green were used to highlight where students verbalized and visualized patterns. Yellow was used to denote the place in the problem-solving process where challenges occurred and an explanation of what the researcher did to aid or hinder the student at that moment. Additional aids included moving on to the next scaffolded question or suggesting the student access an algebraic reasoning tool.

Since each of the six students were presented with the same tasks, each student's strategies could be compared to the other students. What did the students do similarly and what were the differences? How many different explicit patterns can one task have, and will each student notice the same pattern? Also, were the students encountering the same challenges as they reasoned

algebraically and did the scaffolded questions and the accessibility of algebraic reasoning tools eventually help each student come to the same level of understanding? Focus was taken to follow each student's thought process and determine strategies he/she found to answer the questions. The researcher also looked for the similarities in how each student progressed through the stages of generalization and if the abstract generalization they recognized tied back to the informal analyzation of the pattern's mathematical structure.

Summary

This study was conducted to answer two research questions: 1) What was the nature of students' algebraic reasoning through tasks involving generalizing? and 2) What were the challenges students encountered while reasoning algebraically through tasks involving generalizing? In order to have valuable qualitative data to analyze, eighteen 45-minute interviews were conducted which means a total of thirteen hours of algebraic reasoning was collected. This action research study included watching videos and reading transcripts from interviews of six, thirteen to fourteen-year old algebra students to have a record of the nature of their algebraic reasoning. The analysis of the data focused on tracking when and if students independently transitioned through the three stages of generalization: concrete, informal abstraction, and formal abstraction. When guidance was necessary for the student to continue problem-solving, this instance was recorded as a challenge he/she encountered while generalizing the mathematical pattern. The results of the themes that surfaced from the interviews will be discussed in the following chapter.

CHAPTER IV: FINDINGS

The purpose of this study was to observe and understand the nature of students' algebraic reasoning through tasks involving generalizing. After students' algebraic reasoning was collected through three interviews with six students, the data of all eighteen interviews was analyzed. The analysis of the data included reading the interview transcripts to find and highlight any discussion of key algebraic concepts, and recognition of recursive or explicit mathematical patterns. An observational protocol was used to document when and if students were successfully transitioning through the stages of generalization. Successful transition from stage to stage of the generalization process was defined as times where students independently reasoned without aid from the researcher. Then, the challenges students encountered while reasoning were analyzed. Challenges were defined as struggles that kept students from advancing in the problem-solving process in which the researcher had to offer guidance to help the student proceed. The findings of this study were valuable to provide evidence that students already possess algebraic reasoning tools, and accessing additional tools helped them transition from the concrete to the abstract.

This chapter starts with a discussion of research question one: What was the nature of students' algebraic reasoning through tasks involving generalizing? The discussion begins with an overview of the stages of generalization and several examples of students independently transitioning through the stages. Then, the discussion focuses on the reasoning tools students used to transition from one stage to the next. The analysis of the data revealed three themes: 1) it was possible for students to reach stage two (informal abstraction) and have an abstract understanding of the mathematical pattern even if they were not transitioning to stage three (formal abstraction), 2) students relied heavily on visualizations of the tasks as reasoning tools to

reach stage two (informal abstraction), and 3) using the context of the task to understand the mathematical patterns proved to be the most powerful way to reach stage two (informal abstraction). Students who transitioned from stage one to stage two had bridged the gap from the concrete to the abstract and were able to use their informal abstraction to answer other questions related to the given task. Research question two related to the challenges students encountered when reasoning algebraically through tasks involving generalizing. The analysis of the data provided one theme: students had challenges transitioning from one stage to the next of the generalization process. Students struggled to transition from stage one (concrete) to two (informal abstraction) when they did not independently access the available reasoning tools such as visualizations to understand the contexts of the tasks. Though students could informally explain the abstract generalization, most of them were not making the transition to formal algebraic symbols without aid from the researcher to transition from stage two (informal abstraction) to three (formal abstraction).

Research Question One: Nature of Algebraic Reasoning

This section discusses the nature of students' algebraic reasoning through tasks involving generalizing. The collected data was analyzed to search for discussion of key algebraic concepts, recognition of recursive and explicit patterns (Lannin, 2004), and generalization of mathematical patterns (Dougherty et al., 2015). The students in this study were given three different generalization tasks (Appendices A, B, and C) with additional scaffolded questions related to each task. The scaffolded questions forced students to independently analyze the mathematical relationships on a deeper level and therefore initiated discussions leading up to strategies how each student would approach abstract generalizations of the tasks. The algebraic reasoning concepts and reasoning tools discussed have transcended all tasks regardless of the nature of the

task. To be able to more easily understand the discussions of students' algebraic reasoning, it is suggested that the reader try solving the tasks just as the students were asked to do prior to the reading of chapter four. The well task proved to be a difficult problem for the students for two reasons. First, students' initial intuitions involved using the frog's rate of climb to be two feet every two hours or one foot every hour, which are both correct. However, this strategy did not give them correct answers. The students' discussions in this chapter will guide the reader through the details of the mathematical pattern related to this task. Secondly, the main idea of the well task is to find which hour the frog would exit the well, not the exact minute. Some students got too hung up on the details and lost focus of the big picture which meant the researcher should have been more specific with the scaffolded questions. Thus, when reading the questions to the well task, think of the questions as asking which hour the frog would exit a 10-foot well. This clarification should help the reader when trying to follow the discussion of the well task.

Overview of the Stages of Generalization. Similar but slightly revised from Mason's (1996) stages of generalization: articulating, making sense of, and manipulating, this study referred to the three stages of generalization as concrete, informal abstraction, and formal abstraction. While Mason's stages discuss the activity of students attempting to understand and explain a mathematical pattern, it was important for this study to determine when and if students were transitioning from arithmetic to algebra. Therefore, stage one (concrete) of generalization was referred to as the stage where a student discussed and visualized the task to gain a better understanding of a mathematical pattern. Stage two (informal abstraction) was recognized as the moment a student explained the abstract generalization of the mathematical pattern in his/her own words and stage three (formal abstraction) was determined to be the moment a student wrote his/her generalization in a symbolic form. The researcher of this study was aware that

attempting to create stages of generalization was itself a generalization, and it is quite possible that student reasoning was not meant to be categorized in three simple bullet points. However, for the importance of interpreting data for analysis, the organization of the stages was necessary.

Stage one was labeled the concrete stage because many of the beginning conversations were not focused on abstract generalizations of mathematical patterns but rather occurrences of students intently discussing and using visualizations to understand the tasks. With all three of the generalization tasks being of a linear nature, students' conversations revolved around how the number of border tiles was changing from frame to frame, how far the frog was moving every hour, or how the number in the sequence was changing in relation to the previous number in the sequence. Lannin (2004) referred to these conversations as important stepping-stones to lead students to understand the mathematical patterns using contexts of the tasks to justify the underlying structure of the mathematical patterns. Mason (1996) explained this stage as the 'articulation' stage. Visualizing the task was another key stepping-stone students needed to solve the tasks. Visualization of a task was a valuable tool to search for patterns and justifications of those patterns (Lannin, 2003; Stump, 2011). Visualization could be in the form of creating charts, tables, diagrams or pictures to search for mathematical patterns. Mason (1996) referred to manipulating the concrete numbers to search for mathematical patterns as the 'making sense of' stage. Even though key algebraic concepts had already been discussed, it was clear the patterns the students were describing were related to how they saw and understood the mathematical patterns. Recognition of the abstraction of mathematical patterns did not happen until students were asked to extend their patterns to find the number of border tiles surrounding a garden of length 25, the time it would take the frog to climb out of a 20-foot well, or the 50th term in the number sequence. This analysis also showed the value of asking students to find a number with

significant magnitude in which a recursive model seemed inefficient (Lannin, 2004). This forced students to start analyzing the mathematical relationships on a deeper level and therefore initiated discussions leading up to how students would approach abstract generalizations of the tasks.

Stage two (informal abstraction) of the generalization process was the recognition of an abstract generalization of the task's underlying mathematical structure. This stage consisted of evidence that students were transitioning from the concrete to the abstract. For example, students started recognizing how to find the number of border tiles for a garden of any length, the number of hours to exit a well of any depth, or the value of any number in sequence. This transition involved informal algebraic procedures (Kriegler, 2007) like verbal explanations of the abstract generalizations of the mathematical patterns and would eventually lead either to more formal algebraic processes or frustrations as they tried transitioning from the verbal recognition to a symbolic one. In the informal abstraction stage, the nature of how students were making a connection between the key algebraic concepts they discussed and the numerical patterns they visualized became evident and led them to abstract generalizations of the patterns.

Stage three (formal abstraction) of the generalization process included writing the abstract generalization in a symbolic form. This stage of generalization involved taking the recognized mathematical pattern and translating it from words to an algebraic expression which meant the students were now transitioning from informal to formal algebraic procedures. Of the eighteen conducted interviews, only one interview showed evidence of a student transitioning through all three stages of generalization independently and will be shared now. Some of the breaks in independent problem-solving were not the fault of the student but rather due to the nature of the task or the scaffolded questions associated within each task.

An overview of two students' transitions through the garden task and one student's transition through the well task will be presented to the reader at the beginning of this chapter to demonstrate examples of the nature of students' algebraic reasoning through the stages of generalization. However, since the research of this study was to look for algebraic reasoning tools across all tasks, the discussion of transitions from one stage to the next will weave back and forth from task to task discussing overarching themes.

Transition through All Three Stages of Generalization. Ben's conversation with the researcher while solving the garden task represented the only example of a student independently transitioning through all three stages of generalization. The beginning of the conversation demonstrated stage one (concrete) of generalization. When asked what he was counting as he drew his own picture of a garden of length four in the empty frame, the researcher said,

> Researcher: "So can you tell me why you were counting the last one and what you were doing there?"
> Ben: "So I noticed they were all growing in length by one. So I'm just Adding up and then I'm going to shade all the inside ones."
> Researcher: "Ok."
> Ben: "What pattern do you notice about how the gardens grow? It is growing by like one row every time."

Ben was simply answering the question by explaining what he noticed about how the garden grew from frame to frame. This specific scaffolded question did not ask anything about what happened to the border tiles, so he did not mention them at this point.

> Researcher: "Ok."
> Ben: "Find the number of border tiles of the garden of length 25. So of Length 25, the inside would be 25, that means there would be 25 on bottom and top and then this side, there would be three on each side. So, 25 times 2 and 6, so 50….56. Yeah. Wait, border tiles, never mind, 25. Inside not outside."

Now because Ben was asked to find the number of border tiles, he explained how he was noticing the pattern of this task by discussing how he visualized the garden in comparison to the border tiles. He even drew a representation of this on his paper (see Figure 7).

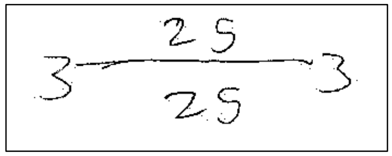


Figure 7. Ben's Depiction of a Garden of Length 25.

Researcher: "Read that sentence one more time."
Ben: "Find the number of border tiles of the garden of length 25. Never mind, I was right."

Even if Ben did not draw the entire picture of the garden, drawing this depiction of the garden of length 25 showed he had the understanding of the context of the task to independently transition from stage one (concrete) to stage two (informal abstraction) and he used the pattern he was visualizing to answer additional questions related to the task.

Ben visualized the garden of length 25 and even though he was still using concrete numbers to calculate a garden of a particular length, his verbal explanation of how he visualized the garden in comparison to the border tiles showed his transition to stage two (informal abstraction). Now Ben was asked how he would find the number of border tiles surrounding a garden of any length.

Ben: "Exactly! I always correct myself and make it wrong. How would you find the number of border tiles of a garden of any length? ….. of border tiles!"
Researcher: "Yes."
Ben: "Uh… It's just the number of length."
Researcher: "But this is asking for border tiles also. So like on the last one."
Ben: "So, it'd be the length equals x, so it would be 2x + 6."

Instead of using words (informal abstraction) to explain his generalization of this pattern, he wrote his generalization in a symbolic form (formal abstraction). However, when asked to use his pattern to find the number of border tiles for a garden of length 100 and to find the length of a garden surrounded by 70 border tiles (see Figure 8), he took a more informal approach to answer the questions. Being able to solve these two questions independently showed Ben had recognized

the informal abstraction (stage two) of the generalization of this mathematical pattern and could also translate the generalization to a symbolic form (stage three).

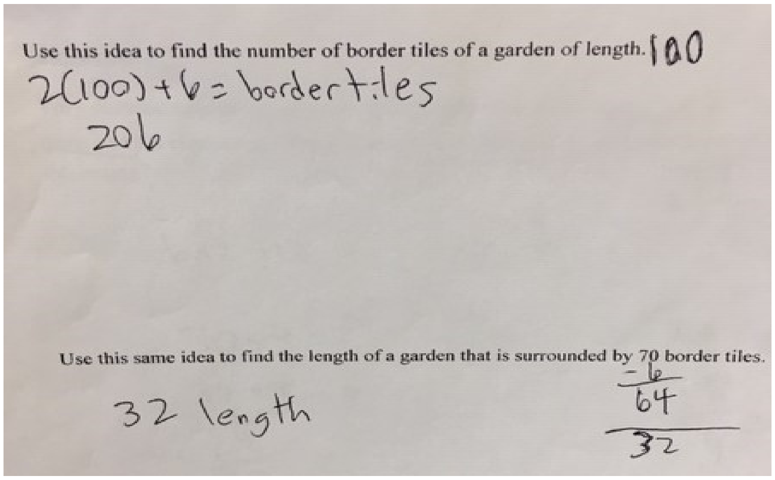


Figure 8. Ben’s Garden Work.

<u>Transition through the First Two Stages of Generalization.</u> The following example illustrates how Susan transitioned through the first two stages of generalization while solving the garden task. Susan did not draw a visualization on her paper; however, after analyzing the conversation she had with the researcher, it is clear she had the same understanding and visualization of the mathematical pattern as Ben. Susan stated an informal abstraction (stage two) of the mathematical pattern but was not able to translate her verbal explanation to a symbolic form (stage three). She started by discussing the pattern she was visualizing.

> Susan: “What patterns do you notice about the garden, how the garden grows? Like every time it just gets one bigger? I don’t really know what this is asking.”
> Researcher: “Kind of what it is asking is what is happening to the garden each time?”
> Susan: “It is just getting one bigger.”
> Researcher: “Is anything else growing besides the garden?”
> Susan: “So like every time there are blocks here (referring to the garden) that’s how many blocks are here (referring to the border tiles).”

Susan explained the pattern exactly how she saw it. Excluding the ends of the garden, she was noticing that as the garden grew, there were the same number of border tiles above or below the garden.

Researcher: "Ok. Great. Well, we can talk about that more later. I just wanted to make sure I understood what you were saying."

Asking Susan if she could find the number of border tiles surrounding a garden of length 25 was valuable to determine if she would extend the pattern she had already discussed and visualized.

Susan: "Ok. Find the number of border tiles for a garden of length 25. That's four (referring to her picture of the garden of length four), and border tiles. Wait, like 25 + 25 + 3 + 3?"
Researcher: "Is that what you think?"
Susan: "I think."
Researcher: "How did you get 25 + 25 + 3 + 3?"

When the researcher asked her why she thought the answer would be found by adding 25, 25, 3 and 3, she gave a verbal explanation of how she recognized the garden pattern. Susan was making a connection between the actual numbers and what they represented in the context of the garden.

Susan: "Well, it's always 3 and 3 on the sides so 3 + 3, and it's always the same as the garden you have so if you have a garden of length 25, you would have 25 on the bottom and 25 on the top."
Researcher: "Great. That is exactly right."
Susan: "So it would be 56."
Researcher: "Very good."
Susan: "How would you find the number of border tiles for a garden of any length? Oh, well it's kind of like what I was just telling you. It's the same amount of the little garden and then three and three."

She was asked if she could find the number of border tiles for a garden of any length and she was confident enough with the pattern she had noticed to explain to the researcher the informal abstraction (stage two) of this pattern was exactly what she had already mentioned, "Oh, well it's kind of like I was just telling you. It's the same amount of the little garden and then three and three." Then the researcher asked if she could write her generalization in a symbolic form (stage three). This was a challenge for Susan, even though in the midst of the conversation; she clarified once again that she understood the underlying mathematical structure of the task.

Researcher: "All right. Now let's go back and read this question one more time."
Susan:" Ok. How can we find the number of border tiles of a garden

of any length? Do you want me to read my answer too?"
Susan: "Yeah. It is the same amount of the garden and then 3 on both sides."
Researcher: "Ok. Do you still agree with that? It's the same amount as the garden and then 3 on both sides?"
Susan: "Oh wait! Times 2."

Even though Susan had initially written the surrounding border tiles were found by adding the length of the garden and then three on either end, she corrected her mistake independently when the researcher asked her to check her answer. Susan saw the mathematical pattern of the garden task right away (informal abstraction) and stated,

Susan: "It's the same amount as the garden, times two then three on both sides."

However, it was much more difficult for her to write her conceptual understanding in a formal way (stage three). After being encouraged to write that same phrase as an algebraic expression, she said,

Susan: "Wait, I'm confused."

The researcher rephrased the question and asked if she could write her verbal pattern as a variable expression and she said,

Susan: "So you would do, so like *l*, oh, would it be like $l = g$, for gardens, and then….. I don't really know, I don't really know how to do this."

After, the researcher restated all the concrete examples she had calculated, she said,

Susan: "Oh, so I'm just going to throw a letter in there?"

Before the researcher stepped in for guidance, her final attempt was on the right track when she said,

Susan: "3+2*g*, for garden, and then + 3. I don't know why the g is in there?"

She could not understand why a variable was necessary when the garden length would always be given to her and then she could just double that number and add three for each end.

Researcher: "Ok, so now let's say the next one is going to be x. This one is garden length 3, this one is garden length 25, this one is garden length 100, the next one is garden length x."
Susan: "It would be $x + x + 3 + 3$ or……"
Researcher: "What is another way to write it?"
Susan: "x, isn't it if it is times 2 you could, can't you do an exponent?"
Researcher: "No, plus x is different than times x".
Susan: "Oh, yes it is. So, it would be $2x + 2$ times 3."

The reason Susan was used as an example as a transition through the generalization process was, even though she did not reach stage three (formal abstraction), when she was asked to use her pattern to find the number of border tiles surrounding a garden of length 100, she could.

Susan: "Ok. Use this idea to find the number of border tiles for a garden of length 100. So it would be 100 + 100 + 3 + 3. Right?"

Additionally, when she was asked if she could find the length of the garden when it was surrounded by 70 border tiles, she could.

Susan: "So use this same idea to find the length of a garden that is surrounded by 70 border tiles. So it would be 70 – 3, no minus 6."
Researcher: "And why did you subtract 6?"
Susan: "Because you are taking away the three from both sides."
Researcher: "Great."
Susan: "So it would be 64. And 64 divided by 2. Because it would be on the top. So it would be 32."
Researcher: "And what does that 32 represent?"
Susan: "The garden tiles."

This example demonstrates a student who had discussed algebraic concepts, visualized the pattern (stage one) and recognized the informal abstraction (stage two) to the point she could answer additional questions related to the task. She had justified and validated the recognition of her pattern with words, but she was challenged to translate her words into symbols (stage three). Susan's reasoning showed an example of a student who transitioned from the concrete to the abstract, but not from informal to formal abstraction.

George's approach to solve the well task will be used as another example of a student naturally transitioning from the concrete (stage one) to an informal abstraction (stage two).

George did not transition through all three stages of generalization, but his strategy was unique, practical and efficient. The approach, steps and calculations were all his ideas followed by some additional aid from the researcher to keep him focused to search and find the mathematical pattern.

The first scaffolded question was to determine how long it would take the frog to exit a 10-foot well. In stage one (concrete) of generalization, George chose to reverse the operations and used the arithmetic pattern of subtracting three and adding one to determine how long it would take the frog to exit the well (see 1st column of Figure 9). This was George's way of trying to understand the mathematical pattern of the well task.

> George: "If a frog is at the bottom of a 10-foot well, poor frog, if he climbs three feet in the first hour and slides back one foot in the second hour and repeats this pattern, how long will it take him to get out? 10, so he climbs three feet then plus one, that'd be 8, minus three, that'd be five, plus one is six, minus three is three, plus one is four, minus three is three."
> Researcher: "Four minus three is what?"
> George: "It's one. I can math. And plus one is two, and minus three is negative one. (Then he went back and wrote the hours beside the depth of the frog.) That's the first hour, second hour, third hour, fourth hour, fifth hour, sixth hour, seventh hour, eighth hour, ninth hour, tenth hour. Ok."

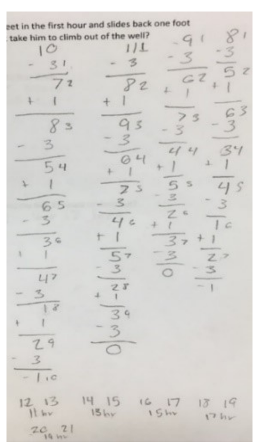


Figure 9. George's Well Calculations.

Instead of using a visualization of a well to move the frog up and down, George chose to manipulate the numbers to gain an understanding of the numerical pattern. He began calculating

the distance the frog still had left to climb after each hour. The small numbers next to each column of calculations represented the time it took the frog to reach that depth after each hour of movement. He also ended his first round of calculations with a negative one which meant he needed to reconcile what ending with a negative number meant and how that affected his answer to this question.

> Researcher: "Where would you say he climbed out of the well? Because this is an awesome pattern, no one else has done this. I like it."
> George: "Uh huh."
> Researcher: "Because what you are showing is how much he has left to go each time. But where did he climb out? Cuz here (pointing to where the frog had one foot left) and then you added one back to it."
> George: "So he would have climbed out in the tenth hour. Well, he would have climbed out, sort of in the middle of here (referring to the ninth hour)."
> Researcher: "I would have to say it would have to be here (during the ninth hour) because you are at a depth of negative one here (meaning after he subtracted three for the tenth hour) which means he has been out for a while."
> George: "Yeah."
> Researcher: "So if you use this pattern you need to say it was here (the ninth hour)."
> George: "So he climbed out during the ninth hour."

Now that George had worked through one concrete example, the next scaffolded question, What patterns do you notice?, would determine if George could transition to stage two (informal abstraction) or if he would need more time to develop his understanding.

> George: "Sweet. So what patterns do you notice? Ummm. Add three, subtract one."
> Researcher: "Ok. Then why on here did you subtract three and add one?"
> George: "Oh whoops. I meant the opposite way around. Subtract three, add one."

George's answer of 'add three, subtract one' revealed he had not arrived at an informal abstraction yet. He was still repeating the pattern he used to arrive at his previous answers and not an overarching generalization as to how to find the length of time it would take the frog to

exit a well of any depth. The next two questions asked for the time it would take the frog to exit an 11- and 9-foot well. Instead of guessing or looking at his first set of calculations to determine an answer, he remained in stage one (concrete) by calculating the exit time two additional times (see 2nd and 3rd columns of Figure 9). Then he started making connections between the depth/hour patterns he had calculated.

> George: "Hmmm. He would have climbed out in the same amount of time. The ninth hour. Interesting!"
> Researcher: "Uh huh. Why do you think it is the same answer as the other one? Can you put your finger on that?"
> George: "Ummm. I think because he was climbing three. But maybe if it was 13, it would be the tenth hour."
> Researcher: "Ok."
> George: "But because it is the 11th hour, it's like the same thing. He had an hour left here (referring to after the frog exited the 10-foot well.) He might even have less that one (hour) at 12."

George was close to transitioning to stage two (informal abstraction) because he understood the frog climbing three feet per hour made it possible for him to exit varying well depths during the same hour. The next question asked him to find the number of hours it would take a frog to exit a 20-foot well.

> Researcher: "So what I do to you now is give you a deeper one and another even deeper one to see if there is a way that you can generalize this so you don't have to do this adding and subtracting thing every time. Cuz the next one is 20-feet deep."
> George: "Ok. We got the 9th hour both of these times (referring to the 10- and 11-foot wells). I am wondering if I do 8, will that work? If I do 8 will I get the 7th hour?"
> Researcher: "Ok."
> George: "If we go forward or backward, we would get two every time."

George's idea to find the time it would take the frog to exit an 8-foot well (see 4th column of Figure 9) was valuable and one of the steps he needed to justify the pattern he was noticing. Now, George was beginning to summarize his concrete answers to look for a generalization to the mathematical pattern he had discovered at the bottom of his page (Figure 9). Organizing his

numbers in this way led him to discuss and visualize the mathematical pattern by saying it would take the frog eleven hours to exit a 12- and 13-foot well and thirteen hours to exit a 15- and 16-foot well and so on.

After George created the calculations for finding the number of hours it would take the frog to exit varying well depths, he was asked if he could find the number of hours it would take the frog to exit a well of any depth?

> George: "Didn't I just do that?"
> Researcher: "Well you did a particular depth. You did 20 feet."
> George: "Ok."
> Researcher: "So, if you want to do this one first you can (referring to the 48-foot well) because you are going to take your pattern and are you going to keep doing the same thing until you get to 48? Or is there something that you can do now that you know multiple answers to different depth wells. Can you generalize and say I already know the answer to this without having to write all of the numbers out?"
> George: "Could it be……could I divide 48 by two? Maybe. No. Hmmmm. How could I do this faster?"

Instead of asking George why he would divide 48 by two, which is not a strategy he had used to find the number of hours for any other well depth, the researcher directed him to the pattern he had already noticed and written at the bottom of his page.

> Researcher: "Look at the 8- and 9-foot well was seven, and 10- and 11-foot well was 9. So what do you notice about the depths compared to the number of hours?"
> George: "Oh! That is interesting. Dang it. It's sort of behind it one."

Mentioning the comparison of the well depths to the number of hours was the only necessary recommendation George needed to transition to stage two (informal abstraction).

> Researcher: "Hmm. Maybe that would help you so that you don't have to….."
> George: "So if it was 48 feet deep then it could be forty-seven hours or….."
> Researcher: "That's a really good guess."
> George: "Or…….wait. Hold on. It starts with an equal number every time so I would assume…."
> Researcher: "Did you say equal number or even number?"
> George: "Even number. So it (his pattern) starts with an even number every time so that is 12 and that is the start of 11 and 14 is the start of 13

so…."

Researcher: "That's a good pattern to notice."

George: "So this one would be 47 hours."

George was now making a connection between the well depth and the number of hours it would take the frog to exit. He had worked with the concrete examples until he arrived at an understanding of the mathematical pattern. George was ready to transition from stage one (concrete) to stage two (informal abstraction) and using his own words he explained the mathematical pattern.

Researcher: "Yes. So now that you have that idea you could do a 99-foot well, 100-foot well or something like that so that is what I am saying about any depth. Could you put that generalization in words and put that right here (referring to the question about finding the number of hours it would take the frog to exit a well of any depth)?"

George: "Yes. For every two feet with the even foot, go back one hour."

Researcher: "Ok. That is good. And what if it's an odd number?"

George: "You would go back two."

George visually organized the numbers of the well task which created an understanding of how the frog's movement up and down the well gave context to the well task's mathematical pattern. This allowed him to recognize the pattern and answer the questions that followed no matter the depth of the well. His numerical representation was powerful because it highlighted the pattern showing two well depths with the same exit time. George was able to validate his generalization using the most efficient method to answer this question by looking at his own number pattern. George did not reach stage three (formal abstraction) but none of the students found a symbolic representation of the well task's mathematical pattern. The most practical abstraction of this mathematical pattern was not an elaborate symbolic representation, but rather a simple explanation of how to calculate the time it would take the frog to exit depending on whether the well depth was odd or even.

As previously mentioned, the analysis of the data showed only one instance of a student independently transitioning through all three stages (concrete – informal abstraction – formal abstraction). However, it is important to recognize that although Susan and George did not reach stage three (formal abstraction), they transitioned from the concrete to the abstract with a solid understanding of the mathematical pattern.

Keys to Transition through the Stages of Generalization. After reading through three students transitioning through the stages of generalization from beginning to end, the study now turned its focus to what tools students were naturally using to generalize mathematical patterns throughout all eighteen interviews. Analysis of the data showed visualizations of the tasks and confidence using symbols were two key tools valuable to a successful transition through the stages of generalization. Visualizations aided students in analyzing the patterns during stage one (concrete), and confidence using symbols aided students during stage two (informal abstraction). The following examples provide support for the importance of using reasoning tools from all six students while solving various tasks and will be discussed in the next section.

<u>Accessing Visualizations of the Tasks.</u> As seen from the previous examples, students relied heavily on visualizations of the tasks as a reasoning tool to independently search for mathematical patterns. Initial discussions during the interviews revolved around how the garden was growing from frame to frame, how far the frog was moving every hour, and how each number in the sequence compared to the previous number. However, asking students to find the number of border tiles surrounding a garden of length 25, the time it would take the frog to climb out of a 20-foot well, or the 50th term in the number sequence forced students to clarify how they were understanding the patterns. Then, the conversations shifted to how students were seeing the garden and the frog's movement up and down the well. If students could not easily access a

mental picture to find a pattern, they began to independently use visualizations of the tasks by creating tables, charts or diagrams to gain a better understanding of the numerical patterns of the tasks. Not every student was able to independently transition from the concrete to the abstract without aid from the researcher; however, the following examples are students who naturally used visualizations of the tasks to analyze the mathematical patterns.

During Ben and Susan's conversations about the garden task it was evident they were both visualizing the garden in the same way (see Figure 10). When Jasmine was asked how the

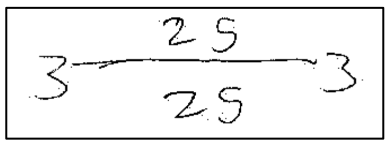


Figure 10. Ben and Susan's Visualization of a Garden of Length 25.

garden grew, she did not discuss how the garden itself was growing or how many border tiles were added to each frame. She said,

> Jasmine: "So this one is a 3 by 3, then it goes 3 by 4, then 5, a 3 by 5, so I'm assuming that one (meaning the fourth frame) is a 3 by 6, because the pattern slowly, gradually gets bigger."

Based on Jasmine's description, she was seeing the garden and the border tiles as a rectangular unit. However, when Jasmine was asked how many border tiles surrounded a garden of length 25, the first thing she said was,

> Jasmine: "I'm going to draw a garden length 25."

Then, the researcher asked if she would rather use algebra tiles, so Figure 11 is a picture of her garden representation.

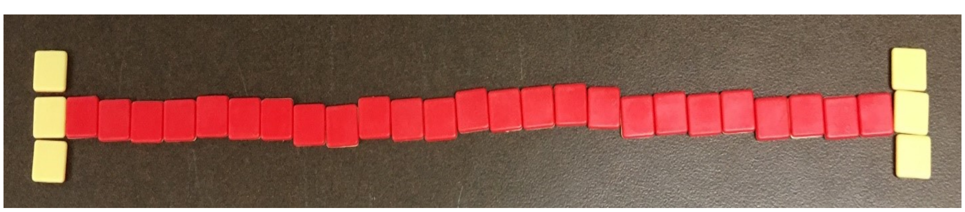


Figure 11. Algebra Tile Visualization of a Garden of Length 25.

Jasmine: "Ok. So it would be fifty-six because it is twenty-five and twenty-five. That equals fifty. Plus three and three. That equals 56."
Researcher: "So where did you get three and three?"
Jasmine: (She points to the ends of the garden) "These tiles."

Independently, Jasmine knew she needed a reasoning tool to see the structure of the garden. Even though she did not lay down all the algebra tiles, she was able to state the correct answer for the number of border tiles surrounding a garden of length 25. Her explanation of how to calculate the number of border tiles surrounding this garden showed she was seeing there would be the same number of border tiles above and below the garden of length 25 and then an additional three on either end. This was the same visualization Ben and Susan had seen to solve this problem and this was a change from how she initially explained the garden's growth as a 3 by 3, 3 by 4, and 3 by 5 and so on. Now she was using the context of the garden to make a connection between the length of the garden compared to its surrounding border tiles.

When answering questions about the well task, Chad independently knew visualizing the well would help him understand the mathematical pattern and did not feel confident guessing the answer to how long it would take the frog to exit a 10-foot well until he accessed a visualization of the well (see Figure 12).

Chad: "A frog is at the bottom of a 10-foot well. He climbs three feet the first hour and slides back down one foot the second hour and repeats the pattern. How long will it take him to climb out of the well? So, I think I might draw like a number line, a vertical line representing each foot of the well. To just give me a visual representation."
Researcher: "Ok."
Chad: "I'm no artist which is why I am in band. (Starts counting to mark off ten feet) Ok. So he goes three feet and then back one. So he will be at two. Then he goes another three feet and back one so he will be at four. So we don't really need to go up and down, we can just add two each time.

He might have already suspected this pattern but drawing the well helped him justify the pattern. At this point, he was generalizing how far the frog climbed every two hours. Also, looking

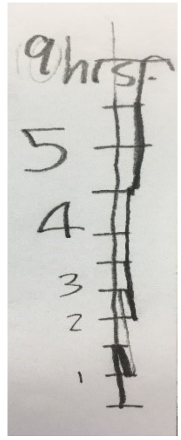


Figure 12. Chad's 10-Foot Well.

closely, it is evident he used his pencil to draw marks up three and down one until he reached the top of the well as an additional justification of this mathematical pattern.

> Researcher: "Ok."
> Chad: "So then up to six. And that would be three. . . three time lapses."
> Researcher: "And what is a time lapse?"
> Chad: "It's a . . . . if he goes three feet up and then back one in two hours, then one time lapse would be two hours."
> Researcher:" Ok."
> Chad: "Then I will just multiply the number I have at the end by two."
> Researcher: "Ok."
> Chad: "So then he goes from 6 to 8. (Recounts on his number line.) Yeah. So then he goes from 6 to 8 in another time lapse and then from 9 to 10. So it will take him 10 hours to get to the top of the well. So what patterns do I notice? So like I said you don't have to go up three and down, you just add two. It takes the same amount of hours as there is feet in the well. So, I guess that can count as a pattern, I guess. I don't know."

Again, Chad was explaining the frog's movement up the well. At this point, he was not able to state a generalization of the mathematical pattern which meant he would need help transitioning to stage two (informal abstraction). However, he used his well visualization to understand if the frog moved at 2 feet every hour, the frog should exit a 10-foot well in 10 hours. This would have been true if it were not for the crucial fact the frog does not slide back down during the tenth hour because he has already exited the well.

Chad had already drawn a visualization of the well and had used it as a reference to answer all the scaffolded questions up to this point. This was the beginning of the conversation that occurred when Chad was asked if he could find the number of hours it would take a frog to exit a well of any depth.

> Chad: "Ok. Could you find the number of hours needed to climb out of a well of any depth? Yes, because you would just be adding two each time and then watching closely at the end to make sure he doesn't go over before he drops."
> Researcher: "Ok."
> Chad: "Cuz that could mess with the just adding two. So, yes. Use this idea to find the number of hours, that's deep (Chuckles at the next question which is a well 48-feet deep)!!"
> Researcher: "And the reason I picked 48 is to try to get you to answer without a picture this time. Can you do that?"
> Chad: "Yeah. I am going to use a mental picture though."
> Researcher: "Ok. Because what did you say the concern was?"
> Chad: "Once he got to the very end he could go over. So I am just going to skip toward the very end."

Therefore, when Chad was asked to summarize how to find the number of hours it would take the frog to exit a well of any depth, this is what he said,

> Chad: "…………But you could find an even number that's at most 3 feet below the depth of the well and that's your amount of hours plus one. If that makes sense."
> Researcher: "Ok. So let's try it with the 48 and see if that works."
> Chad: "Ok. So I think that is kind of what I did. So, not at least three feet, at most three feet at least one foot. So you go to 46 and that's 46 hours in 46 feet and then you add one."
> Researcher: "So that's how you got 47 hours?"
> Chad: "Yes. So I will write down that method."
> Researcher:" Ok."
> Chad: "Find an even number that is at most three feet and at least one foot that is under the depth of the well. Then add one to your hour number."

Chad was using the idea that he could use the recursive pattern and skip to a specific spot just below the top of the well and move the frog up a foot one last time to arrive at the correct

answer. Chad's visualization of the well helped him solidify his understanding of the underlying mathematical structure of the pattern by using the context of the well.

Jasmine used two different visualizations to approach an understanding of the well task. The first reasoning tool she used to understand the mathematical pattern was a visualization of the calculations (see Figure 13). This approach was similar to George's idea of using the adding-three- and-subtracting-one number pattern. Jasmine wrote the numbers on her paper to represent the frog's distance traveled with the corresponding hour traveled directly below.

> Jasmine: "Ok. So, first question. A frog is at the bottom of a ten foot well. If he climbs three feet in the first hour and slides back down one foot in the second hour and repeats this pattern, how long will it take to climb out? Um. So he climbs up three feet in the first hour and then in the second hour he slides down one foot, so that would be the second hour and then he repeats. So that would be the pattern. What is the pattern you notice? That is the pattern."
>
> Researcher: "Ok."
>
> Jasmine: "How long would it take him to get out of a 10-foot well? 3 minus 1. That'd be three. That'd be 2. And 3 minus 1, and that's 2, 4, 6."

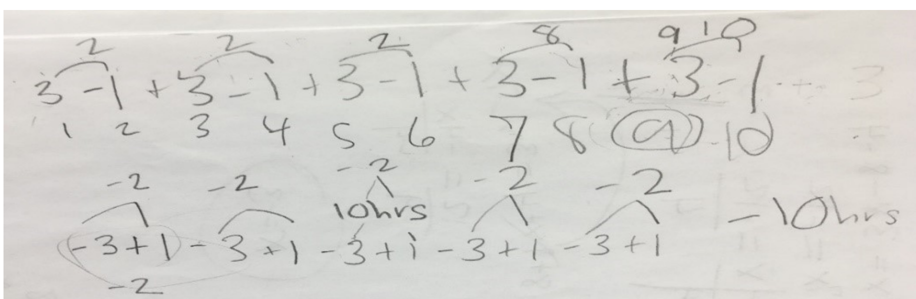


Figure 13. Jasmine's Well Calculations.

> Researcher: "What do the 2's represent?"
>
> Jasmine: "They represent like how far he went up in the first two hours. Cuz he went up three and slid back down one."

Just like Chad, she was generalizing the frog's movement up the well. She was making a connection between the number pattern she was seeing and recognizing the number two was not just an irrelevant number but the rate the frog was traveling every two hours.

> Researcher: "Ok."

Jasmine: "That's third hour, fourth hour, fifth hour, sixth hour, minus one. Plus three, minus one. Yeah. Two, four, six, eight, ten. Six hours, seven hours, eight hours, nine hours, ten hours. Ok. So then that would have been two feet there, and two feet there, so two, four, six, eight, nine, ten. But with plus three, he would have already made it out. So I'm going to draw a well. Wow, I spaced these out beautifully."

Even though Jasmine wrote on her paper that it would take the frog ten hours to exit a 10-foot well, her comment, "But with plus three, he would have already made it out," revealed that Jasmine had insight into the idea that the frog would not slide back after he had already exited the well during the ninth hour. Her visualization of the well provided clarity of understanding while she searched for the mathematical pattern.

Chad and Jasmine both preferred to draw the well (see Figure 14) and physically move their pencils up and down the well; however, Ben chose a different approach. After talking through his initial thoughts about the frog's movement up the well and attempting to guess how long it would take the frog to exit a 10-foot well, he decided it would help him to create a table to

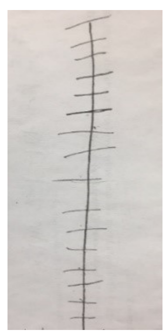


Figure 14. Jasmine's Well Visualization.

organize the movement of the frog so he could verify the mathematical pattern. Here was his explanation and although the passage is lengthy, it shows how his reasoning changed as he gained a deeper understanding of the mathematical pattern by looking at his table (see Figure 15).

Ben**: "**A frog is at the bottom of a ten foot well. He climbs three feet in the first hour and slides back one in the second hour and repeats this pattern. How

long will it take him to get out of the well? So the first one, ten feet so, each two hours, he's gaining two feet and so in order to go ten feet it would take him five times that amount. So it would be ten hours. No. Ok. So first hour he goes up three so he's at three foot. He goes back down to two feet because he slides back down one. So one hour, two hours, at three hours, he's at five feet. And drops back down to four. And goes up to seven and slides back down to six. Goes to nine, slides back to eight. Goes up to eleven. So nine hours. Well, a little less than nine hours. How long will it take him to climb out of the well? Would it be less than nine hours? Yeah, it would be a little less than nine hours. Three feet every hour, so divide one hour by three, so twenty minutes, so it would be like eight hours and 40 minutes."

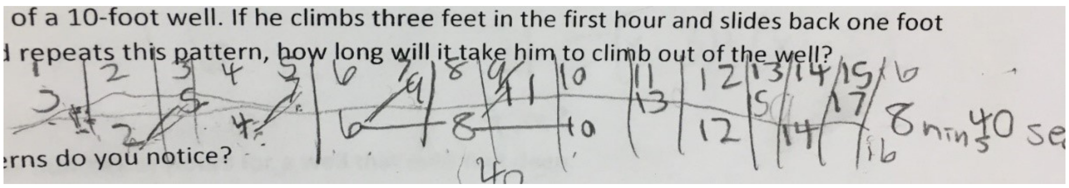


Figure 15. Ben's Table Used to Compare Depth and Time.

Ben used his own table to organize the numbers in this mathematical pattern. While he was talking, you could see how his table helped him change his initial guess of 10 hours, which was a reasonable guess, to find the exact answer which was 8 hours and 40 minutes (Which he said in the discussion even though he wrote 8 mins and 40 sec on his paper). This table was his way of making a connection between the depth of the well and the time it would take the frog to exit.

Researcher: "Ok."
Ben: "What pattern or patterns do you notice? I noticed that he's going to gain two feet every two hours."
Researcher: "Uh huh. Ok. Is that just from reading the instructions where he went up three and slides back down one?"
Ben: "Uh huh. And the next hour he slides back."
Researcher: "So you are just taking that and over time….."
Ben: "Yeah. And making it a cycle. What if the well was nine feet deep? Then he would pass it in seven hours. Get out of it, not pass it."
Researcher: "And so how did you come to that conclusion?"
Ben: "Cuz I have it here, so, I had all of the numbers down so I just looked (at the table he created) cuz I already had nine feet with seven hours above it."

After the researcher asked him to explain how he arrived at this answer, it was clear the top row of his chart represented the number of hours it took the frog to exit the well, the bottom row was

the depth the frog stopped at after climbing and sliding in a two-hour interval, and the middle row was the corresponding depth of the well. Using this table, Ben was able to answer additional questions related to the problem. He had already made the connection that the time it took the frog to exit was dependent on the depth of the well and his table was the strategy that helped him make that connection.

The number sequence problem did not have a visual picture or any related story to use as a context. Although the numbers in the sequence task were the set of odd numbers, the key algebraic concept necessary to reach an eventual abstract generalization of the sequence was realizing that the common difference from each term to the next term was two. Four of the six students made mention of this when they were discussing how they found the next number in the sequence.

> Ben: "They are all odd numbers or skipping two."
> Chad: "It's just adding two."
> George: "Adding two to the previous number."
> Jasmine: "You are always adding two."

After analyzing the interviews of the other tasks, the researcher is convinced students would have used the common difference to find the 50th or 100th term in the sequence if the next scaffolded question had asked them to do so. However, the researcher suggested each student create an x/y chart (see Figure 16) to analyze the numbers in the sequence so the rest of the conversations related to this task will revolve around the mathematical patterns they were noticing while looking at their charts.

While searching for numerical patterns, students leaned heavily on the algebraic reasoning tool, visualizing the task, to analyze mathematical patterns. Students used visualizations in the form of creating charts, tables, diagrams or pictures to search for these patterns (Lannin, 2003;

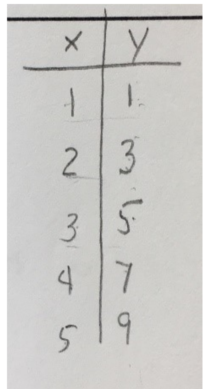


Figure 16. *X* and *Y* Comparison of the Number Sequence.

Stump, 2011). These mathematical patterns found when visualizing the tasks aided students to make successful transitions from stage one (concrete) to two (informal abstraction) as they tried transitioning from the concrete to the abstract.

<u>Confidence Using Variables</u>. Students used many methods to visualize mathematical patterns. Some students drew their own diagrams while others organized their numerical calculations by creating tables. Up to this point, it was natural for students to recognize recursive patterns and they could explain how the patterns were changing from frame to frame. On all three tasks, students were discussing how the garden was growing from frame to frame, how far the frog was moving every hour and how to get the next number in the sequence. However, when students were asked if they could find the number of border tiles for a garden of any length, how long it took the frog to exit a well of any depth or any number in the sequence, to transition from a verbal explanation of the mathematical pattern (informal abstraction) to a symbolic representation (formal abstraction) it was necessary for students to have confidence and an understanding of when and how to use symbols to represent what was varying in the tasks.

Although students demonstrated recognition of the generalization of the mathematical patterns during the problem-solving process, two students independently paired their informal understanding of the mathematical patterns with formal algebra by using symbols (stage three) to

represent the unknowns. They were confident with their understanding of when and where to use variables to explain the pattern of each task by translating their words to an algebraic expression. Using an algebraic representation to generalize is one of the key algebraic reasoning strategies, and some suggest, what separates success from failure in algebra (Center for Algebraic Thinking, 2020). In order to make a transition from informal to formal algebra, it is important that students transition to the third stage of generalization.

When Chad was asked if he could find the number of border tiles for a garden of any length, he initially used words to explain his recognition of the mathematical pattern.

> Chad: "How would you find the number of border tiles for a garden of any length? You could just do the two sides."
> Researcher: "Uh huh."
> Chad: "The two sides. The four corners. And then multiply the number of black tiles by two and add that to the amount of sides and corners."

His explanation of the mathematical pattern explained how he visualized the garden as it was growing.

> Researcher: "So could we let the amount of black tiles be, kind of like we do in algebra, let the amount of black tiles be a variable of some sort."
> Chad: "Uh huh."
> Researcher: "Could you come up with a variable expression?"
> Chad: "Yeah."
> Researcher: "That would represent how many border tiles it would take?"
> Chad: "$2a + 6$"

The researcher had to ask him if he could translate his generalization to a symbolic form. However, when he was asked if he could write a variable expression (stage three) to represent his informal abstraction (stage two), Chad independently wrote "$2a + 6$". Chad did not explain this to the researcher, but he had already noticed since his last explanation of the pattern, if the garden's length is a, then his original pattern $2a + 4 + 2$ (multiply the length of the garden by two, the four corners and the two on either end) could also be written as $2a + 6$.

Using the number sequence task with no context to reference made it difficult for students to have discussions of algebraic concepts or access visualizations of the tasks. Since the scaffolded questions only asked students to list the next three numbers in the sequence and find the pattern, the researcher extended the algebraic reasoning by asking students to find the 50th term of the sequence. Because the researcher had already suggested students create a chart with *x*-values representing the position of the number in the sequence and *y*-values as the value of the number in that position, students used the numerical patterns in their charts to search for a generalization to find any number in the sequence. The students that recognized the patterns found these patterns by comparing the *x*- and *y*-values of each pair to write the explicit pattern of the number sequence.

When solving the number sequence task, Ben saw the pattern (stage one), verbally explained it (stage two) and then translated his idea into a symbolic representation (stage three). It was helpful to him to see the number pairs and to compare each *x* to its respective *y* which led him to this mathematical patterns' explicit formula. When Ben was asked if he could find the pattern explaining the relationship between the *x* and *y* values, this is how he found the pattern.

> Researcher: "Is there an algebraic pattern that you could use to find the 50th term of this sequence without having to add two to the previous number every time?"
> Ben: "I would say……………, no that doesn't work. Never mind."
> Researcher: "What were you thinking? Say what you were thinking."
> Ben: "I was going to say times two and then minus something, but it would change for each one."
> Researcher: "Try it."
> Ben: "Wait! Would it be times two and minus one? I was right."

Ben discovered an informal abstraction for finding any number in the sequence. If he multiplied the *x*-value by two and subtracted one, that would give him the *y*-value of the given *x/y* pair.

Ben had already stated the informal abstraction (stage two) of how to find any number in the number sequence task so the researcher just had to ask if he could write it symbolically (stage three).

> Researcher: "So can you write that as an algebraic equation?"
> Ben: "$2x - 1 = y$"

There were many instances where students discussed and analyzed mathematical patterns in the tasks. However, students who used visualizations of the tasks used the contexts of the tasks to understand the mathematical patterns which allowed them to successfully and independently transition from stage one (concrete) to stage two (informal abstraction), and students who had confidence using variables transitioned from stage two (informal abstraction) to stage three (formal abstraction). There were also many instances where students needed much more aid from the researcher to transition from one stage to next in the generalization process and those challenges in algebraic reasoning will be discussed in the next section of this paper.

Research Question Two: Challenges to Algebraic Reasoning

As can be seen in the last section, students found value in using a visualization of the task, such as charts, tables, diagrams or pictures, to analyze the mathematical relationship. Students needed these reasoning tools to help them understand the mathematical pattern by using the context of the task to make connections about how many border tiles were added as the garden grew, how many hours it took the frog to exit depending on the well's depth, and the value of the number in the sequence compared to its position in the sequence. This next section discusses the analysis of research question two: "What were the challenges students encountered while reasoning algebraically through tasks involving generalizing?" After analyzing eighteen student

interviews there was one theme: students encountered challenges when transitioning from one stage to the next in the generalization process.

Challenges to Transition through the Stages of Generalizations. After reading through the interview transcripts to answer research question one "What was the nature of how students reason algebraically through generalization tasks?", it was evident that students were analyzing mathematical patterns. The challenges occurred when students did not access a visualization of the task to gain an understanding of how the context of the task created the numerical pattern. Although it was natural for students to look for patterns by discussing ideas, students did not always think to use a diagram, table or chart to search for an understanding of the pattern without prompting from the researcher. Additionally, even if students recognized an informal abstraction of the mathematical patterns, there were many instances where students did not have the confidence using variables to transition from an informal to a formal abstraction.

<u>Not Accessing Visualizations of the Tasks</u>. Students naturally discussed key algebraic concepts such as how many border tiles were added to each frame or how far the frog was moving each hour, but some encountered challenges in the problem-solving process when they did not access visualizations of the tasks. Using cues from the students' conversations, students often needed a prompt from the researcher to create a visual way to see the mathematical pattern to correct their own errors. The visualizations gave context to the task and created a solid understanding of the pattern to find the number of border tiles surrounding a garden of length 25, the time it would take the frog to climb out of a 20-foot well, or the 50^{th} number in the sequence.

When solving the garden task Chad, Lance and George made mistakes in calculating the number of border tiles for a garden of length 25 and were prompted by the researcher to use a visualization of the garden to find their errors. George used algebra tiles, Chad drew his own

picture, and Lance created a chart with help from the researcher. Regardless of the reasoning tool, a visualization helped these three students find understanding through the context of the task that was necessary to recognize the mathematical pattern.

Here was the conversation Chad had with the researcher as he tried to calculate the number of border tiles surrounding a garden of length 25. He began by discussing how many border tiles were added for each garden tile. Since Chad noticed the first frame had eight border tiles and two border tiles were added to each frame, he tried to talk through his calculations to find the border tiles for a garden of length 25.

> Chad: "So there is always two more around eight. So that would be 108 border tiles."
> Researcher: "How did you get that?"
> Chad: "So I did…..wait. No. It'd be 58. Because two times twenty-five. Right?"
> Researcher: "Ok. Well. Two times twenty-five gives you what?"
> Chad: "Fifty."
> Researcher: "I know but I mean are you talking about the length of the border tiles?"
> Chad: "No. The number of border tiles."
> Researcher: "Ok."
> Chad: "Since for each new length there are always two new tiles. You just do two times 50 and add it to the original number of border tiles."
> Researcher: "Ok. So you think the answer is 108?"
> Chad: "No, 58."
> Researcher: "Oh, 58."
> Chad: "At first I did four times 25; I don't know why."

At first Chad used a common difference of four, but then independently corrected his mistake to a common difference of two when the researcher asked if he thought 108 was the answer. After rethinking his steps, he changed his answer to 58 border tiles surround a garden of length 25. Chad knew the first frame had eight border tiles and since the border tiles were increasing by two in each frame, he reasoned two border tiles must be added 25 times, which would be 50 additional tiles added to the first frame's original 8 tiles, giving him 58 total border tiles. He was

on the right track but should have only added 2 tiles, 24 times, which would be adding 48 tiles to the original 8 which would mean 56 border tiles surround a garden of length 25.

He had used well-thought-out reasoning strategies; however, because the researcher knew his answer was incorrect, Chad was asked if he thought it would be helpful to draw a picture or use the algebra tiles to check his answer. Chad chose to draw his own picture (see Figure 17). After looking at his visual representation of the garden, here is what he said,

> Chad: "Yeah. So… (begins drawing his own diagram) So these are all…..I'm just going to draw… perfect."

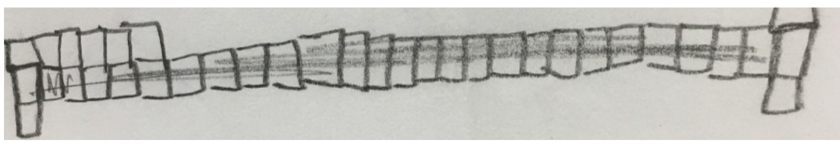


Figure 17. Chad's Garden Drawing.

> Researcher: "Uh huh."
> Chad: "So there would be two on the edges. And four in the corners. And then 50, wait…. And then 25 on the top and the bottom so there would be 56."

Even though Chad did not think to draw the picture on his own, after looking at his visual representation to see the garden, he corrected his mistake independently and changed his justification to one that related to the context of the picture. Instead of correcting his mistake by saying he should have only added two border tiles 24 times rather than 25 times, he now was finding the border tiles by doubling the garden length, adding four tiles for the corners and two more for the tiles at either end. It was not until Chad analyzed his visual representation that he was able to see the pattern to correct his original answer. This is an important discovery for Chad because he used this same idea to recognize the pattern to find the number of border tiles for a garden of any length.

> Chad: "How would you find the number of border tiles of any length? You could just do the two sides."

Researcher: “Uh huh.”
Chad: “The two sides. The four corners. And then multiply the number of black tiles by two and add that to the amount of sides and corners.”

Two students, Lance and George, were also making mistakes while finding the number of border tiles surrounding a garden of length 25. Their errors were not corrected until the researcher prompted them to create a visualization to search for the mathematical pattern. Lance and George were comparing the garden to the bottom row of border tiles. They used this comparison to calculate the number of border tiles surrounding a garden of length 25. Instead of noticing the total number of border tiles in the garden was increasing by two, they both used the idea that the bottom and top row of the border tiles were always two tiles longer than the length of the garden. Even though Lance and George were using a slightly different pattern recognition, they ran into the same error Chad experienced, and that was the common difference should only be added 24 times. Here is the conversation between Lance and the researcher as he calculated the number of border tiles for a garden of length 25.

Lance: “So it starts with three, then 2 -4, 3-5, so it should just add 25 to the 1. Or 24. It should be 28? And then 30 with the sides, and another 28 longways. So that’s going to be . . . “
Researcher: “You can use the calculator.”
Lance: “Ok. I am bad at math.”
Researcher: “You are not bad at math. That’s what the calculator is here for.”
Lance: “30 + 28 which is 58, so 58 border tiles.”

Lance had used the common difference between the bottom row of border tiles and the garden itself, which was two. You can see in the conversation he was waffling back and forth between adding 24 or 25 to the original garden length to find how many border tiles would be in the bottom row of a garden of length 25. He used 25, instead of 24 and this is how he corrected his mistake:

Researcher: “Maybe we should make a chart. (Researcher draws a table, Figure 18, comparing the garden length to the bottom row of

border tiles.) So if you have a garden of length one, we have 3, and when we say 3, we are just talking about the bottom row."

Lance: "Yeah."

Researcher: "So garden of length 1 has 3 tiles at the bottom. A garden of length 2 has 4 tiles at the bottom, a garden of length 3 has 5 tiles at the bottom. Your garden of length 4 has 6 tiles at the bottom."

Lance: "Yep."

Researcher: "We don't want to do this all the way, but you could if you don't see what the pattern is from here to here (first pair to the twenty-fifth pair) until you get to 25."

Lance: "Ok. If you have 4 to 6, you could just take away 4 from 25. That's 21 and then you add 21 to 6. And that's 27."

Researcher: "Ok. So that's 27."

Lance: "So there are 27 on the bottom and the top. 27 + 27 + 2. Which is 56."

Since Lance was already making a connection between the length of the garden and the number of border tiles below the garden, making a table (see Figure 18) to see the number

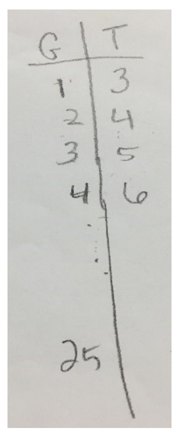


Figure 18. Lance's Table to Compare Garden Length to Border Tiles.

pattern helped him correct his own mistake. This was not an actual visual representation of the garden but a visual representation of the number pattern he needed to recognize the relationship between the length of the garden and the number of border tiles below the garden. Taking careful note of what was given and what was being asked proved to be vital to Lance's algebraic reasoning process. When Lance was asked to find the number of border tiles surrounding a garden of any length, he initially started by comparing the numbers in the table the researcher had created for him.

Lance: “Yes. So probably like the chart. Just draw a chart or something. And just find the pattern of it.”
Researcher: “Uh huh. Because when you found your answer for a garden of length 25, how did you get that? Walk me through that again and maybe we can write something down in just generic words.”
Lance: “Yes.”
Researcher: “How did you get from 25 to 56?”
Lance: “If it goes from 1 to 3, 2 to 4, and 3 to 5, and 4 to 6. You take four from 25 and that gives you 21 and since the total is six here, it’s just going to add one to it.”
Researcher: “Each time?”
Lance: “So you just add one to three each time?”
Researcher: “But what about a relationship not between 1 and 25, but between these two (pointing to the 1 to 3)?”
Lance: “Oh! They are each two bigger each time.”

The recognition of the difference between the garden length and the bottom row of border tiles was more helpful than comparing the number of border tiles from the first garden to the 25th garden. With guidance from the researcher to use the pattern Lance was noticing to help him make the connection between the length of the garden and the surrounding border tiles, he arrived at the informal abstraction (stage two) of the garden task’s mathematical pattern. The pattern he was noticing was how the length of the garden compared to the bottom row of border tiles which was a valuable piece of the puzzle for Lance. Even though he was getting lost in the search for the mathematical pattern, the analyzation he was using could be used to lead him to the informal abstraction (stage two) and it was the pattern he would eventually use to arrive at a symbolic representation (stage three) of the generalization. After he finally arrived at the correct generalization to this task, the researcher asked him,

Researcher: “What do you think tripped you up on trying to figure out what the pattern was?”
Lance: “I don’t really know. I just didn’t see it, I guess. I just kept thinking it was kind of random.”

Even though Lance had a solid understanding of the picture of the garden, he had lost track of which numbers were important to finding the generalization until he organized them in the table

to see the importance of how each garden length compared to the corresponding number of border tiles directly under it.

George used the same numerical and visual pattern combination that Lance had recognized but also made the same mistake as Chad and Lance. Being prompted by the researcher to use a visualization aided him to correct his own mistake. When asked to find the number of border tiles surrounding a garden of length 25, he said,

> George: "I need to start with the first one. So 3 + 25 = 28. 28 border tiles. Wait, all around?"

At this point, George did independently realize he needed to find the number of border tiles all around the garden but before George continued to calculate the number of border tiles with an incorrect number of 28 for the bottom row of tiles, the researcher asked him,

> Researcher: "Would it help to have a picture with these (algebra tiles)? Would that help?"

Using a visualization of the garden, similar to Figure 19, George was able to correct his mistake that the bottom row of border tiles should be 27, not 28.

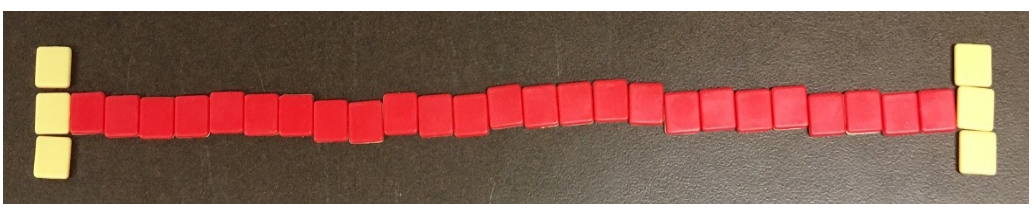


Figure 19. Algebra Tile Visualization of a Garden of Length 25 .

> Researcher: "So if you have 27, do what you were saying earlier. Say what you were going to do to find all the way around it (the garden)."
> George: "Multiply 27 times 2 and then add the extra 2 on the sides. So 27 times 2 which is 54. 54 tiles and then add two to that, so 56 tiles."

Seeing a visualization of the garden of length 25 allowed George to correct his own mistake, and make a connection between the length of the garden and the number of border tiles directly under the garden.

The well task was just challenging in general. Four students knew it would be helpful to use some form of visualization to search for mathematical patterns and did so independently. George manipulated the concrete numbers to look for the pattern. Ben created a chart. Chad and Jasmine drew a well. However, Susan and Lance had to be prompted to draw a sketch of the well (see Figure 20). They did not need much more than a suggestion, but it was an important tool that allowed them to see how the context of the task was creating the mathematical pattern. Here was the conversation after Lance answered the question referring to the length of time it would take the frog to exit a 10-foot well.

> Researcher: "5 hours? Ok. Would it be helpful to have a line representing the well and let it climb and slide back down maybe?"
>
> Lance: "Alright. (He draws a line on his paper to represent the well, Figure 20.) So if he goes up three feet and down one, up three feet, down one, up three feet, down one, up three and down one. That's only four and then it kind of stops and only goes two up."

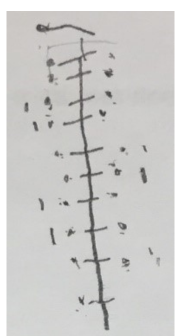


Figure 20. Lance's Well Visualization.

At this point, Lance mentions that it will take the frog 'four' which meant four time lapses or eight total hours. Then he said but it (the frog) kind of stops and only goes two up meaning he has started to recognize that the frog will not slide back down after he has exited. The researcher

asked Lance to move the frog up and down one more time to make sure he knew that going up three happened the first hour and going down one happened the second hour.

> Researcher: "Ok. So let's start back here and I'll count the hours as you go up and down."
> Lance: "Ok."
> Researcher: "Start at the bottom."
> Lance: "Three feet."
> Researcher: "That's the first hour."
> Lance: "And then down one."
> Researcher: "That's the second hour."
> Lance: "Oh, yeah!! (He realizes as the researcher is counting….first hour…. second hour….that the up and down movement is happening over a two hour period of time.) Ok so it should be eight hours."
> Researcher: "I think you are really close. Let's continue this because you are really close."
> Lance: "So he goes up three feet in the first hour and then he goes down one which is the second hour up three feet which is the third hour and down one which is the fourth hour up three which is the fifth hour down one – sixth, seven, eight, nine because he doesn't go down one!"

Lance answered this question correctly and had also discovered the most important part of this mathematical pattern, which was the frog would not slide back down in the tenth hour since he had already exited the well. When Lance was asked if he could explain how to find the time it would take the frog to exit a well of any depth he said, "Sometimes." Here is the conversation he had with the researcher while explaining why he could only find the number of hours it took a frog to exit the well in certain instances.

> Lance: "Yes. You could find the number of hours that a frog could climb out of a well of any depth."
> Researcher: "Ok so how would you explain that? Now that you have done a 10-foot well, a 9-foot well and a 20-foot well? What it's asking is could you tell someone else…..let's say the well is 200 feet deep. Could you give a generalized pattern to get the answer without having to draw a 200-foot well and use your number line?"
> Lance: "I would say yes sometimes. Because you can really use the pattern, two hours for two feet until, like this one (referring to the 20-foot well) where you use two instead of three."

Lance had already found the answers to four concrete examples using his visualization of moving the frog up and down the well and answered them all correctly. The visualization helped him realize you don't have to move the frog up three and down one but rather up two feet for every two hours.

> Researcher: "Ok."
> Lance: "So, I guess, sometimes."
> Lance: "Oh. Ok. So for the 48-foot you would just multiply 48 by two which is 96 and then add one, so 97 hours."
> Researcher: "Is that what you did to go from nine feet to get seven hours?"

Lance was thinking back to the idea that the frog was moving two feet in two-hour intervals but he was getting confused as to how to use that rate to find the time it would take the frog to exit a 48-foot well.

> Lance: "No, it's not. So to get to 48 feet you divide forty-eight by two and that's twenty-four and that would be 25 hours. Right? Because two hours is every two feet and if he goes 24 hours that is 48 feet but add one on the end so 25 hours."
> Researcher: "Is that what you are doing to get this one (9-foot well)? Dividing nine by two and then adding an hour?"
> Lance: "No."
> Researcher: "Or even the even one? Are you dividing twenty by two and then adding an hour?"
> Lance: "So you would get 10 and then it would just be…..hm……no I'm not."

The researcher tried directing Lance back to the questions he had already answered to remind him of how he found the correct answer to the last four questions, but this time without using his visualization.

> Researcher: "I think what is confusing about this is there is a time lapse. So you go up and down is one time lapse but it is still two hours. So if he travels two hours in two feet, four hours he will have gone four feet and six hours he will have gone six feet. So even though it is one full time lapse, it's still really one hour for every foot."
> Lance: "Ok."
> Researcher: "But then at the very end, when he climbs out you don't have to count the sliding back down part."

These ideas the researcher mentioned were all concepts Lance has already discovered and explained earlier in the interview, but he was not seeing those as the mathematical pattern he should be looking for.

> Lance: "Ok. So you,……so the 20-foot is 19 hours, and then the 9-feet is 7 hours and then11-feet is 9 hours. And the 10-feet is 9 hours. So is it like every time you hit an even, like 20 or 10, you just go down one hour? But every time you hit an odd number, like 11 or 9, you go down two hours?"

Now Lance has recognized the mathematical pattern. He is comparing the previous examples and using those to verbalize an informal abstraction of the well task.

> Researcher: "Yeah."
> Lance: "Oh, so for 48, you would go down one? So it would be 47 hours."

Lance is now confident enough with the understanding of the pattern to determine how long it would take the frog to exit a 48-foot well. He stopped multiplying by two and dividing by two and, in his own words, stated the most practical way to explain the pattern.

As was predicted by the students, there was more to the mathematical pattern of this task than just saying the frog moved two feet every two hours or one foot every hour. It was important for the students to look at and use a visualization of the well to discover that after the frog exited the well, the last hour of sliding back down does not occur. Reasoning while visually and physically moving the frog up and down the well to find this mathematical pattern was valuable which will lead us to the importance of understanding the context of the task to find the mathematical pattern. Lance determined it would take the frog 9 hours to exit a 10-foot well with this visualization of the task. He even commented,

> Lance: "Graphs are helpful."

The number sequence task did not have a visual picture or story behind the number pattern to use as a context. Although the numbers in the sequence task were the set of odd numbers, the

key algebraic concept necessary to reach a generalization of the sequence was to realize that the common difference from each term to the next term was two. Four of the six students made mention of this when they were discussing how they found the next number in the sequence.

> Ben: “They are all odd numbers or skipping two.”
> Chad: “It’s just adding two.”
> George: “Adding two to the previous number.”
> Jasmine: “You are always adding two.”

This common difference could have been used to find the mathematical pattern of this arithmetic sequence, which students would have done if the next scaffolded question had asked them to find the 50th number of the sequence. But instead, the researcher asked the students to be more specific about numerical patterns they noticed by suggesting they draw an *x/y* chart. Even though students did not naturally think of displaying the numbers in a table with their respective placement value, once the numbers were compared (see Figure 21) the students began to notice mathematical structure. The *x*-value represented the position of the number in the sequence and the *y*-value represented the number in the sequence. 1, 3, 5, 7, 9, and so on.

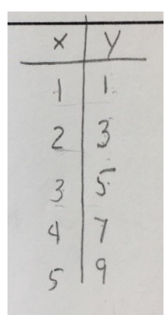


Figure 21. *X* and *Y* Comparison of the Number Sequence.

<u>No Confidence Using Variables</u>**.** Students verbalized informal abstractions (stage two) of the generalization of the tasks but lacked the understanding of when and where to use the variable when attempting to translate their patterns to a symbolic form (stage three).

Chad’s visualization of the number sequence task, through the use of the chart, aided him to transition from informal abstraction (stage two) to formal abstraction (stage three). Chad had

already stated the common difference by noticing each number in the number sequence task was found by adding two to the previous number. As was already mentioned Chad did use the common difference to state the recursive pattern as can be seen in the picture of his work (see Figure 22). It was not his idea to graph the *x/y* pairs either, or to graph the ordered pairs. The researcher thought Chad might remember how to find the linear equation by using the slope and the *y*-intercept. After graphing the points, Chad drew in the rise and the run of the graph but never mentioned its importance in the recursive pattern nor did it lead him to the explicit answer he eventually found which was $y = 2x - 1$. He found the explicit pattern by comparing each x to its respective y and noticed that if you double the x and subtract one that would give you the y-value. Then Chad had an equation that could be used to find any number in the sequence.

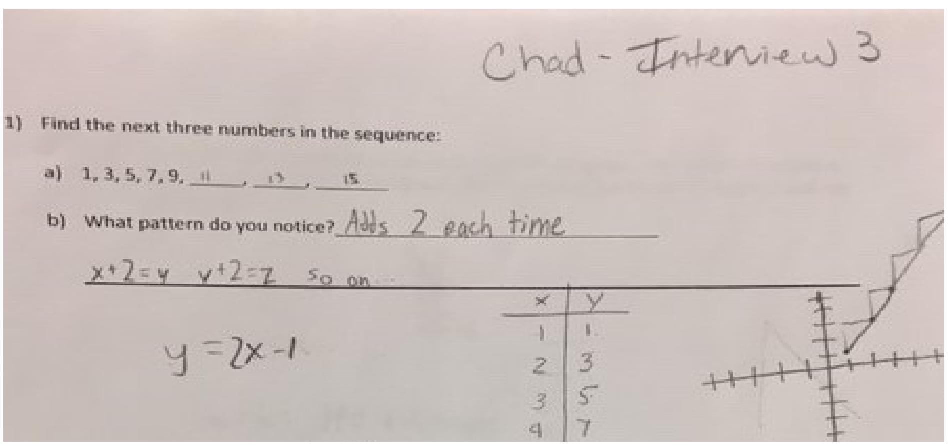


Figure 22. Chad's Work for the Number Sequence Task.

Visualizing the mathematical pattern by using the chart helped Jasmine focus and find the mathematical pattern of the number sequence task. When Jasmine was analyzing the *x/y* pairs (see Figure 23) she had noticed the difference between the *x*- and *y*-values was getting larger each time. Then the researcher wanted to see if she was making a connection between the middle column and the first column and said,

> Researcher: "How does it relate to the other numbers in the table?"
> Jasmine: "It is always one less than the *x*."

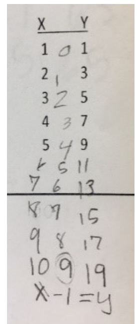


Figure 23. Jasmine's Comparison of the Number Sequence.

Even though she had to be prompted to analyze the patterns shown, her idea of seeing the pattern in a different way would lead her to an equation that was different from the other students.

Jasmine needed a lot of prompting to analyze the mathematical pattern of the number sequence and, due to the nature of the task, she was not the only one. After being asked to create the table and discuss the patterns she was seeing, the equation she found for finding the value of any number in the sequence was '$x + (x - 1) = y$'. In true Jasmine fashion she was seeing and finding patterns differently than the other students in the study.

Ben worked on the well task, trying to transition from stage two to three, for over twenty minutes. He had already created a table to organize and compare the depth of the well to the time it would take the frog to exit. He had already independently answered the concrete questions but now he was asked if he could find the number of hours it would take the frog to exit a well of any depth and here are bits and pieces of his thought process.

> Ben: "The thing is, if I'm writing an algebra equation, the thing that's going to throw it off is when it's one foot deep. I don't know how to do that. Because if it's one foot deep it's going to take him it's going to take him less than an hour. It's going to take him like twenty seconds."

Ben had recognized from looking at the table he created that he might be able to create an algebraic expression for any well depth except a one-foot well. Knowing the frog would exit a one-, two- and three-foot well all in the first hour of climbing was one discovery Ben made when

searching for a mathematical pattern. Then he began discussing how to determine the time the frog would be climbing compared to the time the frog would be sliding.

> Ben: "Because you have that extra plus three, and minus two. So you have to have something like this many times of going up three and this many times of going down."
>
> Ben: "Yes. Cuz it's going to be, half the number, if it's an odd number, it's half the number and one. Well, if you subtract the number, half the number plus one. And half of the first number is going to be the numbers of going up and down. Except it's going to be that number times 3 and that number times negative one."

He had the correct idea that the frog would be moving up thee feet for half of the time and down one foot for one hour less than half of the time. He was using *d* for the depth of the well and *t* for the time it would take the frog to exit. He was dividing half of the depth by two and multiplying it by three to represent the movement up and dividing half of the depth by two and multiplying it by negative one to represent the number of times the frog would slide back down a foot.

> Ben: "I want to try using 48, because I know I just did it."

The numbers all around his equation (see Figure 24) are numbers he was using to check his formula to see if it worked. He had already answered five of the questions correctly using the table he created and now he was using those same numbers he already knew were correct to check the validity of his equation. After working to find the symbolic form for some time, Ben made a big discovery.

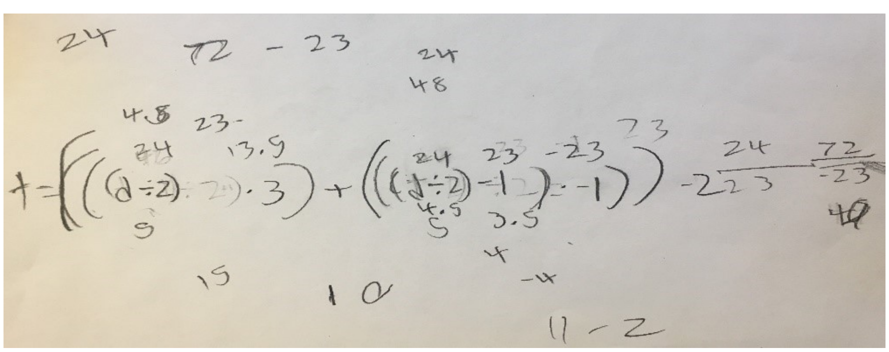


Figure 24. Ben's Work on the Well Task.

Ben: “So I’m going to start with ten which would give you five, then fifteen, five, four, negative four. Which is giving you two more than it so I need to subtract two. I am always getting two more than it!! Ugh. So now I have to minus two at the end of it. But it’s still…… so minus two! After all of that, it’s minus two. Just minus two?”

He had now realized that maybe he was making this process more difficult than it needed to be.

Ben: “See that’s the thing. Odd one’s you’d have to…..”
Ben: “So it’s just common sense, there’s no mathematics.”

After this discovery Ben still continued to discuss the equation he had created. He was having a difficult time leaving it unfinished. The researcher appreciated his determination and perseverance but thought it interesting that he seemed reluctant to write down the commonsense generalization (informal abstraction) to this mathematical task.

The well task was the most challenging task to find and explain its mathematical pattern (informal abstraction), not to mention trying to find a symbolic representation (formal abstraction) of the pattern as well. However, after reading Ben’s approach, it was interesting to see that George’s mathematical pattern, although it was not in symbolic form, was presented in the most efficient and practical way.

George: “To find the time it would take the frog to exit an even-depth well, go back one hour, and if it’s an odd depth, go back two.”

The researcher created a table for Lance and suggested he compare each garden length with its respective bottom row of border tiles. Then with additional help Lance was eventually able to reach stage three (formal abstraction) of the generalization process.

Lance: “How would you find the number of border tiles of a garden of any length?”
Researcher: “But what about a relationship not between 1 and 25, but between these two (pointing to the 1 to 3)?”
Lance: “Oh! They are each two bigger each time.”
Researcher: “Yes! So let the length of the garden be “l”. And to find this bottom number, what did you do to it? If the garden was 3, it was 5, if it was 4, it was 6, if it was 25, it was 27. So let’s let

the garden length be "l", what would…."

Lance: "Ok. So like L + 2 equals answer."

Researcher: "Ok. But L + 2 just gives you the bottom section."

Lance: "Oh."

It was important for Lance to make the distinction between the L and the L + 2 to proceed in finding the symbolic form of the generalization.

Researcher: "So what did you do after that?"

Then the researcher made sure Lance was thinking back to his strategy for finding the number of border tiles for a garden of length 25.

Lance: "I got the bottom row and I added the bottom and top together since they are the same."

Researcher: "Ok."

Lance: "And then I did the sides."

Researcher: "So if the bottom was L + 2, what is the top?"

Lance: "L + 2".

Researcher: "Ok. Put that down."

Lance: "L + 2 + L + 2 + 2."

It was not easy for Lance to reach stage three (formal abstraction) of generalization but using his own reasoning strategies and understanding of the mathematical pattern eventually led him to a symbolic generalization of the garden task.

When Jasmine was asked to translate her verbal explanation of the pattern into an algebraic expression she said,

Researcher: "So what it is kind of asking here is, um, you figured out and you see what the pattern is and what is happening with it but could you come up with a generalization as to what is my …… Well, let's go to the next one and then we will come back to here. Because the next one asks a similar question and it asks you to use your pattern but I think you might want to…. Just read this one and don't use the pattern yet just see if you can figure this one out."

Jasmine: "Ok. Use this idea to find the number of border tiles of a garden of length 100. Ok so it's 100, so that would be 100, 200, because 100 plus 100 is 200. And then plus six so it would be 206."

Researcher: "Go ahead and put that down. Sounds to me like you have the

idea because you did it here and you did it here. So you are doing the same thing for this one. Because what was this one's length and how did you get that?"

Jasmine: "25 + 25 + 6"

Researcher: "Ok. And this one?"

Jasmine: "100 + 100 + 6"

Researcher: "Ok. So let's go back here."

Jasmine: "So it would be length times… wait… yeah. Two times any variable. So two times x. So 2x + two times…. I need to use a w. Can I use w?"

She determined that rectangular patterns usually have a length and a width and therefore this expression must contain an 'l' and a 'w'.

Researcher: "Is the width changing?"

Jasmine: "No. Wait no. Length. Width. No."

Researcher: "Ok."

Jasmine: "It's always 3. So you could do 3 + 3?"

Researcher: "Yeah."

Jasmine: "Ok. That makes it more simple."

Researcher: "If the garden was growing in length and width, then you might want to change that and let it be a variable. Ok, so now let's go back and do that one and let's see if you can use that idea. If you did use this idea…."

Jasmine: "Ok."

Researcher: "To find the garden length of 100."

Jasmine: "So use this idea to find the number of border tiles for garden of length 100**.** So we did 100 times 2, which would be 200 and then we did the two sides which would be 3 + 3, which would be 206."

Jasmine did not make a transition from the concrete to the abstract until the previous examples were presented in a way (see Figure 25) that she could see the transition from the concrete to the abstract to help her not just arrive at the recognition of the generalization (informal abstraction) but the symbolic form (formal abstraction) of that generalization $2l + 3 + 3$.

All six students eventually arrived at a symbolic representation of the pattern of the garden task; however, this could not have happened without the use of a table of concrete examples, or the table of the comparison of the garden's length to the bottom row of border tiles, or a visualization of the task. These visualizations gave context to the task and helped student

Garden Length	Concrete Examples	Border Tiles
1	1+1+3+3	8
2	2+2+3+3	10
3	3+3+3+3	12
4	4+4+3+3	14
25	25+25+3+3	56
g	g+g+3+3	2g + 6

Figure 25: Concrete to Formal Abstraction for the Garden Task.

understand the mathematical patterns and connect their patterns with symbols. These organizational tools eventually equipped all students to transition from arithmetic to algebra.

Summary

This study's data was collected from six students during eighteen different interviews. Each interview was watched and transcribed, notes were taken, algebraic reasoning was recognized, and the stages of generalization were tracked with an observational protocol. In reference to research question number one, what was the nature of students' algebraic reasoning through tasks involving generalizing?, the analysis of the data showed three main themes in students' natural algebraic reasoning: 1) it was possible for students to reach an abstract generalization of the mathematical pattern even if they did not reach stage three (formal abstraction), 2) students relied heavily on visualizations as reasoning tools to generalize the patterns, and 3) using the context of the task proved to be the most powerful way to reach a generalization. Students who were able to discuss, visualize and recognize the mathematical patterns had transitioned from the concrete to the abstract and were able to use their generalization to answer other questions related to the given task. Research question two which related to the challenges students

encountered when reasoning algebraically through tasks involving generalizing provided one theme: students had challenges transitioning from one stage to the next of the generalization process. Students struggled to generalize independently when they did not access the available reasoning tools, such as visualizations which provided contexts for the tasks. Though many of the students could informally explain the abstract generalization, most of them were not making the transition to formal algebraic symbols without aid from the researcher.

CHAPTER V: DISCUSSION AND RECOMMENDATIONS

Summary of the Study

The purpose of this study was to observe and understand the nature of students' algebraic reasoning through generalization tasks. After students' algebraic reasoning had been analyzed, the challenges they encountered while reasoning was also analyzed. Six 45-minute interviews were conducted with eighth grade students while solving generalization tasks. Then the interviews were watched and analyzed to find overarching themes in students' algebraic reasoning. This chapter will be a summary of the findings from the interviews, followed by recommendations for future research, concluding with recommendations for future practice of algebraic reasoning.

Summary of the Findings

The first research question was "What was the nature of students' algebraic reasoning through tasks involving generalizing?". The researcher watched the interviews to look for what reasoning tools students independently accessed, such as visualizations and contexts of the tasks to search for mathematical patterns. The analysis of the data revealed three themes: 1) it was possible for students to reach an informal abstraction (stage two) of the mathematical pattern even if they did not reach stage three (formal abstraction), 2) students relied heavily on visualizations as reasoning tools to generalize the patterns, and 3) using the context of the task proved to be the most powerful way to reach a generalization. The second research question was "What were the challenges do students encountered when reasoning algebraically through tasks involving generalizing?" To find challenges in algebraic reasoning, the researcher watched the

interviews to find places the students needed guidance to proceed in the generalization process. Where did the researcher step in to offer a suggestion to help the students move on to the next stage of generalization? The analysis of the data provided one theme: students had challenges transitioning from one stage to the next of the generalization process. Students struggled to generalize independently when they did not access the available reasoning tools such as visualizations and the contexts of the tasks. Though many students could informally explain the abstract generalization, they were not making the transition from an informal algebraic explanation to a symbolic representation.

Discussion

One discovery of this study was the importance of students discussing and visualizing mathematical patterns to arrive at an abstract generalization. Kriegler (2007) and Lannin (2003) would say algebraic reasoning is a combination of mathematical thinking with formal algebraic procedures so students can conceptualize mathematical patterns. This study confirmed Kriegler's and Lannin's research by finding students do rely heavily on mathematical thinking procedures, or as this study called them, reasoning tools. These would be tools, such as visualizations or contexts of the tasks, one may or may not find in an algebra textbook but aided students in solving the tasks.

An example of reasoning tools would include the students drawing a diagram of the well and physically moving the frog up and down to find the hours it took him to exit. These diagrams helped the students find the crucial point of this mathematical pattern which was the frog would exit the well before he slid back down, meaning the answer to this task's generalization had to be explained as two different cases.

Another example of students using reasoning tools included using algebra tiles or students' drawings of the garden task to help analyze the mathematical pattern. Even though only six student interviews were analyzed for this study, the mathematical patterns were seen and recognized in four different ways (see Figures 26-29).

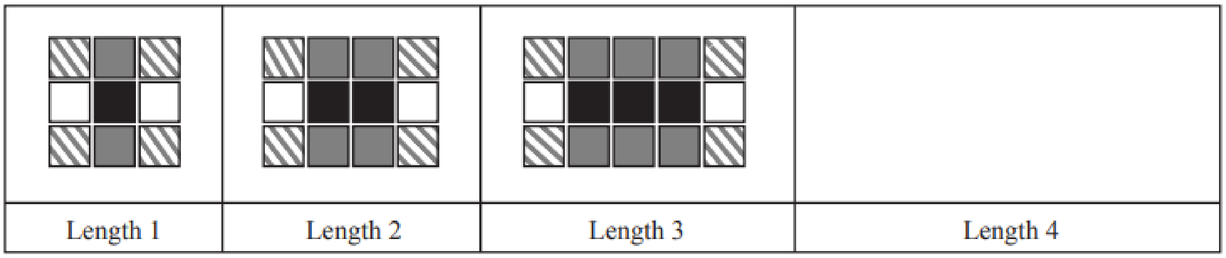


Figure 26. Chad's Recognition of the Garden Task.

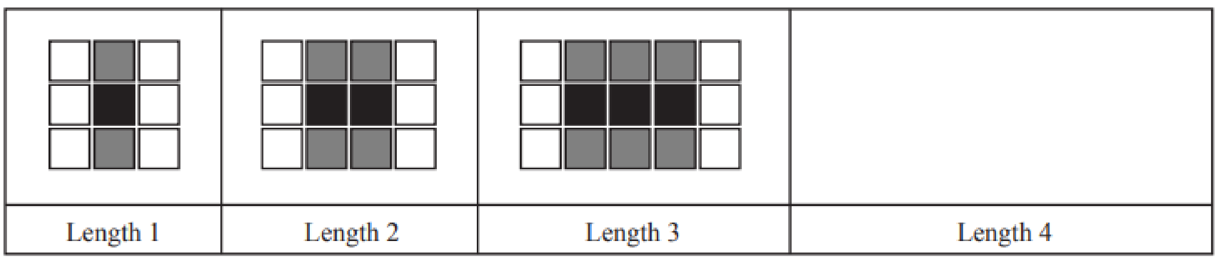


Figure 27. Susan and Ben's Recognition of the Garden Task.

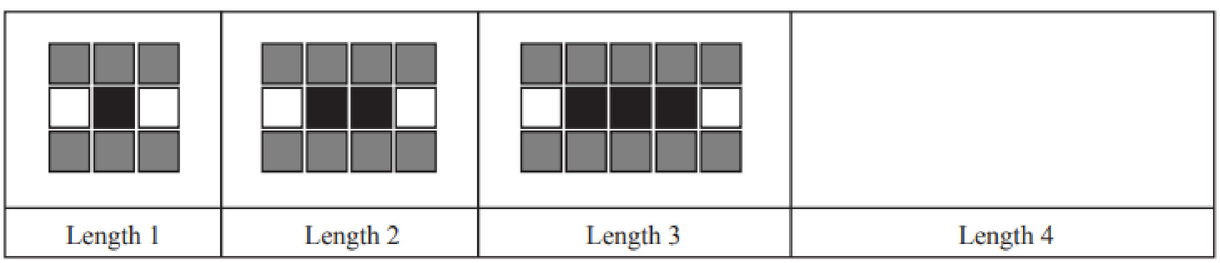


Figure 28. George and Lance's Recognition of the Garden Task.

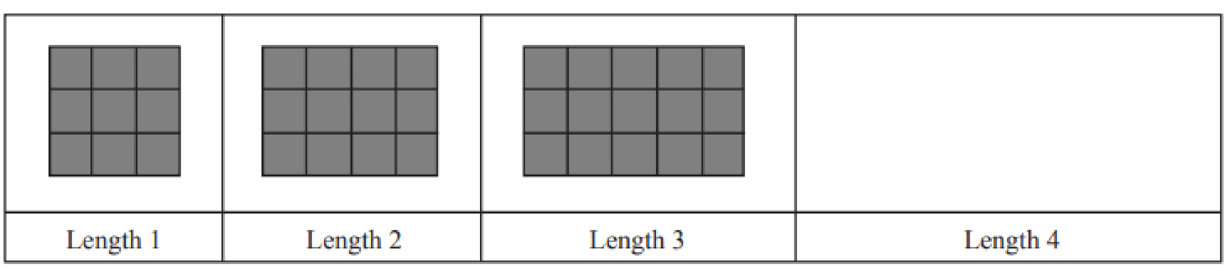


Figure 29. Jasmine's Recognition of the Garden Task.

It was important to let the pattern each student saw lead him or her to a generality (Mason, 1996) because using a visualization of the task would eventually lead to a more accurate and efficient generalization of each task (Lannin, 2004). Not every student naturally drew the garden, but as Kriegler (2007) and Lannin (2004) suggest, once the students had access to a visualization of the garden, they were able to understand the context of the task to search for the mathematical pattern.

Another discovery of this study was students did go through stages trying to generalize mathematical patterns. According to Mason (1996) and the Center for Algebraic Thinking (2014), there are four stages of generalization: seeing, saying, recording and symbolic form. Although the students in this study did transition through stages, which was important in reaching a generalization of the mathematical pattern, the findings of this study emphasized the difference between the concrete and the abstract, and therefore referred to the stages as concrete, informal abstraction, and formal abstraction.

Stage one was called the concrete stage where students were discussing algebraic concepts about how the garden was changing from frame to frame, how far the frog climbed each hour, and how each number in the sequence changed from the previous number. Student's initial discussions of the given tasks related to conversations about the patterns they were noticing. At this point, they were simply discussing ideas or perhaps brainstorming what might be happening in the pattern. However, once they accessed visualizations of the patterns, students began to see and understand the mathematical patterns using the contexts of the tasks. During stage two, informal abstraction, students began explaining the abstract generalizations of the mathematical patterns. Stage three, formal abstraction, was representing the patterns in symbolic forms. Figure

30 is an outline with examples of students' discussions to highlight the stages of generalization through the garden task. Subdividing the stages in this way not only emphasized the importance

I. Concrete

a) Discussing: "The garden and tiles are growing by one, but the width stays the same." (George)

b) Visualizing the context of the garden: "They kind of grow by one space and there is kind of an extra black square and two extra white ones." (Chad)

II. Informal Abstraction: To find the number of border tiles of a garden of any length, "It's the same as the garden times two. Then three on both sides." (Susan)

III. Formal Abstraction: "Let x represent the length of the garden, then $2x + 6$ would be used to find the total number of border tiles." (Ben)

Figure 30. The Three Stages of Generalization

of using the concrete to reach the abstract, but also showed algebraic reasoning was not happening until students were able to generalize the mathematical patterns.

Another discovery of this study was the importance of using visualizations of the given tasks to analyze the mathematical patterns. Therefore, the findings of this study confirm Lannin's (2004) work that, to reach an abstract generalization of the pattern, students use diagrams, charts or tables. Students needed reasoning tools to transition from the concrete to the abstract. Through the organization of concrete examples students began to understand the mathematical patterns and were able to recognize the abstract generalizations.

An example of this was observed when George independently used his adding-three and subtracting-one pattern searching to understand the well task. Other examples of students having to be prompted to access a generalization occurred during the garden task. George recognized the mathematical pattern after the researcher suggested visualizing the garden with the use of algebra tiles. The researcher prompted Lance to create a table comparing the length of the garden to the bottom row of border tiles which helped him recognize the mathematical pattern.

Kriegler (2007) and Lannin (2003) would say algebraic reasoning is a combination of mathematical thinking with formal algebraic procedures so students can conceptualize mathematical patterns. Dougherty et al. (2015) would say algebraic reasoning includes analyzing, noticing structure, and generalizing, and then, later goes on to summarize it "……is being be able to analyze quantitative relationships, generalize, model, justify, prove, predict, problem-solve, and notice structure" (p. 274). The findings of this study would agree that algebraic reasoning does involve all of these components. Even though the focus of this study was to understand how students generalize patterns, it was evident students were also analyzing, justifying, proving, predicting, problem-solving, and noticing structure as they transitioned through the stages of generalization. When generalizing the garden task, students were also being asked to predict the number of border tiles for much larger gardens, having to justify the reasons for their guesses, and asked how to find the length of a garden surrounded by 70 border tiles.

Now I can confidently explain why students struggle to transition from the concrete to the abstract and know what to give my students that I did not get. For the last year, I have given tasks like the ones in this study to my pre-algebra and algebra classes and we call it 'finding the hidden code'. I love to see my students excited about learning a subject I love. They enjoy searching for patterns. Students are interested to see what patterns their classmates have discovered and are proud to share their pattern, especially if it is different.

Recommendations for Future Research

One recommendation for future research would be to use participants from different levels of algebra. Interviewing some students who have not had algebra, some students who are in algebra, and some students in an upper level algebra class might give different perspectives on

the stages of generalization. Also, it would be interesting to have two or more students in the interviews so the researcher could be more of an observer rather than a participant in the problem-solving process. It might be best to give each student time to read the question and start looking for patterns and then let the students share ideas with each other as they corporately try to find the mathematical patterns.

Another recommendation for future research is to carefully choose the generalization task and its related scaffolded questions. Some tasks should elicit the value of finding the recursive pattern while other tasks should encourage students to find an abstract generalization of the mathematical pattern. The garden task was borrowed from an article read during the literature review and it worked well in this study because it provided a variety of different students' visualizations of the garden. The well and number sequence tasks were found on the EdTech Leaders Online website. Both the garden and well tasks were interesting and thought-provoking, but the garden task had already been studied, by other researchers, and the well task was challenging which frustrated the students. The number sequence task, with no context and limited scaffolded questions, did not produce conversations of algebraic reasoning like the study would have hoped. There are many generalization tasks available as resources so choose a few tasks and try the tasks first, and with small groups of students, before asking an entire class to solve it and make sure the task has the appropriate difficulty level for the students in your class. It would make the study even more interesting if the researcher would make up new and different tasks that have never been studied before.

Recommendations for Future Practice

The rationale for this study was to figure out why students are not transitioning from arithmetic to algebra. If arithmetic is analyzing numbers you can easily discuss and see and algebra is noticing the structure in order to make an abstract generalization of the mathematical patterns you can discuss and see, then letting students reach the abstraction using the strategies they visualized was important in the generalization process. "Generality is not a single notion, but rather is relative to an individual's domain of confidence and facility. What is symbolic or abstract to one many be concrete to another" (Mason, 1996, p. 74). What strategy seemed obvious to one student seemed foreign to another.

The challenge students faced when reasoning algebraically happened when they were not using their own intuitions and reasoning processes but rather trying to use ideas or strategies presented by the researcher. Looking back on the interviews, I thought I was being helpful, but I was frustrating some of the students by trying to get him/her to see it my way. Students may see the pattern in a different way from their classmate or teacher, and it is important for each student to find a way to generalize that makes sense to him/her. Then, after solving the task, they should be open to listen to other suggestions or methods to learn to solve the task in a different way which might be valuable for future tasks. Then when students come up with different symbolic representations, a discussion can be had about the equivalence of several algebraic expressions. There were times during the interviews where guidance from the researcher was instrumental in referring students back to the context of the situation to find the mathematical pattern. When students were prompted to access a diagram or chart of the task or look back at previous concrete examples, this often allowed students to develop further connections between the context and the variables (Lannin, 2005). Students need guidance and practice using these reasoning tools until

they begin to see these strategies can be used in similar situations which will lead them to mathematical independence.

Students already have ability to analyze and justify mathematical patterns. This study showed many instances where students independently transitioned from the concrete to an informal abstraction. This should be celebrated. They were answering the questions of how to find the border tiles for a garden of any length, the number of hours it would take the frog to exit a well of any depth, and any number in the sequence. They answered the questions, even if it was not using formal algebra. Students need help and practice transitioning from informal to formal algebra. Teachers should search for generalization tasks even if they are not found in the school curriculum and present students with generalization tasks with strategically scaffolded questions that will guide them independently from the concrete to the abstract. Along with this practice, students will be having algebraic discussions and be accessing visualizations by the use of tables, charts and pictures of the patterns, so they will gain confidence and learn when to access reasoning tools to become independent successful algebra students (Kriegler, 2007).

This research study's conversations were not like any discussions that happened in my algebra class in the 1980s. Students were analyzing, generalizing, and noticing structure in mathematical patterns. If my algebra curriculum asked my students to solve the number sequence task, first of all, I would not start like my algebra teacher did, by giving students the formula and telling them how to use it. I would scaffold the questions related to the task in such a way that the students could discover the formula for themselves. I would ask students to find the next three terms in the sequence, and then the 25th, 50th and 100th term in the sequence. Then I would ask them to explain how they would find any term in the sequence. Scaffolding questions in this manner gives students time to independently analyze the mathematical patterns to search

for understanding. I would give each student ten minutes to think about the task (Lannin, 2004) or maybe even assign this one problem for homework. Then, after each student had taken time to develop strategies on his/her own, I would pair each student with a partner to let them discuss their strategies. Discussing their ideas and strategies with peers forces them to formulate justifications for their patterns to convince classmates of the accuracies of their generalizations or to notice inaccuracies in other generalizations. These discussions allow students to help each other make the transition from the concrete to the abstract. Depending on the level of algebra class, I would also make sure students transitioned to a symbolic form of the abstraction as well. It is my preference that generalization tasks have a context. There is nothing wrong with students looking for patterns from a sequence of numbers, but the real world does not work that way. If we are preparing students for life after graduation, it is important that we present real-world tasks and preferably tasks students can relate to, so they have a vested interest, not just in the task but the mathematics as well.

Summary

To mathematics educational researchers none of this study's findings should come as a surprise, but to the teacher on the front lines, trying to make a difference but too busy to know how: this is for you! Emphasizing students' own natural algebraic reasoning strategies should awaken teachers to never give up on the struggling algebra student. Seeing the challenges students have connecting the underlying mathematical structure and formal algebra should motivate us to implement strategies to bridge the gap. They may come to us unprepared which is unlikely their fault. They may appear to be unmotivated which is more than likely a result of years of failure. Let's help them transition from the concrete to the abstract. Let's help equip

these students with algebraic reasoning tools to recognize and algebraically represent an abstract generalization. Let's undergird their mathematical foundations to strengthen their mathematical independence so they will not just be problem-solvers or procedural robots, but rather students with the capability to tackle this difficult subject with confidence.

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APPENDICES

Appendix A: Observational Protocol

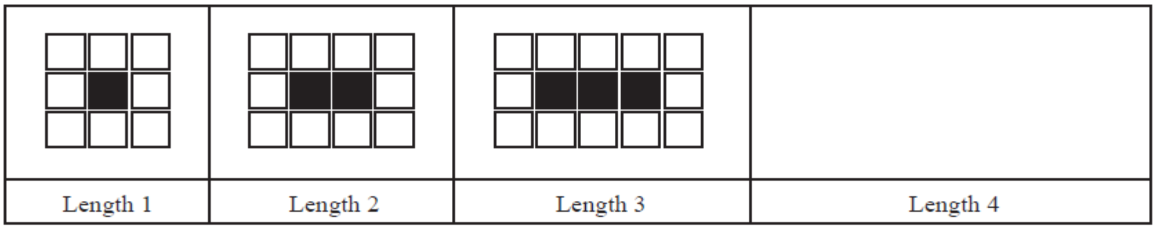
I. **Stage One (Concrete) -** Beginning conversations were not focused on abstract generalizations of mathematical patterns but rather occurrences of students intently discussing and using visualizations of the tasks to understand the context of the tasks.

II. **Stage Two (Informal Abstraction) –** A verbal explanation of the abstract generalization of the task's underlying mathematical structure. This stage consisted of evidence that students were transitioning from the concrete to the abstract. For example, students started recognizing how to find the number of border tiles for a garden of any length, the number of hours to exit a well of any depth, or the value of any number in sequence.

III. **Stage Three (Formal Abstraction) –** Translating the informal abstract generalization to a symbolic form. This stage of generalization involved taking the recognized mathematical pattern and translating it from words to an algebraic expression which meant the students were now transitioning from informal to formal algebraic procedures.

Appendix B: The Generalization Tasks

Generalization Task #1: The Garden Task *(modified from Kriegler, 2007, p. 5)*



Gardens are framed with a single row of border tiles as illustrated here.

a) *In the space provided, draw the garden of length 4.*
b) *What patterns do you notice about how the gardens grow?*
c) *Find the number of border tiles of the garden of length 25.*
d) *How would you find the number of border tiles of a garden of any length?*
e) *Use this same idea to find the length of a garden surrounded by 70 border tiles.*

Generalization Task #2: The Well Task (from the article *Three Components of Algebraic Thinking: Generalization, Equality, Unknown Quantities*, EdTech Leaders Online)

A frog is at the bottom of a 10-foot well. If he climbs three feet in the first hour and slides back one foot in the second hour and repeats this pattern, how long will it take him to climb out of the well?

a) *What pattern or patterns do you notice?*
b) *How long would it take the frog to exit an 11-foot well?*
c) *How long would it take the frog to exit a 20-foot well?*
d) *Could you find the number of hours needed to climb out of a well of any depth?*
e) *Can you use this idea to find the depth of the well if it takes the frog 15 hours to climb out?*

Generalization Task #3: The Number Sequence Problem (from the article *Three Components of Algebraic Thinking: Generalization, Equality, Unknown Quantities*, EdTech Leaders Online)

1, 3, 5, 7, 9, ____, _____, _____

a) *Find the next three numbers in the sequence.*
b) *What pattern do you notice?*

Appendix C: Informed Consent Agreement (School)

MISSOURI STATE UNIVERSITY
INFORMED CONSENT (School)

Consent to Participate in a Research Study
Missouri State University
College of Natural and Applied Sciences
College of Education

Exploring the Nature of Similarities and Differences in Middle School Students' Algebraic Thinking.

Dr. Patrick Sullivan
Dr. Gay Ragan
Andrea Martin

Introduction
Your school has been asked to participate in a research study. Before you agree to participate in this study, it is important that you read and understand the following explanation of the study and the procedures involved. The investigators will also explain the project to you in detail. If you have any questions about the study or your students' role in it, be sure to ask the investigator. If you have more questions later, Dr. Patrick Sullivan or Dr. Gay Ragan, will answer them for you. You may contact Dr. Sullivan at: PatrickSullivan@MissouriState.edu or Dr. Ragan at GayRagan@missouristate.edu.

You will need to sign this form giving us your permission to be involved in the study. Taking part in this study is entirely your choice. If you decide to give permission to take part in the study but later change your mind, you may stop at any time. If you decide to stop, you do not have to give a reason and there will be no negative consequences for participation in the study.

Purpose of this Study
The purpose of this teaching experiment is to explore secondary mathematics students' ability to think algebraically by studying their potential to access skills such as generalization, problem-solving, and flexibility to work with unknown quantities, in order to develop practical and realistic strategies to help all mathematics students master difficult content.

Description of the Study
If your school agrees to be part of this study each student participant will be interviewed and recorded for three separate 45-minute sessions over a three-week period of time. During the interview sessions, the students will be asked to perform mathematical activities without any help from the interviewer and will be asked to think out-loud in order to have a record of their algebraic thoughts.

What are the risks?
One possible risk that may apply to this research study is during the first interview students may feel uncomfortable solving mathematics problems while being recorded. The students are already comfortable discussing mathematics with me because I am currently their teacher, so I think that after the first few minutes of the interview session, students will become more comfortable. In addition, the students who are not as confident in their mathematics ability may be uncomfortable explaining the methods and procedures they are using to solve the mathematics problems.

What are the benefits?
The benefits of this study is pinpointing what is holding students back from being successful in algebra. The results of this study with be shared with colleagues to access current methods of teaching mathematics and give guidance in making curriculum changes.

How will my privacy be protected?
The name of the school and students will not appear on any information that we share with others. None of these identifiers will be identified by name in any publications that result from this research. We will use pseudo-names to identify any written work or video that may involve the image of the student. The information gathered will be accessible only by the investigators and it will be in a locked cabinet. Any data saved electronically will be secured on a password-protected computer. All information not used in publications will be destroyed three years after the study ends

Consent to Participate
If you wish for your school and students to participate in the study, *Exploring the Nature and Similarities and Differences in Middle School Students' Algebraic Thinking,* we ask that you sign below indicating your willingness to allow them to participate:

I have read and understand the information in this form. I have been encouraged to ask questions and all of my questions have been answered to my satisfaction. I have also been informed that I can withdraw from the study at any time. By signing this form, I voluntarily agree to allow the students at my school to participate in this study. I have received a copy of this form for my own records.

____
Printed Name of PrincipalDate

____
Signature of PrincipalDate

Andrea Martin
Signature of InvestigatorDate
____
Signature of InvestigatorDate

Appendix D: Informed Consent Agreement (Student)

MISSOURI STATE UNIVERSITY
INFORMED ASSENT (Student)

Consent to Participate in a Research Study
Missouri State University
College of Natural and Applied Sciences
College of Education

Exploring the Nature of Similarities and Differences in Middle School Students' Algebraic Thinking.

Dr. Patrick Sullivan
Dr. Gay Ragan
Andrea Martin

Introduction

Your classroom teacher is conducting a research study. Before you agree to participate in this study, it is important that you read and understand the following explanation of the study and the procedures involved. The investigators will also explain the project to you in detail. If you have any questions about the study or your students' role in it, be sure to ask the investigator. If you have more questions later, Dr. Patrick Sullivan or Dr. Gay Ragan, will answer them for you. You may contact Dr. Sullivan at: PatrickSullivan@MissouriState.edu or Dr. Ragan at GayRagan@missouristate.edu.

You will need to sign this form giving us your permission to be involved in the study. Taking part in this study is entirely your choice. If you decide to give permission to take part in the study but later change your mind, you may stop at any time. If you decide to stop, you do not have to give a reason and there will be no negative consequences for participation in the study.

Purpose of this Study

The purpose of this teaching experiment is to explore secondary mathematics students' ability to think algebraically by studying their potential to access skills such as generalization, reversibility, and flexibility to work with unknown quantities, in order to develop practical and realistic strategies to help all mathematics students master difficult content.

Description of the Study

If your child agrees to be part of this study he/she will be interviewed and recorded for three separate 45-minute sessions over a six-week period of time. During the interview sessions, he/she will be asked to perform mathematical activities without any help from the interviewer and will be asked to think out-loud in order to have a record of their algebraic thoughts.

What are the risks?

One possible risk that may apply to this research study is your child may feel uncomfortable solving mathematics problems while being recorded. Since your son/daughter is already comfortable discussing mathematics with me because I am currently his/her teacher, I think he/she will quickly become more comfortable as the interview continues. In addition, if your child is not confident in his/her mathematics ability, he/she may be uncomfortable explaining the methods and procedures he/she is using to solve the mathematics problems. I will be sure to continually remind your child that I am not looking for correct answers, but rather patterns in algebraic thinking.

What are the benefits?

The benefits of this study are pinpointing what is holding students back from being successful in algebra. The results of this study will be shared with colleagues to assess current methods of teaching mathematics and give guidance in making curriculum changes.

How will my privacy be protected?

The name of the school and students will not appear on any information that we share with others. None of these identifiers will be identified by name in any publications that result from this research. We will use pseudo -names to identify any written work or video that may involve the image of the student. The information gathered will be accessible only by the investigators and it will be in a locked cabinet. Any data saved electronically will be secured on a password-protected computer. All information not used in publications will be destroyed three years after the study ends

INSTRUCTIONS FOR COMPLETING FORM AND RETURNING TO RESEARCHERS.

With permission from you and your parents we would like to audiotape and videotape your interview session. The use of audiotapes and videotapes assist us in our analysis of the data. For example, we are able to use the audiotapes and videotapes to capture events exactly as they happened as well as listen and observe your algebraic thinking.

Please print your child's first and last name below, check the appropriate box and sign two copies of the parental consent forms provided. Then, please return one signed copy to your teacher. Please keep one copy for your reference or future use.

Print First and Last Name of Your Child

______I **do not** wish for my child to participate in the research study.

______I wish for my child to participate in the research study, but **do not** wish my child to be videotaped or audiotaped.

______I wish for my child to participate in the research study and I am willing to allow my child to be videotaped and audiotaped.

CONSENT: Parent or Guardian SignatureDate

ASSENT: Age 13 and Older Child's SignatureDate

Videotaping of sessions: It is possible that audio and video excerpts could be used in conference presentations, articles submitted to professional journals and teacher training. Audio and video excerpts of students at work will be used in professional capacities only if I have explicit parental and student consent from each participant in the group. I will keep these recordings in a password-protected file. The only people who will have access to these recordings will be the principal investigators, Dr. Patrick Sullivan and Dr. Gay Ragan, and the researcher, Andrea Martin.

May the researcher use your child's video or voice records for future research?
<u>Please check</u> two options:

________ I *do not* give permission for my child's image and recorded voice to be **archived** for **future research**, reports, and publications. The records will be destroyed by December 31, 2021.

________ I *do not* give permission for my child's image and recorded voice to be **archived** for **educational** and **training** purposes. The records will be destroyed by December 31, 2021.

________ I give permission for my child's image and recorded voice to be **archived** for use in **future research** reports and publications.

________ I give permission for my child's image and recorded voice to be **archived** for **educational** and **training** purposes

Parent SignatureDate

I, the undersigned, verify that the above informed consent procedures will be followed.

Person Obtaining Consent – ResearcherDate
Dr. Patrick Sullivan

Person Obtaining Consent – ResearcherDate
Andrea Martin

Appendix E: IRB Approval

Date: 5-5-2020

IRB #: IRB-FY2019-244
Title: Exploring the Nature of Similarities and Differences in Middle School Students' Algebraic Thinking
Creation Date: 10-6-2018
End Date: 1-13-2021
Status: Approved
Principal Investigator: Patrick Sullivan
Review Board: MSU
Sponsor:

Member	Gay Ragan	Role	Co-Principal Investigator	Contact	gayragan@missouristate.edu
Member	Andrea Martin	Role	Investigator	Contact	wic71@live.missouristate.edu

Study History

Submission Type	Initial	Review Type	Expedited	Decision	Approved
Submission Type	Renewal	Review Type	Expedited	Decision	Approved

Key Study Contacts

Member	Patrick Sullivan	Role	Principal Investigator	Contact	patricksullivan@missouristate.edu
Member	Patrick Sullivan	Role	Primary Contact	Contact	patricksullivan@missouristate.edu
Member	Gay Ragan	Role	Co-Principal Investigator	Contact	gayragan@missouristate.edu
Member	Andrea Martin	Role	Investigator	Contact	wic71@live.missouristate.edu

Appendix F: Analysis of the Generalization Tasks

Garden Task	Stage 1	Stage 1	Stage 2	Stage 3
	Discussing	Visualizing	Informal	Formal
Ben	Context	Context	Skipped Stage 3	Context 2x + 6
Chad	Recursive	Context	Verbal	X 2a + 6 (Prompt)
Susan	Context	Context	Verbal	X 2x + 2 x 3 (Chart)
George	Recursive	X Context (Tiles)	X Discussion	X L +2 x 2 + 2 (Previous Examples)
Lance	Recursive	X Context (Chart)	X Context (Chart)	X L + 2 + L + 2 + 2 (Chart)
Jasmine	Ratios	X Context (Tiles)	X Discussion	X 2L + 3 + 3 (Examples)

Well Task	Stage 1	Stage 1	Stage 2	Stage 3
	Discussing	Visualizing	Informal	Formal
Ben	Common Difference (10 hours)	Depth versus Time Chart	Did not want to use words to explain this pattern.	He tried!! T = (d/2)x3)) + ((d/2)x -1)
Chad	Common Difference	Drew a Well (Context)	X "Find an even number that is at most 3 feet under, at least one foot under the depth of the well. Then add one to your found number."	No
Susan	Common Difference	X Drew a Well (Context)	X "Seeing where you stopped previously and going from there."	No
George	Concrete Addition and Subtraction Calculations	Number patterns.	X "Every two feet, starting with the even foot, go back one hour. And if it's an odd number, go back two hours."	No
Lance	Common Difference	X Drew a Well (Context)	X "If it's odd, you go down two and if it's even, go down one."	No
Jasmine	Concrete Addition and Subtraction Calculations	Drew a Well and moved the Frog up and down. (Context)	X Did not recognize the pattern.	No

Number Sequence Task	Stage 1	Stage 1	Stage 2	Stage 3
	Discussing	Visualizing	Informal	Formal
Ben	→	X →	→	→ 2x – 1 = y “Writing the table helped me visualize the pattern.”
Chad	→ Recursion	X →	X (Compare the x and y) “x + 2 = y, y +2 = z, So “x + (z + (2times z))”	→ y = 2x - 1
Susan	→	X →	X →	X → x + x -1 = y
George	→ Recursion	X →	→ “x+ y = z + x = y”	None
Lance	→	X →	X → “The distance between x and y gradually gets bigger.”	X → “y= x(2) – 1”
Jasmine	→	X →	X →	X → “x + x – 1 = y”