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On Elliptic Curves

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ON ELLIPTIC CURVES

A Master's Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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Mathematics

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ABSTRACT

An elliptic curve over the rational numbers is given by the equation $y^2 = x^3 + Ax + B$. In our thesis, we study elliptic curves. It is known that the set of rational points on the elliptic curve form a finitely generated abelian group induced by the secant-tangent addition law. We present an elementary proof of associativity using Maple. We also present a relatively concise proof of the Mordell-Weil Theorem.

KEYWORDS: elliptic curves, plane curves, associativity, groups, Mordell, Weil, Fermat, Andrew Wiles, Maple

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I never could have imagined my graduate studies would coincide with an unprecedented global pandemic. Yet through it all, each of my professors rose to the challenge and adapted to the never-ending changes. For that, I want to thank each of them. It is through their commitment to my success that the graduate school experience was still engaging, challenging, and fulfilling. I especially want to thank my advisor, Dr. Shah, for all his helpful input, insight, and guidance on this endeavor. Also, thank you to Dr. Kemp and Dr. Stanojevic for serving on the thesis committee.

I want to thank my parents – Mike and Gretchen Miller. Though they may not understand the mathematics in the pages that follow, it is only by their love, support, and prayers that I am able to. Even on the hardest days, they have always reminded me: *This is the day the Lord has made. We will rejoice and be glad in it. Psalm 118:24.*

I dedicate this thesis to my parents.

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CHAPTER 1: ELLIPTIC CURVES

The theory of elliptic curves is a masterful culmination of algebra and geometry. Not only are elliptic curves intriguing in the abstract sense, but they also have profound applications in other other areas of mathematics such as number theory and cryptography. Consequently, the study of elliptic curves has become a subject in its own right. As a result, we make no effort or attempt to cover every facet of the topic. Instead, we aim to highlight some important theorems and applications of these fascinating mathematical objects.

The nomenclature here is somewhat unfortunate as elliptic curves are not really ellipses. Instead, they arise from elliptic integrals which were traditionally used to describe the circumference of an ellipse [4]. To illustrate, suppose $a > b > 0$ and consider the ellipse given by $x^2/a^2 + y^2/b^2 = 1$ (graphed in Figure 1). As there is no known algebraic formula

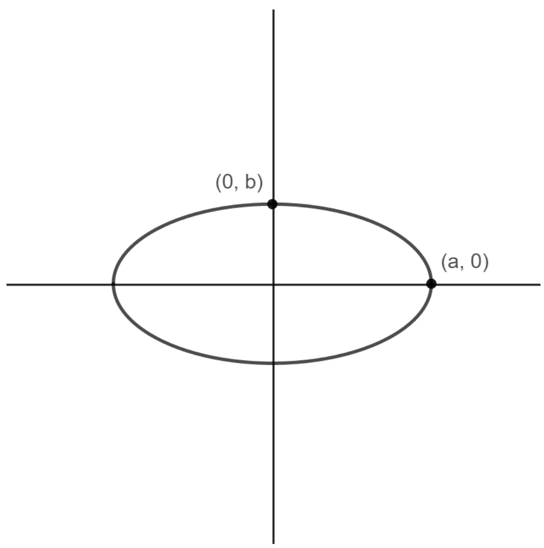

Figure 1: Graph of $x^2/a^2 + y^2/b^2 = 1$.

to compute the circumference of the ellipse, we must rely on the tools of calculus to set up an arc length integral. Since the ellipse is symmetric about both axes, the circumference of the ellipse is four times the length of the arc in the first quadrant. Solving for y in the

upper half of the ellipse, we obtain

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \text{ with } -a \leq x \leq a.$$

After some algebraic manipulations, we find that the circumference is given by

$$4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^a \frac{\sqrt{a^2 - k^2x^2}}{\sqrt{a^2 - x^2}}\,dx$$

where $k = \sqrt{1 - b^2/a^2}$ is the eccentricity of the ellipse. This integral can be further simplified by making the substitution $x = at$ which yields

$$4a\int_0^1 \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}}\,dt = 4a\int_0^1 \frac{1 - k^2t^2}{\sqrt{(1 - t^2)(1 - k^2t^2)}}\,dt.$$

The complexity of this integral lies in the denominator, which is the square root of a quartic polynomial in t. If we set $v = \sqrt{(1 - t^2)(1 - k^2t^2)}$, then $v^2 = (1 - t^2)(1 - k^2t^2)$. After further algebraic manipulations, it can be shown that

$$\left(\frac{v}{(t-1)^2}\right)^2 = k^2\left(1 + \frac{2}{t-1}\right)\left(1 + \frac{1 - 1/k}{t-1}\right)\left(1 + \frac{1 + 1/k}{t-1}\right).$$

Labeling $\alpha = 2$, $\beta = 1 - 1/k$, $\gamma = 1 + 1/k$, $\overline{y} = v/(t-1)^2$ and $\overline{x} = 1/(t-1)$, the equation above becomes

$$\begin{aligned}
\overline{y}^2 &= k^2(1 + \alpha\overline{x})(1 + \beta\overline{x})(1 + \gamma\overline{x}) \\
&= k^2\alpha\beta\gamma\overline{x}^3 + k^2(\alpha\beta + \alpha\gamma + \beta\gamma)\overline{x}^2 + k^2(\alpha + \beta + \gamma)\overline{x} + k^2
\end{aligned}$$

which has the form $\overline{y}^2 = A\overline{x}^3 + B\overline{x}^2 + C\overline{x} + D$ for constants A, B, C, and D, and $A \neq 0$. This equation defines a curve in the plane. Due to its connection to the elliptic integral, such curves are called elliptic curves (see [4]). In the sections that follow, we will formally introduce elliptic curves and discuss some of their important properties.

Section 1.1: Preliminaries

We assume elementary projective geometry [5].

Definition 1.1. We say that a projective curve $C : F(X, Y, Z) = 0$ is *singular* provided there is a point $[X_0, Y_0, Z_0] \in C$ such that

$$\frac{\partial F}{\partial X}(X_0, Y_0, Z_0) = \frac{\partial F}{\partial Y}(X_0, Y_0, Z_0) = \frac{\partial F}{\partial Z}(X_0, Y_0, Z_0) = 0.$$

Such points are called *singular points.* If no such points exists on C, then C is said to be *non-singular.*

Definition 1.2 ([7])**.** An *elliptic curve* E defined over a field K is a non-singular projective curve given by

$$E : Y^2Z + aXYZ + bYZ^2 = X^3 + cX^2Z + dXZ^2 + eZ^3 \text{ (projective curve } E)$$

where $a, b, c, d, e \in K$.

Remark. Observe that $[0, 1, 0] \in E$.

While E is a projective curve, we will consider E in its affine form. Since the projective points $[x, y, 1]$ are in one-to-one correspondence with the points $(x, y) \in \mathbb{R}^2$, the affine form of E is

$$E : y^2 + axy + by = x^3 + cx^2 + dx + e \text{ (affine curve } E).$$

This process comes at the expense of omitting the projective point $[0, 1, 0]$ from E. We define $[0, 1, 0]$ to be the *point at infinity*, denoted $\mathcal{O}$, and remain cognizant of the fact that, while $\mathcal{O}$ does not have a visual representation in the affine plane, it remains a valid point on the curve E. While we cannot visualize $\mathcal{O}$ in $\mathbb{R}^2$, we can interpret what it means to connect a point $(a, b) \in \mathbb{R}^2$ with $\mathcal{O}$. Consider the vertical line $x = a$ passing through

the point (a, b). Then $\mathcal{O}$ is a solution of the homogenized form of $x = a$, namely $X = aZ$. Therefore, we "connect" (a, b) with $\mathcal{O}$ by simply drawing the vertical line $x = a$.

The affine curve E can be simplified even further provided that $\text{char}(K) \neq 2$. In particular, we can complete the square by adding $[(ax + b)/2]^2$ to both sides of E. Upon simplification, we obtain

$$\left(y + \frac{ax+b}{2}\right)^2 = x^3 + \left(\frac{a^2}{4} + c\right)x^2 + \left(\frac{ab}{2} + d\right)x + \frac{b^2}{4} + e.$$

Labeling $\hat{y} = y + (ax + b)/2$, $f = a^2/4 + c$, $g = ab/2 + d$, and $h = b^2/4 + e$, we have

$$E : \hat{y}^2 = x^3 + fx^2 + gx + h.$$

Furthermore, if $\text{char}(K) \neq 3$, we let $x = \hat{x} - f/3$. Upon simplification, we find

$$\hat{y}^2 = \hat{x}^3 + \left(g - \frac{f^2}{3}\right)\hat{x} + \frac{2f^3}{27} - \frac{fg}{3} + h.$$

Relabeling once more, we see that E takes on the form

$$E : \hat{y}^2 = \hat{x}^3 + A\hat{x} + B.$$

It follows that every affine elliptic curve E defined over a field K with $\text{char}(K) \neq 2, 3$ can be written in this form, called the *Weierstrass form*. For the remainder of our discussion, we will assume our curves are in the Weierstrass form. We still require $E : y^2 = x^3 + Ax + B$ to be non-singular. That is, for all $(x_0, y_0) \in E$

$$\frac{\partial f}{\partial x}(x_0, y_0) \neq 0 \text{ and } \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$$

where $f(x, y) = y^2 - x^3 - Ax - B$. This is equivalent to the condition that $4A^3 + 27B^2 \neq 0$. To see this, we begin with a lemma.

Lemma 1.3. *Suppose $4A^3 + 27B^2 = 0$ with $A, B \neq 0$ and let $C : y^2 = x^3 + Ax + B$. Then the following hold:*

1. *If $B > 0$, then $(\sqrt{-A/3}, 0) \in C$.*
2. *If $B < 0$, then $(-\sqrt{-A/3}, 0) \in C$.*

Proof. The assumption that $4A^3 + 27B^2 = 0$ together with the fact that $A, B \neq 0$ implies that $A < 0$. Now suppose that $B > 0$. Solving for B in the equation $4A^3 + 27B^2 = 0$ yields

$$B = \frac{2|A|}{3}\sqrt{\frac{-A}{3}} \implies B = -\frac{2A}{3}\sqrt{\frac{-A}{3}}.$$

Then

$$\left(\sqrt{\frac{-A}{3}}\right)^3 + A\left(\sqrt{\frac{-A}{3}}\right) + B = \frac{2A}{3}\sqrt{\frac{-A}{3}} + B = \frac{2A}{3}\sqrt{\frac{-A}{3}} - \frac{2A}{3}\sqrt{\frac{-A}{3}} = 0.$$

Hence $(\sqrt{-A/3}, 0) \in C$, so (1) holds. On the other hand, if $B < 0$, we find that

$$B = -\frac{2|A|}{3}\sqrt{\frac{-A}{3}} \implies B = \frac{2A}{3}\sqrt{\frac{-A}{3}}.$$

Then

$$\left(-\sqrt{\frac{-A}{3}}\right)^3 + A\left(-\sqrt{\frac{-A}{3}}\right) + B = -\frac{2A}{3}\sqrt{\frac{-A}{3}} + B = -\frac{2A}{3}\sqrt{\frac{-A}{3}} + \frac{2A}{3}\sqrt{\frac{-A}{3}} = 0.$$

Hence $(-\sqrt{-A/3}, 0) \in C$, so (2) holds. □

Proposition 1.4. *The curve $E : y^2 = x^3 + Ax + B$ is non-singular if and only if $4A^3 + 27B^2 \neq 0$.*

Proof. $\Longleftarrow$ Assume that $4A^3 + 27B^2 \neq 0$ and express E as $E : f(x, y) = y^2 - x^3 - Ax - B = 0$. For the sake of contradiction, suppose that E is singular. Then there exists $(x_0, y_0) \in E$

such that

$$\frac{\partial f}{\partial x}(x_0, y_0) = -3x_0^2 - A = 0 \text{ and } \frac{\partial f}{\partial y}(x_0, y_0) = 2y_0 = 0.$$

This implies that $A = -3x_0^2$ and $y_0 = 0$. Since $(x_0, y_0) \in E$, we have $0 = x_0^3 + (-3x_0^2)x_0 + B$, and hence $B = 2x_0^3$. Then

$$4A^3 + 27B^2 = 4(-3x_0^2)^3 + 27(2x_0^3)^2 = -108x_0^6 + 108x_0^6 = 0$$

which contradictions the assumption that $4A^3 + 27B^2 \neq 0$. Therefore E is non-singular.

$\Longrightarrow$ We proceed by considering the contrapositive. That is, we assume that $4A^3 + 27B^2 = 0$ and wish to show that E is singular. If $A = B = 0$, the curve E is given by $E : f(x, y) = y^2 - x^3 = 0$. In this case, E is singular since $\partial f/\partial x$ and $\partial f/\partial y$ vanish at $(0, 0)$. Now suppose $A, B \neq 0$. It follows from Lemma 1.3 that either $(\sqrt{-A/3}, 0)$ or $(-\sqrt{-A/3}, 0)$ is a point on E. In either case, $\partial f/\partial x$ and $\partial f/\partial y$ vanish at these points. Hence E is singular. □

There are three types of singular points: cusps, nodes, and isolated points. To illustrate each, we will use the fact that a curve is singular whenever it contains a repeated root. Consider the curve

$$C : y^2 = x^3 + Ax + B = (x - a)^2(x - b) = x^3 - (b + 2a)x^2 + (a^2 + 2ab)x - a^2b.$$

Equating coefficients, we see that $b + 2a = 0$, so $b = -2a$. Then $A = a^2 + 2ab = -3a^2$ and $B = -a^2b = 2a^3$. So we have

$$C : y^2 = x^3 + Ax + B = x^3 - 3a^2x + 2a^3.$$

When $a = 0$, we obtain the curve $C : y^2 = x^3$, which has triple root at $(0, 0)$. This results

in a cusp at $(0,0)$ as seen in Figure 2.

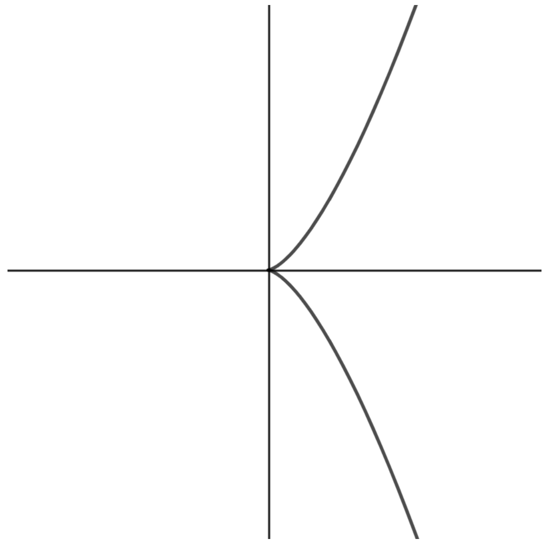

Figure 2: Cusp singularity

If $a > 0$ the resulting curve has a node at $(a, 0)$ as seen in Figure 3.

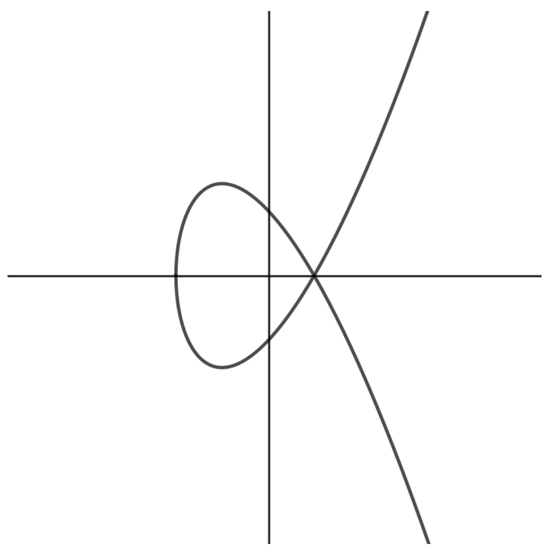

Figure 3: Node singularity

Lastly, for $a < 0$ the resulting curve has an isolated point at $(a, 0)$ as seen in Figure 4.

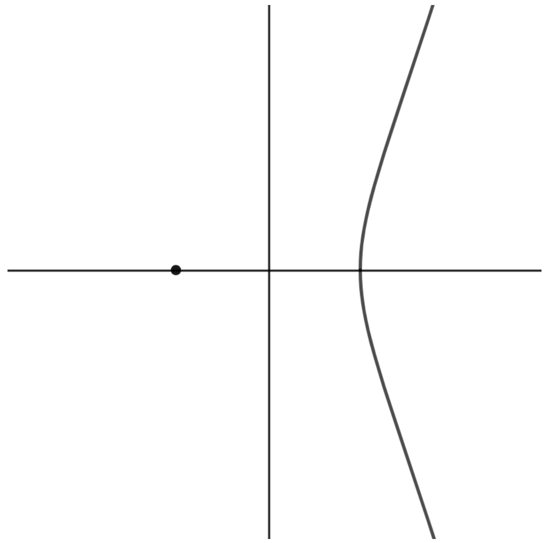

Figure 4: Isolated point singularity

Section 1.2: The Group Law

Suppose $E : y^2 = x^3 + Ax + B$ is an elliptic curve defined over $\mathbb{Q}$. What makes the study of elliptic curves particularly interesting is the fact that we can define an operation that makes the set of rational points on the curve E into a group. To do this, we use the secant-tangent process which we describe below.

Suppose P and Q are rational points on an elliptic curve E, say as in Figure 5.

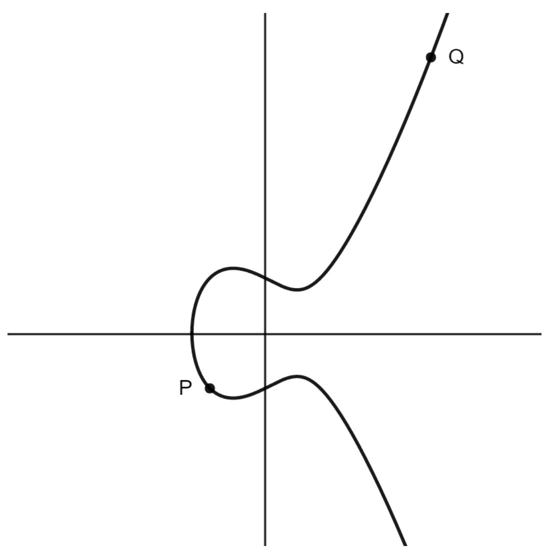

Figure 5: P and Q on the curve E

For any point $T = (x, y)$ in the coordinate plane, we define $-T$ to be the reflection of T across the x-axis. That is, $-T = (x, -y)$. Now let R denote the third point of intersection of E and $\overline{PQ}$. Define $P + Q = -R$, the reflection of R across the x-axis (see Figure 6 below).

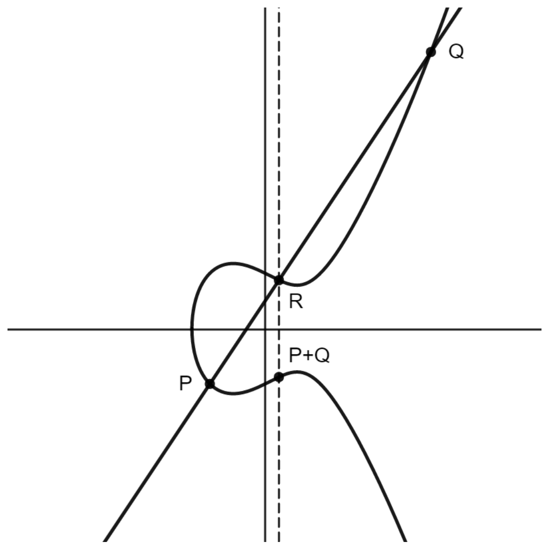

Figure 6: The sum $P + Q$

It could happen that $P = Q$. In this case, we follow a similar procedure using the tangent line as illustrated in Figure 7.

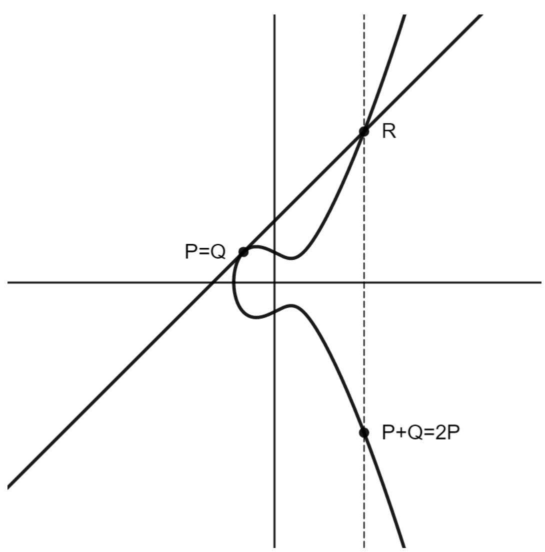

Figure 7: The case $P = Q$

There are a few other cases we will address shortly. For now, the next natural step is to derive an explicit formula for this process. Supposing $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, as in the examples above, the line $\overline{PQ}$ can be written as

$$y = \lambda x - \lambda x_1 + y_1$$

where λ denotes the slope of $\overline{PQ}$ (the tangent slope if $P = Q$). Substituting this equation into $E : y^2 = x^3 + Ax + B$ leads to the equation

$$x^3 - \lambda^2 x^2 + (2x_1\lambda^2 - 2y_1\lambda + A)x - x_1^3 - x_1^2\lambda^2 + 2x_1y_1\lambda - Ax_1 = 0.$$

Using the fact that $(x - x_1)$ and $(x - x_2)$ are factors of the left-hand side, we find that the above equation factors as

$$(x - x_1)(x - x_2)(x - \lambda^2 + x_1 + x_2) = 0.$$

Hence the x-coordinate of the third point of intersection is $\lambda^2 - x_1 - x_2$. Now substituting this result into the equation of the line $\overline{PQ}$, we find that the y-coordinate is given by $\lambda^3 - 2\lambda x_1 - \lambda x_2 + y_1$. It follows that

$$(x_1, y_1) + (x_2, y_2) = (\lambda^2 - x_1 - x_2, -\lambda^3 + 2\lambda x_1 + \lambda x_2 - y_1).$$

As we mentioned earlier, there are a few additional cases that must be addressed. First, if P and Q are distinct and the line connecting them is vertical, the formula above does not apply. However, recall that every vertical line intersects $\mathcal{O}$, the point at infinity. Since the reflection of the projective point $\mathcal{O}$ is again $\mathcal{O}$, we simply define $P + Q = \mathcal{O}$. Likewise, if $P = Q$ and the tangent line is vertical, we define $P + Q = \mathcal{O}$. In the same way, if $P = Q = \mathcal{O}$, we define $P + Q = \mathcal{O}$. Definition 1.5 below formalizes the secant-tangent

process using the formula we have derived.

Definition 1.5. Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve defined over $\mathbb{Q}$ and let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be two points on E.

1. If $x_1 = x_2$ and $y_1 = -y_2$, define $P + Q = \mathcal{O}$.

2. Otherwise, let

$$\lambda = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} & \text{if } x_1 \neq x_2 \\ \\ \dfrac{3x_1^2 + A}{2y_1} & \text{if } x_1 = x_2 \text{ and } y_1 \neq -y_2. \end{cases}$$

   Define

$$(x_1, y_1) + (x_2, y_2) = (\lambda^2 - x_1 - x_2, -\lambda^3 + 2\lambda x_1 + \lambda x_2 - y_1).$$

Proposition 1.6. *Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve defined over $\mathbb{Q}$ and let*

$$E(\mathbb{Q}) := \{(a, b) \in E : a, b \in \mathbb{Q}\} \cup \{\mathcal{O}\}$$

where $\mathcal{O}$ denotes the point at infinity. Then $E(\mathbb{Q})$ is an additive abelian group under the secant-tangent process.

Remark. Recall, we consider the point (a, b) synonymous with the point $[a, b, 1]$ in the projective plane.

Proof. Apart from associativity, the proof is fairly straightforward. From Definition 1.5, it is clear that $E(\mathbb{Q})$ is closed under addition of points. It is also easy to see that the secant-tangent process is commutative, since the line $\overline{PQ} = \overline{QP}$ intersect E at the same point, say R. Then $P + Q = -R = Q + P$. Moreover, $\mathcal{O}$ acts as the identity element, and for any point $P \in E(\mathbb{Q})$, its inverse is simply $-P$. Much less obvious is associativity. We aim to present an elementary proof of associativity.

As there are many cases to check, we will be content with demonstrating a proof that utilizes Maple [6] for the computations. Assume that $P, Q, R \neq \mathcal{O}$ are distinct points, such that $P + Q \neq \mathcal{O}$, $Q + R \neq \mathcal{O}$, $P + Q \neq \pm R$ and $Q + R \neq \pm P$. Suppose $P = (x_1, y_1), Q = (x_2, y_2)$, and $R = (x_3, y_3)$. We wish to show that

$$(P + Q) + R = P + (Q + R).$$

Let $(P + Q) + R = (X_1, Y_1)$ and $P + (Q + R) = (X_2, Y_2)$. Then associativity is equivalent to showing that $X_1 - X_2 = 0$ and $Y_1 - Y_2 = 0$.

Define

$$\lambda := \frac{y_2 - y_1}{x_2 - x_1}.$$

It follows that Definition 1.5 that

$$P + Q = (x_1, y_1) + (x_2, y_2) = (\lambda^2 - x_1 - x_2, -\lambda^3 + 2\lambda x_1 + \lambda x_2 - y_1).$$

Label $\alpha := \lambda^2 - x_1 - x_2$ and $\beta := -\lambda^3 + 2\lambda x_1 + \lambda x_2 - y_1$. Now define

$$\mu := \frac{y_3 - \beta}{x_3 - \alpha}.$$

Since we require that $P + Q \neq \pm R$, this ensures that $x_3 \neq \alpha$ and that μ properly represents the slope between $P + Q$ and R. Following from Definition 1.5, we have

$$(P + Q) + R = (\alpha, \beta) + (x_3, y_3) = (\mu^2 - \alpha - x_3, -\mu^3 + 2\mu\alpha + \mu x_3 - \beta).$$

So $X_1 = \mu^2 - \alpha - x_3$ and $Y_1 = -\mu^3 + 2\mu\alpha + \mu x_3 - \beta$.

Now define

$$\epsilon = \frac{y_3 - y_2}{x_3 - x_2}.$$

Then

$$Q + R = (x_2, y_2) + (x_3, y_3) = (\epsilon^2 - x_2 - x_3, -\epsilon^3 + 2\epsilon x_2 + \epsilon x_3 - y_2).$$

Label $\phi := \epsilon^2 - x_2 - x_3$ and $\psi := -\epsilon^3 + 2\epsilon x_2 + \epsilon x_3 - y_2$. Now define

$$\delta := \frac{\psi - y_1}{\phi - x_1}.$$

As before, since $Q + R \neq \pm P$, δ is well-defined. Then

$$P + (Q + R) = (x_1, y_1) + (\phi, \psi) = (\delta^2 - x_1 - \phi, -\delta^3 + 2\delta x_1 + \delta\phi - y_1).$$

So $X_2 = \delta^2 - x_1 - \phi$ and $Y_2 = -\delta^3 + 2\delta x_1 + \delta\phi - y_1$. The numerators of $X_1 - X_2$ and $Y_1 - Y_2$ (see appendix for Maple computation) share a common factor of

$$J = x_1^3 x_2 - x_1^3 x_3 - x_1 x_2^3 + x_1 x_3^3 + x_2^3 x_3 - x_2 x_3^3 + x_1 y_2^2 - x_1 y_3^2 - x_2 y_1^2 + x_2 y_3^2 + x_3 y_1^2 - x_3 y_2^2.$$

Now since $P, Q, R \in E$, we have

$$\begin{aligned} y_1^2 &= x_1^3 + Ax_1 + B \\ y_2^2 &= x_2^3 + Ax_2 + B \\ y_3^2 &= x_3^3 + Ax_3 + B. \end{aligned}$$

Substituting y_1^2, y_2^2, y_3^2 into J, we find that $J = 0$ and hence $X_1 - X_2 = 0$ and $Y_1 - Y_2 = 0$. Therefore associativity holds in this case. Similar computations in Maple can verify that associativity holds in the other cases, while some cases are even more straightforward. $\square$

Remark. The inspiration for our associativity proof is found in [3]. Whereas the proof in [3] uses Mathematica, we have used Maple. We have also shown a more direct argument that $X_1 - X_2 = Y_1 - Y_2 = 0$.

Section 1.3: The Points of $E(\mathbb{Q})$

Consider the elliptic curve $E : y^2 = x^3 - x + 1$ which contains the rational point $(3, 5)$. Since the set of rational points is closed, we can obtain other rational points on E by computing

$$n(3,5) = \underbrace{(3,5) + (3,5) + \cdots + (3,5)}_{n-\text{times}}.$$

Using the point $(3, 5)$, we find the following points are also in $E(\mathbb{Q})$:

$$2(3,5) = \left(\frac{19}{25}, \frac{103}{125}\right), 3(3,5) = \left(-\frac{223}{784}, \frac{24655}{21952}\right), 4(3,5) = \left(-\frac{350701}{265225}, \frac{13919407}{136590875}\right).$$

However, observe that

$$\begin{aligned}
\left(\frac{19}{25}, \frac{103}{125}\right) &= \left(\frac{19}{5^2}, \frac{103}{5^3}\right) \\
\left(-\frac{223}{784}, \frac{24655}{21952}\right) &= \left(-\frac{223}{28^2}, \frac{24655}{28^3}\right) \\
\left(-\frac{350701}{265225}, \frac{13919407}{136590875}\right) &= \left(-\frac{350701}{515^2}, \frac{13919407}{515^3}\right)
\end{aligned}$$

This gives rise to the following proposition about the points of $E(\mathbb{Q})$.

Proposition 1.7. *Every element of $E(\mathbb{Q})$ has the form has the form $(m/t^2, n/t^3)$ where $t > 0$ and $\gcd(m,t) = \gcd(n,t) = 1$.*

Proof. Suppose that $(x, y) \in E(\mathbb{Q})$. Then we can write $x = m/M$ and $y = n/N$ where $m, n > 0$ and $\gcd(m, M) = \gcd(n, N) = 1$. So

$$\frac{n^2}{N^2} = \frac{m^3}{M^3} + \frac{Am}{M} + B.$$

Multiplying the above equation by N^2M^3, we obtain

$$\begin{aligned} n^2M^3 &= m^3N^2 + AmN^2M^2 + BN^2M^3 \\ &= N^2(m^3 + AmM^2 + BM^3) \end{aligned}$$

and hence $N^2 \mid n^2M^3$. But since $\gcd(n, N) = 1$, it follows that $N^2 \mid M^3$. On the other hand, the equation above can be rearranged as

$$\begin{aligned} m^3N^2 &= n^2M^3 - AmN^2M^2 - BN^2M^3 \\ &= M^2(n^2M - AmN^2 - BN^2M). \end{aligned}$$

So $M^2 \mid m^3N^2$. But since $\gcd(m, M) = 1$, it follows that $M^2 \mid N^2$, and hence $M \mid N$. That is, $N = Mk$ for some $k \in \mathbb{Z}$. Making the appropriate substitution in the previous equation, we find

$$\begin{aligned} m^3N^2 &= n^2M^3 - Amk^2M^4 - BN^2M^3 \\ &= M^3(n^3 - Amk^2M - BN^2). \end{aligned}$$

Thus $M^3 \mid m^3N^2$, and so $M^3 \mid N^2$. Since $M^3 \mid N^2$ and $N^2 \mid M^3$, it follows that $N^2 = M^3$. Let $t = N/M$. Then

$$\begin{aligned} t^2 &= N^2/M^2 = M^3/M^2 = M \\ t^3 &= N^3/M^3 = N^3/N^2 = N \end{aligned}$$

So $(x, y) = (m/t^2, n/t^3)$. $\square$

Remark. In the proceeding proof, we have closely followed the proof found in [8]. However, we have filled in additional details.

Proposition 1.8. *Let*

$$E(\mathbb{Q})_T := \{P \in E(\mathbb{Q}) : nP = \mathcal{O} \text{ for some } n \in \mathbb{N}\}$$

be the set of rational points of finite order. Then $E(\mathbb{Q})_T$ is a subgroup of $E(\mathbb{Q})$.

Remark. Points of finite order are called *torsion points*, hence the subscript T in $E(\mathbb{Q})_T$.

Proof. Clearly $\mathcal{O}$ has finite order, so $\mathcal{O} \in E(\mathbb{Q})_T$ and hence $E(\mathbb{Q})_T$ is non-empty. Let $P, Q \in E(\mathbb{Q})_T$. Then there exists $m, n \in \mathbb{N}$ such that $nP = mQ = \mathcal{O}$. Then

$$\begin{aligned} mn(P+Q) &= \underbrace{(P+Q)+\cdots+(P+Q)}_{mn-\text{times}} = \underbrace{P+\cdots+P}_{mn-\text{times}} + \underbrace{Q+\cdots+Q}_{mn-\text{times}} \\ &= m(nP) + n(mQ) = \mathcal{O}. \end{aligned}$$

So $P+Q$ has finite order and hence $E(\mathbb{Q})_T$ is closed under addition.

We also have

$$n(-P) = nP + n(-P) = n(P + (-P)) = \mathcal{O}.$$

Hence $-P$ has finite order, so $E(\mathbb{Q})_T$ is closed under inverses. Therefore $E(\mathbb{Q})_T$ is a subgroup of $E(\mathbb{Q})$. $\square$

Section 1.4: Mordell-Weil Theorem

The Mordell-Weil theorem is a fundamental result in the theory of elliptic curves. In the general case, the theorem states that for any elliptic curve E defined over a field K, the group $E(K)$ is finitely generated. The proof is a descent argument and it utilizes the following fact:

Theorem 1.9 (Weak Mordell-Weil Theorem)**.** *Given an elliptic curve E defined over a field field K, the group $E(K)/mE(K)$ is finite for all $m \geq 2$.*

Remark. Here we define $mE(K) = \{mP : P \in E(Q)\}$, which is a subgroup of $E(\mathbb{Q})$. The proof of this theorem can be found in [7].

The Mordell-Weil theorem was first conjectured by Henri Poincare in 1901 and it was proved by Louis Mordell in 1922 for elliptic curves defined over $\mathbb{Q}$ [10]. A few years later, André Weil extended Mordell's result. In the context of curves defined over $\mathbb{Q}$, the Mordell-Weil theorem states that there is a finite set of points $\{P_1, P_2, ..., P_m\}$ such that every point $P \in E(\mathbb{Q})$ can be written as

$$P = \sum_{i=1}^{m} a_i P_i$$

for some $a_1, ..., a_m \in \mathbb{Z}$. In this section, we aim to prove this fact under the assumption that $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite – a special case of Theorem 1.9. The proof that $E(\mathbb{Q})/2E(\mathbb{Q})$ is finite is rather tedious, and can be found in [8].

We argue that it is enough to only consider elliptic curves of the form $y^2 = x^3 + Ax + B$ where $A, B \in \mathbb{Z}$. We have already seen that elliptic curves defined over the field $\mathbb{Q}$ can be written in Weierstrass form $E : y^2 = x^3 + Ax + B$ where $A, B \in \mathbb{Q}$. However, if we let d denote the least common multiple of the denominators of A and B and define $X = d^2x$ and $Y = d^3y$, we see that $x = X/d^2$ and $y = Y/d^3$. Thus

$$\frac{Y^2}{d^6} = \frac{X^3}{d^6} + \frac{AX}{d^2} + B.$$

Multiplying through by d^6, we obtain

$$Y^2 = X^3 + d^4AX + d^6B$$

which is of the form $Y^2 = X^3 + aX + b$ where $a, b \in \mathbb{Z}$. This means that for any curve $E : y^2 = x^3 + Ax + B$ with $A, B \in \mathbb{Q}$, we can map $(x, y) \mapsto (X = d^2x, Y = d^3y)$ and analyze the group $C(\mathbb{Q})$ of the elliptic curve $C : Y^2 = X^3 + aX + b$ (see [8]).

Definition 1.10. Given a rational number $r = m/n$ with $\gcd(m, n) = 1$, define the *height* of r as

$$H(r) = \max\{|m|, |n|\}.$$

Define the *small height* of r as

$$h(r) = \log H(r).$$

Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve with $A, B \in \mathbb{Z}$. Given a point $P = (x, y) \in E(\mathbb{Q})$, define the height H and small height h of P as follows:

$$H(P) = H(x) \quad \text{and} \quad h(P) = h(x).$$

With this definition in mind, we are ready to lay the groundwork of the Mordell-Weil Theorem. The proof will rely on three lemmas (see [8]) which we present now.

Lemma 1.11. *For all $C \in \mathbb{R}$, the set*

$$E(\mathbb{Q})_{h \leq C} := \{P \in E(\mathbb{Q}) : h(P) \leq C\}$$

is finite.

Proof. Let $C \in \mathbb{R}$ and suppose $P = (x, y) \in E(\mathbb{Q})_{h \leq C}$. Since $x \in \mathbb{Q}$, we can write $x = m/n$ where $\gcd(m, n) = 1$. So

$$h(P) = \log H(x) = \log H\left(\frac{m}{n}\right) = \log(\max\{|m|, |n|\}).$$

Since $P \in E(\mathbb{Q})_{h \leq C}$, it is required that $\log |m| \leq C$ and $\log |n| \leq C$. This implies that $|m| \leq \exp C$ and $|n| \leq \exp C$. Since $|m|$ and $|n|$ are bounded, it follows that there are only finitely many choices $m, n \in \mathbb{Z}$ for x. Moreover, for each possible x, there are only two possible values for y. Therefore the set $E(\mathbb{Q})_{h \leq C}$ is finite. $\square$

Lemma 1.12. *Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve with $A, B \in \mathbb{Z}$. Suppose $P_0 = (x_0, y_0) \in E(\mathbb{Q})$. Then for all $P = (x, y) \in E(\mathbb{Q})$, there is a constant k_0 depending solely on x_0, y_0, A, and B such that*

$$h(P + P_0) \leq 2h(P) + k_0.$$

Proof. It follows from Definition 1.5 and Definition 1.10 that

$$H(P_0 + P) = H\left(\frac{(y - y_0)^2}{(x - x_0)^2} - x - x_0\right).$$

Expanding this expression and using the fact that $y^2 - x^3 = Ax + B$, we have

$$\begin{aligned} H\left(\frac{(y - y_0)^2}{(x - x_0)^2} - x - x_0\right) &= H\left(\frac{(y - y_0)^2 - (x - x_0)^2(x + x_0)}{(x - x_0)^2}\right) \\ &= H\left(\frac{y^2 - x^3 + x_0x^2 + x_0^2x - 2y_0y - x_0^3 + y_0^2}{x^2 - 2x_0 - x_0^2}\right) \\ &= H\left(\frac{x_0x^2 + (A + x_0^2)x - 2y_0y + (B - x_0^3 + y_0^3)}{x^2 - 2x_0x - x_0^2}\right) \end{aligned}$$

Observe that the coefficients on x and y, along with the constant terms, depend solely on x_0, y_0, A, and B. We can force the coefficients and constants to be integers by multiplying by the least common denominator among all of the terms. Thus, after a suitable relabeling, we can write

$$H\left(\frac{x_0x^2 + (A + x_0^2)x - 2y_0y + (B - x_0^3 + y_0^3)}{x^2 - 2x_0x - x_0^2}\right) = H\left(\frac{ax^2 + bx + cy + d}{ex^2 + fx + g}\right)$$

where $a, b, c, d, e, f, g \in \mathbb{Z}$ depend solely on x_0, y_0, A and B. By Proposition 1.7, we can write $x = m/t^2$ and $y = n/t^3$ for some $n, m, t \in \mathbb{Z}$. So we have

$$H\left(\frac{ax^2 + bx + cy + d}{ex^2 + fx + g}\right) = H\left(\frac{am^2 + bmt^2 + cnt + dt^4}{em^2 + fmt^2 + gt^4}\right).$$

Since $\gcd(am^2 + bmt^2 + cnt + dt^4, em^2 + fmt^2 + gt^4)$ need not be 1, it follows that

$$H\left(\frac{am^2 + bmt^2 + cnt + dt^4}{em^2 + fmt^2 + gt^4}\right) \leq \max\{|am^2 + bmt^2 + cnt + dt^4|, |em^2 + fmt^2 + gt^4|\}$$
$$\leq \max\{|am^2| + |bmt^2| + |cnt| + |dt^4|, |em^2| + |fmt^2| + |gt^4|\}.$$

Now since $H(P) = \max\{|m|, t^2\}$, we have $|m| \leq H(P)$ and $t \leq H(P)^{1/2}$. Moreover, since $(m/t^2, n/t^3)$ satisfies E, we have

$$\frac{n^2}{t^6} = \frac{m^3}{t^6} + \frac{Am}{t^2} + B \implies n^2 = m^3 + Amt^4 + Bt^6.$$

In particular, we find that

$$\begin{aligned}|n^2| &\leq |m^3| + |Amt^4| + |Bt^6| \\ &\leq H(P)^3 + |A|H(P)^3 + |B|H(P)^3 \\ &\leq H(P)^3(1 + |A| + |B|)\end{aligned}$$

Thus $|n| \leq kH(P)^{3/2}$ where $k = \sqrt{1 + |A| + |B|}$. Now on one hand,

$$\begin{aligned}|am^2| + |bmt^2| + |cnt| + |dt^4| &\leq |a|H(P)^2 + |b|H(P)^2 + |ck|H(P)^2 + |d|H(P)^2 \\ &= H(P)^2(|a| + |b| + |ck| + |d|).\end{aligned}$$

On the other hand,

$$|em^2| + |fmt^2| + |gt^4| \leq |e|H(P)^2 + |f|H(P)^2 + |g|H(P)^2 = H(P)^2(|e| + |f| + |g|).$$

So it follows that

$$H(P_0 + P) = H\left(\frac{ax^2 + bx + cy + d}{ex^2 + fx + g}\right) \leq \max\{|a| + |b| + |ck| + |d|, |e| + |f| + |g|\}H(P)^2.$$

Let $k_0 := \log \max\{|a| + |b| + |ck| + |d|, |e| + |f| + |g|\}$. By taking logarithms, we obtain

$$h(P + P_0) \leq 2h(p) + k_0.$$

$\square$

Remark. This proof is similar to the proof found in [8], however we have added clarifying details as well as streamlined the arguments.

Lemma 1.13. *Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve with $A, B \in \mathbb{Z}$. Then there is a constant k depending solely on A and B such that for all $P \in E(\mathbb{Q})$*

$$h(2P) \geq 4h(P) - k.$$

Proof. Suppose $P = (x, y) \in E(\mathbb{Q})$. Using Definition 1.5 to compute $P + P = 2P$, it follows that

$$H(2P) = H\left(\frac{x^4 - 2Ax^2 - 8Bx + A^2}{4x^3 + 4Ax + 4B}\right)$$

Since E defines an elliptic curve, we also have that $\Delta = 4A^3 + 27B^2 \neq 0$. Viewing $x^4 - 2Ax^2 - 8Bx + A^2$ and $4x^3 + 4Ax + 4B$ as polynomials in the indeterminate x, we argue that they are coprime in the ring $\mathbb{Q}[x]$. We use the Euclidean algorithm to verify this fact. First assume that $A \neq 0, B \neq 0$. Then

$$\begin{aligned}
x^4 - 2Ax^2 - 8Bx + A^2 &= \left(4x^3 + 4Ax + 4B\right)\left(\frac{1}{4}x\right) + (-3Ax^2 - 9Bx + A^2) \\
4x^3 + 4Ax + 4B &= (-3Ax^2 - 9Bx + A^2)\left(-\frac{4}{3A}x + \frac{4B}{A^2}\right) + \left(\frac{4\Delta}{3A^2}x\right) \\
-3Ax^2 - 9Bx + A^2 &= \left(\frac{4\Delta}{3A^2}x\right)\left(-\frac{9A^3}{4\Delta}x - \frac{27A^2B}{4\Delta}\right) + (A^2) \\
\frac{4\Delta}{3A^2}x &= (A^2)\left(\frac{4\Delta}{3A^4}\right) + 0
\end{aligned}$$

Since the last non-zero remainder is a constant, it follows that the polynomials are coprime. If $B = 0$, the assumption that $\Delta \neq 0$ forces $A \neq 0$. In this case, we can simply remove all terms containing a factor of B in the Euclidean algorithm above. It is easy to see that we will reach the same conclusion. Lastly, if $A = 0$, then $B \neq 0$. In this case, we see that

$$\begin{aligned} x^4 - 8Bx &= (4x^3 + 4B)\left(\frac{1}{4}x\right) + (-9Bx) \\ 4x^3 + 4B &= (-9Bx)\left(-\frac{4B}{9}x^2\right) + (4B) \\ -9Bx &= (4B)\left(-\frac{9}{4}x\right) + 0. \end{aligned}$$

Again, since the last non-zero remainder is a constant, the polynomials are coprime. For ease of notation, let $\phi(x) = x^4 - 2Ax^3 - 8Bx + A^2$ and $\psi(x) = 4x^3 + 4Ax + 4B$. Now let $x = m/n \in \mathbb{Q}$ with $\gcd(m, n) = 1$ and set

$$\begin{aligned} \Phi(m, n) &= n^4\phi(m/n) = m^4 - 2Am^2n^2 - 8Bmn^3 + A^2n^4 \\ \Psi(m, n) &= n^4\psi(m/n) = 4m^3n + 4Amn^3 + 4Bn^4. \end{aligned}$$

Since $\gcd(\phi(x), \psi(x)) = 1$, by Bezout's identity there exist unique polynomials $f(x), g(x) \in \mathbb{Q}[x]$ such that

$$f(x)\phi(x) + g(x)\psi(x) = 1$$

with $\deg f(x) < \deg\psi(x) = 3$ and $\deg g(x) < \deg\phi(x) = 4$. Now fix $r \in \mathbb{Z}$ so that $rf(x), rg(x) \in \mathbb{Z}[x]$. Evaluating the equation above at $x = m/n$ and multiplying by n^3r we obtain

$$[n^3rf(m/n)][\Phi(m, n)] + [n^3rg(m/n)][\Psi(m, n)] = n^3r.$$

But since $\max\{\deg f(x), \deg g(x)\} \leq 3$ and since $rf(x)$ and $rg(x)$ have integer coefficients,

it follows that $n^3 rf(m/n)$ and $n^3 rg(m/n)$ are both integers. Now let $\gamma = \gcd(\Phi(m,n), \Psi(m,n))$. Since γ divides both $\Phi(m,n)$ and $\Psi(m,n)$, it follows from the previous equation that $\gamma | n^3 r$. However, we argue that $\gamma | r$. To see this, observe that

$$\gamma | nr\Phi(m,n) = m^4 nr - 2Am^2 n^3 r - 8Bmn^4 r + A^2 n^5 r.$$

But since γ divides each term proceeding $m^4 nr$, this forces $\gamma | m^4 nr$. It follows that

$$\gamma | \gcd(n^3 r, m^4 nr) = nr \gcd(n^2, m^4) = nr.$$

Likewise,

$$\gamma | r\Phi(m,n) = m^4 r - 2Am^2 n^2 r - 8Bmn^3 r + A^2 n^4 r.$$

But since we have just shown that $\gamma | nr$, it follows that γ divides each term proceeding $m^4 r$. This forces $\gamma | m^4 r$. Hence

$$\gamma | \gcd(n^3 r, m^4 r) = r \gcd(n^3, m^4) = r.$$

That is, there exists a fixed integer r such that $\gcd(\Phi(m,n), \Psi(m,n)) \leq r$ for any choice of relatively prime integers m and n. Thus,

$$\begin{aligned} H(2P) = H(\phi(x)/\psi(x)) = H(\Phi(m,n)/\Psi(m,n)) &\geq \frac{1}{r} \max\{|\Phi(m,n), \Psi(m,n)\} \\ &= \frac{1}{r} \max\{|n^4 \phi(m/n)|, |n^4 \psi(m/n)|\} \\ &\geq \frac{1}{2r} (|n^4 \phi(m/n)| + |n^4 \psi(m/n)|). \end{aligned}$$

Since $H(P)^4 = \max\{|m|^4, |n|^4\}$, it follows that

$$\begin{aligned}\frac{H(2P)}{H(P)^4} &\geq \frac{1}{2r} \cdot \frac{|n^4\phi(m/n)| + |n^4\psi(m/n)|}{\max\{|m|^4, |n|^4\}} \\ &= \frac{1}{2r} \cdot \frac{|\phi(m/n)| + |\psi(m/n)|}{\max\{|m/n|^4, 1\}}.\end{aligned}$$

Define

$$p(t) = \frac{|\phi(t)| + |\psi(t)|}{\max\{t^4, 1\}} = \frac{|t^4 - 2At^3 - 8Bt + A^2| + |4t^3 + 4At + 4B|}{\max\{|t|^4, 1\}}$$

Then $\lim_{t\to\infty} p(t) = 1$. Hence $p(t)$ is bounded outside of some closed interval I. Moreover, since $p(t)$ is continuous on the closed interval I, it achieves a minimum. However, since we have shown that $\phi(t)$ and $\psi(t)$ are coprime, $\phi(t)$ and $\psi(t)$ have no common roots. Thus $|\phi(t)| + |\psi(t)| > 0$ for all $t \in I$, so the minimum of $p(t)$ on I must be positive. Therefore $p(t)$ is bounded on both I and $\overline{I}$, hence there exists $c > 0$ such that $p(t) \geq c$ for all $t \in \mathbb{R}$. It follows that

$$\frac{H(2P)}{H(P)^4} \geq \frac{1}{2r} \cdot \frac{|\phi(m/n)| + |\psi(m/n)|}{\max\{|m/n|^4, 1\}} = \frac{1}{2r} p\left(\frac{m}{n}\right) \geq \frac{c}{2r}.$$

Multiplying both sides of the inequality by $H(P)^4$, we find that

$$H(2P) \geq \frac{c}{2r} H(P)^4$$

Let $k = \log(2r/c)$ so that $-k = \log(c/2r)$. By taking logarithms of the inequality above, we obtain

$$h(2P) \geq 4h(P) - k.$$

□

Remark. Again, a similar proof is found in [8]. However, our proof has a few key differences. For example, our argument uses the Euclidean algorithm to show coprimality of two

polynomials. This process reveals the importance that E be non-singular since the quantity Δ appears in the denominator of a quotient while performing the algorithm. Also, in an effort to make the proof more concise, we have condensed Lemma 1.3 to a single proof.

Now that we have proved Lemmas 1.11, 1.12, and 1.13 we are ready to prove the Mordell-Weil theorem.

Theorem 1.14. *Let $E : y^2 = x^3 + Ax + B$ be an elliptic curve with $A, B \in \mathbb{Z}$. Then $E(\mathbb{Q})$ is finitely generated.*

Proof. By the Weak Mordell-Weil Theorem, we have that $[E(\mathbb{Q}) : 2E(\mathbb{Q})]$ is finite, so suppose $[E(\mathbb{Q}) : 2E(\mathbb{Q})] = n$. That is, there are n cosets of $2E(\mathbb{Q})$ in $E(\mathbb{Q})$ which can be represented by $Q_1, Q_2, ..., Q_n$. Then for any point $P \in E(\mathbb{Q})$, there is a representative Q_{i_1} from the list $Q_1, Q_2, ..., Q_n$ for which

$$P - Q_{i_1} \in 2E(\mathbb{Q}).$$

In particular

$$P - Q_{i_1} = 2P_1 \implies P = Q_{i_1} + 2P_1$$

for some $P_1 \in E(\mathbb{Q})$. But we can repeat this procedure for P_1. That is, there is a representative Q_{i_2} from the list $Q_1, Q_2, ...Q_n$ for which

$$P_1 - Q_{i_2} \in E(\mathbb{Q})$$

and so

$$P_1 - 2Q_{i_2} = P_2 \implies P_1 = Q_{i_2} + 2P_2$$

for some $P_2 \in E(\mathbb{Q})$. By iterating this process, we obtain a sequence of equation

$$P = Q_{i_1} + 2P_1$$
$$P_1 = Q_{i_2} + 2P_2$$
$$P_2 = Q_{i_3} + 2P_3$$
$$\vdots$$
$$P_{m-1} = Q_{i_m} + 2P_m$$

From these equations, we can write P as

$$\begin{aligned}
P &= Q_{i_1} + 2(Q_{i_2} + 2P_2) \\
&= Q_{i_1} + 2Q_{i_2} + 4(Q_{i_3} + 2P_3) \\
&= Q_{i_1} + 2Q_{i_2} + 4Q_{i_3} + 8(Q_{i_4} + 2P_4) \\
&\vdots \\
&= Q_{i_1} + 2Q_{i_2} + 4Q_{i_3} + 8Q_{i_4} + \cdots + 2^{m-1}(Q_{i_m} + 2P_m) \\
&= Q_{i_1} + 2Q_{i_2} + 4Q_{i_3} + 8Q_{i_4} + \cdots + 2^{m-1}Q_{i_m} + 2^m P_m
\end{aligned}$$

This shows that P is in the group generated by $\{Q_1, Q_2, ..., Q_n, P_m\}$. It follows from Lemma 1.12 that for any point $P \in E(\mathbb{Q})$

$$h(P - Q_i) \leq 2h(P) + k_i$$

for all $1 \leq i \leq n$ where k_i are constants. However, by fixing $k' = \max\{k_1, ..., k_n\}$, we have that

$$h(P - Q_i) \leq 2h(P) + k'$$

for all $1 \leq i \leq n$. Furthermore, following from Lemma 1.13, we have that

$$4h(P_j) \leq h(2P_j) + k.$$

But recall that $P_{j-1} = Q_{i_j} + 2P_j$, so $2P_j = P_{j-1} - Q_{i_j}$ and hence

$$4h(P_j) \leq h(P_{j-1} - Q_{i_j}) + k.$$

But we have just shown that for any arbitrary point $P \in E(\mathbb{Q})$, $h(P - Q_i) \leq 2h(P) + k'$ for any Q_i. In particular, we have

$$4h(P_j) \leq 2h(P_{j-1}) + k' + k.$$

Dividing both sides of the inequality by 4, we find that

$$\begin{aligned} h(P_j) &\leq \frac{1}{2}h(P_{j-1}) + \frac{1}{4}(k' + k) \\ &= \frac{3}{4}h(P_{j-1}) - \frac{1}{4}h(P_{j-1}) + \frac{1}{4}(k' + k) \\ &= \frac{3}{4}h(P_{j-1}) - \frac{1}{4}(h(P_{j-1}) + k' + k) \end{aligned}$$

If $h(P_{j-1}) \geq k + k'$, the inequality above guarantees that $h(P_j) \leq 3h(P_{j-1})/4$. Thus in the list of of points $P, P_1, P_2, ...$, if $h(P_{j-1}) \geq k + k'$, then the height of the next point in the sequence is smaller. Therefore, the sequence of point heights must approach zero. In particular, there is a point P_m in the list for which $h(P_m) \leq k' + k$. This implies that for any point $P \in E(\mathbb{Q})$, we can write

$$P = a_1Q_1 + a_2Q_2 + \cdots + a_nQ_n + 2^mP_0$$

for $a_1, ..., a_n \in \mathbb{Z}$ and $h(P_0) \leq k' + k$. Thus, $E(\mathbb{Q})$ is generated by $\{Q_1, Q_2, ..., Q_n\} \cup \{P_0 \in$

$E(\mathbb{Q}) : h(P_0) \leq k' + k\}$. Both of these are finite sets, which completes the proof. $\square$

Remark. The Mordell-Weil theorem is well-known and the proof we have just presented is a traditional descent argument (see [8]). In furnishing the lemmas and proof of the Mordell-Weil theorem, we have added many clarifying details and modifications that help streamline and organize the arguments.

Section 1.5: Applications of Elliptic Curves

Elliptic curves have many useful applications, most notably in realm of cryptography. To give a very general idea as to why, consider the point $(3, 5)$ from earlier on the elliptic curve $E : y^2 = x^3 - x + 1$. Now consider the equation

$$n(3,5) = \left(\frac{1301099730561}{1906277262400}, -\frac{2098030206970736191}{2631958890650432000}\right).$$

It is a very difficult feat, even for a computer, to find the value of n. This concept plays a central role in elliptic curve cryptography. Moreover, elliptic curves can be used for primality testing. Elliptic curves are especially useful for testing the primality of large numbers where other sieving techniques are insufficient or impractical. Later, we will see how elliptic curves relate to Fermat's Last Theorem. The application of elliptic curves to cryptography and primality testing appear in [8].

CHAPTER 2: ELLIPTIC FUNCTIONS

We now shift our attention to the study of elliptic functions. From the onset, elliptic functions may seem unrelated to the study of elliptic curves. However, the concepts are inextricably linked. We will outline their profound connection in this section. Before we formally introduce elliptic functions, we will review a few important ideas from complex analysis.

Section 2.1: Lattices

Definition 2.1. Given two non-zero complex numbers $\omega_1 = u_1 + v_1 i$ and $\omega_2 = u_2 + v_2 i$ such that (u_1, v_1) and (u_2, v_2) are linearly independent in $\mathbb{R}^2$, we say that the set

$$L := \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$$

is a *lattice* and ω_1 and ω_2 are its *generators*.

Perhaps the most rudimentary example of a lattice is the set of Gaussian integers $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$. Since 1 and i generate $\mathbb{Z}[i]$, we can write $\mathbb{Z}[i] = \langle 1, i \rangle$.

Definition 2.2. The *fundamental domain* $\mathcal{F}$ of a lattice $L = \langle \omega_1, \omega_2 \rangle$ is defined as

$$\mathcal{F}_L = \{\lambda\omega_1 + \mu\omega_2 | 0 \leq \lambda < 1, 0 \leq \mu < 1\}$$

It follows from this definition that the fundamental domain of the Gaussian integers is given by the set

$$\mathcal{F}_{\mathbb{Z}[i]} = \{\lambda + \mu i | 0 \leq \lambda < 1, 0 \leq \mu < 1\}.$$

In other words, the fundamental domain of the Gaussian integers is the indicated square in the complex plane (see Figure 8).

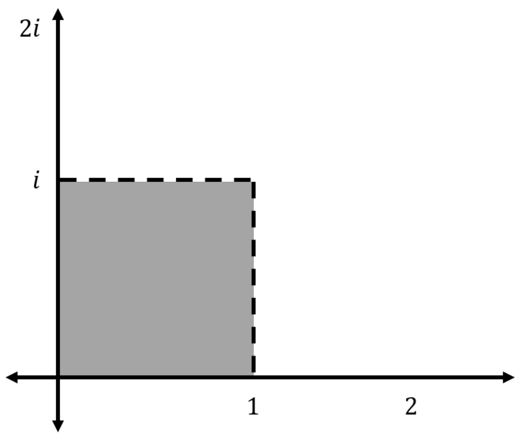

Figure 8: Fundamental domain of the Gaussian integers.

Given a lattice L with fundamental domain $\mathcal{F}_L$, we can translate $\mathcal{F}_L$ by any element $\ell \in L$. That is,

$$\mathcal{F}_L + \ell = \{\ell + \lambda w_1 + \mu\omega_2 | 0 \leq \lambda < 1, 0\lambda\mu < 1\}.$$

To illustrate what this means geometrically, consider $F_{\mathbb{Z}[i]}$ translated by $1 + i \in \mathbb{Z}[i]$. So,

$$\mathcal{F}_{\mathbb{Z}[i]} + (1 + i) = \{(\lambda + 1) + (\mu + 1)i | 0 \leq \lambda < 1, 0 \leq \mu < 1\}.$$

Now the shaded region has shifted (see Figure 9).

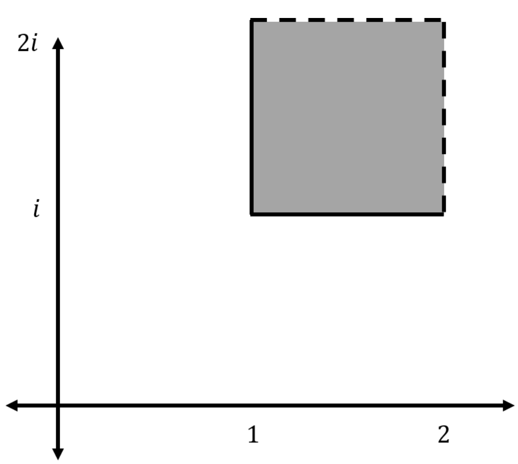

Figure 9: Translate of the fundamental domain of $\mathbb{Z}[i]$.

It becomes intuitively obvious that the fundamental domain of a lattice and its lattice point translates form a disjoint cover of the complex plane. In particular, any complex number lies in exactly one translate of the fundamental domain of lattice.

Section 2.2: Complex Functions

Definition 2.3 (see [5]). A function $f : \mathbb{C} \to \mathbb{C}$ defined for all $z \in \mathbb{C}$ is said to be *analytic* on $\mathbb{C}$ provided it is complex differentiable at all points $z_0 \in \mathbb{C}$.

For example, consider the function $f(z) = z$. Then for all $z_0 \in \mathbb{C}$, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = 1$$

That is, the derivative is defined on all of $\mathbb{C}$. Hence $f(z) = z$ is analytic on $\mathbb{C}$. More generally, all complex polynomial functions are analytic on $\mathbb{C}$.

Definition 2.4. A function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is said to be *meromorphic* in $\mathbb{C}$ provided it is analytic on $\mathbb{C}$ except for set of isolated poles.

Take, for instance, the function $g(z) = 1/z$ which has a pole at $z = 0$. Then for $z_0 \neq 0$, we have

$$\lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{1/z - 1/z_0}{z - z_0} = \lim_{z \to z_0} \frac{z_0 - z}{zz_0(z - z_0)} = -\lim_{z \to z_0} \frac{1}{zz_0} = -\frac{1}{z_0^2}.$$

Since the derivative of $g(z)$ is analytic on $\mathbb{C} \setminus \{0\}$ (that is, the derivative is defined for all of $\mathbb{C}$ except at the pole), it follows that $g(z)$ is meromorphic in $\mathbb{C}$.

Definition 2.5. Let L be a lattice. An *elliptic function* is a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ such that for all $\ell \in L$ and $z \in \mathbb{C}$, $f(z + \ell) = f(z)$.

It follows from this definition that elliptic functions are doubly periodic. That is, if L is a lattice generated by ω_1 and ω_2, then $f(z + \omega_1) = f(z + \omega_2) = f(z)$. In other

words, f has two periods. One of the most important elliptic functions is the Weierstrass $\wp$-function, which we will discuss now.

Section 2.3: The Weierstrass $\wp$-function

Definition 2.6. Let L be a lattice. Then the Weierstrass $\wp$-function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{\ell \in L \setminus \{0\}} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} \right).$$

The Weierstrass $\wp$-function has profound connections to elliptic curves. This function, as well as its derivative, converge absolutely for all $z \in \mathbb{C} \setminus L$ (see [7]). It is not difficult to see that $\wp(z)$ is an even function, but somewhat less obvious is that for any $\ell' \in L$, $p(z+\ell') = p(z)$. We verify this fact below.

$$\begin{aligned}
\wp(z+\ell') &= \frac{1}{(z+\ell')^2} + \sum_{\ell \in L \setminus \{0\}} \left(\frac{1}{(z+\ell'-\ell)^2} - \frac{1}{\ell^2} \right) \\
&= \frac{1}{(z+\ell')^2} + \sum_{\ell \in L \setminus \{0\}} \frac{1}{(z+\ell'-\ell)^2} - \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^2} \\
&= \frac{1}{(z+\ell')^2} - \frac{1}{(z+\ell')^2} + \sum_{\ell \in L} \frac{1}{(z+\ell'-\ell)^2} - \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^2} \\
&= \sum_{\ell \in L} \frac{1}{(z-(\ell-\ell'))^2} - \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^2}
\end{aligned}$$

Now let $\ell'' = \ell - \ell'$. So we have

$$\begin{aligned}
\sum_{\ell \in L} \frac{1}{(z-(\ell-\ell'))^2} - \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^2} &= \sum_{\ell'' \in L} \frac{1}{(z-\ell'')^2} - \sum_{\ell'' \in L \setminus \{0\}} \frac{1}{(\ell'')^2} \\
&= \frac{1}{z^2} + \sum_{\ell'' \in L \setminus \{0\}} \frac{1}{(z-\ell'')^2} - \sum_{\ell'' \in L \setminus \{0\}} \frac{1}{(\ell'')^2} \\
&= \frac{1}{z^2} + \sum_{\ell'' \in L \setminus \{0\}} \left(\frac{1}{(z-\ell'')^2} - \frac{1}{(\ell'')^2} \right) \\
&= \wp(z).
\end{aligned}$$

The derivative of $\wp(z)$ is given by

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\ell \in L \setminus \{0\}} \frac{-2}{(z-\ell)^3} = -2 \sum_{\ell \in L} \frac{1}{(z-\ell)^3}.$$

It is clear that the derivative $\wp'(z)$ is also doubly periodic. Now consider the following:

$$\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2} = \frac{(-1/\ell)^2}{(-1/\ell)^2(z-\ell)^2} - \frac{1}{\ell^2} = \frac{1/\ell^2}{[(-1/\ell)(z-\ell)]^2} - \frac{1}{\ell^2} = \frac{1}{\ell^2}\left(\frac{1}{(1-z/\ell)^2} - 1\right).$$

Using that fact that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

we can write

$$\frac{1}{\ell^2}\left(\frac{1}{(1-z/\ell)^2} - 1\right) = \frac{1}{\ell^2}\left(\sum_{n=0}^{\infty}(n+1)\frac{z^n}{\ell^n} - 1\right) = \frac{1}{\ell^2}\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\ell^n} = \sum_{n=1}^{\infty}(n+1)\frac{z^n}{\ell^{n+2}}.$$

Thus,

$$\wp(z) = \frac{1}{z^2} + \sum_{\ell \in L \setminus \{0\}} \left(\frac{1}{(z-\ell)^2} - \frac{1}{\ell^2}\right) = \frac{1}{z^2} + \sum_{\ell \in L \setminus \{0\}} \left(\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\ell^{n+2}}\right).$$

But since $\wp(z)$ is an even function, any terms containing an odd power of z must vanish. That is, we only need to consider the sum when $n = 2k$ for $k \in \mathbb{N}$. So,

$$\wp(z) = \frac{1}{z^2} + \sum_{\ell \in L \setminus \{0\}} \left(\sum_{n=1}^{\infty}(n+1)\frac{z^n}{\ell^{n+2}}\right) = \frac{1}{z^2} + \sum_{\ell \in L \setminus \{0\}} \left(\sum_{k=1}^{\infty}(2k+1)\frac{z^{2k}}{\ell^{2k+2}}\right)$$

To ease notation, we define

$$G_{2k} := \sum_{\ell \in L \setminus \{0\}} \frac{1}{\ell^{2k}}.$$

So

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k}.$$

This is referred to as the Laurent series for $\wp(z)$ about $z = 0$. We will use this series in the following proof to illustrate the connection between the Weierstrass $\wp$-function and elliptic curves.

Theorem 2.7. *For all $z \in \mathbb{C}$*

$$\left(\frac{\wp'(z)}{2}\right)^2 = \wp(z)^3 - 15G_4\wp(z) - 35G_6.$$

Proof. We have shown that the Laurent series for $\wp(z)$ is given by

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}z^{2k} = \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots$$

Then we have

$$\begin{aligned}
\wp(z)^3 &= \left(\frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots\right)\left(\frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots\right)\left(\frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots\right) \\
&= \left(\frac{1}{z^4} + 6G_4 + 10G_6z^2 + \cdots\right)\left(\frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + \cdots\right) \\
&= \frac{1}{z^6} + \frac{9G_4}{z^2} + 15G_6 + \mathcal{O}(z^2)
\end{aligned}$$

where $\mathcal{O}(z^2)$ represents a series with terms of degree 2 and higher. On the other hand, we can use the Laurent series for $\wp(z)$ to compute the derivative $\wp'(z)$.

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=1}^{\infty}2k(2k+1)G_{2k+2}z^{2k-1} = -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + \cdots$$

Then we have

$$\begin{aligned}\left(\frac{p'(z)}{2}\right)^2 &= \left(-\frac{1}{z^3} + 3G_4z + 10G_6z^3 + \cdots\right)\left(-\frac{1}{z^3} + 3G_4z + 10G_6z^3 + \cdots\right)\\ &= \frac{1}{z^6} - \frac{6G_4}{z^2} - 20G_6 + \mathcal{O}(z^2).\end{aligned}$$

Again, $\mathcal{O}(z^2)$ represents a series with terms of degree 2 and higher. Upon simplifying $(\wp'(z)/2)^2 - \wp(z)^3 + 15G_4\wp(z) + 35G_6$, we find that

$$\left(\frac{\wp'(z)}{2}\right)^2 - \wp(z)^3 + 15G_4\wp(z) + 35G_6 = \mathcal{O}(z^2).$$

That is, every term cancels except those of degree 2 or higher. Now since $\wp(z)$ and $\wp'(z)$ are doubly periodic, it follows that left hand side, and hence $\mathcal{O}(z^2)$, are doubly periodic. But doubly periodic functions assume all of their values in the the fundamental domain of the lattice L. That is, $\mathcal{O}(z^2)$ must be bounded. Moreover, since $\mathcal{O}(z^2)$ is a series with terms z^2 and higher, it is analytic on $\mathbb{C}$. Hence $\mathcal{O}(z^2)$ must be a constant by Louiville's theorem. In particular, $\mathcal{O}(z^2) = 0$. So

$$\left(\frac{\wp'(z)}{2}\right)^2 - \wp(z)^3 + 15G_4\wp(z) + 35G_6 = 0.$$

Therefore,

$$\left(\frac{\wp'(z)}{2}\right)^2 = \wp(z)^3 - 15G_4\wp(z) - 35G_6.$$

$\square$

Remark. We have added many clarifying details to the proof found in [9].

This result implies that the elliptic curve $E : y^2 = x^3 - 15G_4x - 35G_6$ can be parameterized by $\wp(z)$ in the sense that $(\wp(z), \wp'(z)/2)$ is a point on E. Furthermore, for any elliptic curve $y^2 = x^3 + Ax + B$ defined over $\mathbb{C}$, there is a lattice L for which $A = -15G_4$

and $B = -35G_6$. That is to say, any elliptic curve given in Weierstrass form can be $\wp$-parameterized. Proofs of these facts can be found in [7].

CHAPTER 3: ELLIPTIC CURVES AND FERMAT'S LAST THEOREM

In the previous section we saw that an elliptic function can be used to parameterize an elliptic curve. In this section we will focus how elliptic curves play an important role in Fermat's Last Theorem. The idea will be similar to the previous section in the sense that there is another type of function, a modular function, that can be used to parameterize elliptic curves defined over $\mathbb{Q}$.

Section 3.1: Historical Remarks

The centuries-long journey that ultimately led to the proof Fermat's Last Theorem is one marked by the extraordinary fortitude of the mathematicians whose ideas and contributions forever changed the study of elliptic curves and modular forms. We would be remiss if we failed to offer at least some historical insights to help motive this this section.

We begin with Diophantus of Alexandria – a mathematician whose historical narrative remains as much of an enigma as some of the problems he influenced. His famous work *Arithmetica* concerns rational solutions to polynomial equations. For example, finding rational solutions of the Pythagorean equation $x^2 + y^2 = z^2$ is regarded as a classical Diophantine problem. Such problems have long been the source of great mathematical inspiration. Indeed, it was in the margin of *Arithmetica* where Fermat wrote his cryptic note:

> "To resolve a cube into [the sum of] two cubes, a fourth power into fourth powers, or in general any power higher than the second into two of the same kind, is impossible; of which fact I have found a remarkable proof. The margin is too small to contain it..." [9].

Put symbolically, Fermat claimed, without proof, that the equation $x^n + y^n = z^n$ has no positive integral solutions for $n > 2$. A formal proof of this claim evaded mathematicians for centuries until Andrew Wiles and Richard Taylor settled the matter in 1995. While

the detailed proof of Fermat's Last Theorem is well beyond the purview of our study, we endeavor to highlight the strategy behind Wiles' use of elliptic curves (see [9]).

Fermat's Last Theorem is a rare and quintessential example of a problem whose breadth spans from the Pythagoreans to the present century and whose solution encompasses so many facets of both ancient and modern mathematics. As we just discussed, finding rational solutions of the Pythagorean equation $x^2 + y^2 = z^2$ is of particular interest. From our perspective, we are particularly interested in finding integral solutions of the Pythagorean equation, such as $(3, 4, 5)$ and $(5, 12, 13)$. It is well known that the formula for generating such triples can be obtained via the parameterization of a circle. For the sake of completeness, we will illustrate this process.

Consider the unit circle $x^2 + y^2 = 1$ together with a line intersecting the circle at $(-1, 0)$ and an arbitrary point (x, y) (see Figure 10).

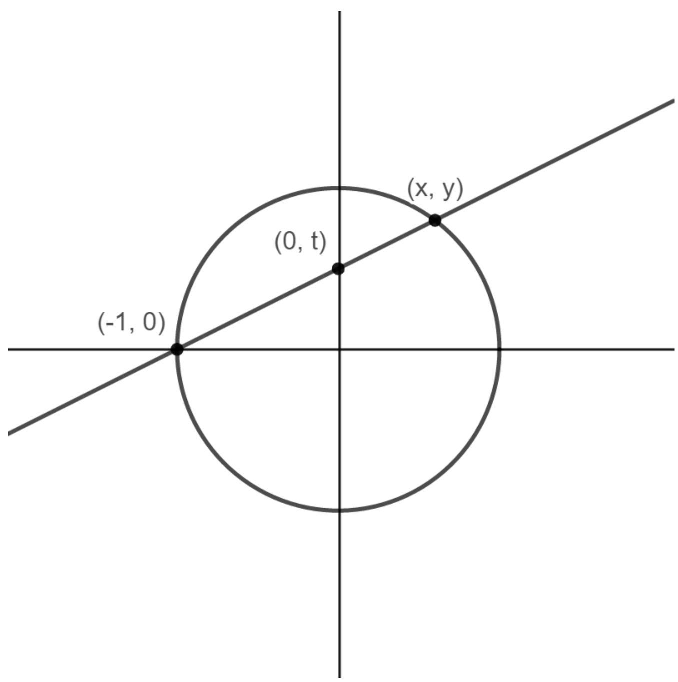

Figure 10: The circle $x^2 + y^2 = 1$ and the line $y = tx + t$

The line is given by the equation $y = tx + t$. Substituting this into the equation

$x^2 + y^2 = 1$, we obtain the quadratic equation

$$(1 + t^2)x^2 + 2t^2 x + t^2 - 1 = 0.$$

Applying the quadratic formula, we find a solution

$$x = \frac{1 - t^2}{1 + t^2}.$$

Since $y = tx + t$, we can write y in terms of t by substituting our solution above. We find that

$$y = \frac{2t}{1 + t^2}.$$

Thus the unit circle $x^2 + y^2 = 1$ can be parameterized as

$$\left(\frac{1 - t^2}{1 + t^2}, \frac{2t}{1 + t^2} \right).$$

Now let $t = v/u$. Then

$$x = \frac{1 - v^2/u^2}{1 + v^2/u^2} = \frac{u^2 - v^2}{u^2 + v^2}$$

and

$$y = \frac{2v/u}{1 + v^2/u^2} = \frac{2uv}{v^2 + u^2}.$$

Since $x^2 + y^2 = 1$, we find that

$$\frac{(u^2 - v^2)^2}{(u^2 + v^2)^2} + \frac{(2uv)^2}{(u^2 + v^2)^2} = 1$$

or equivalently,

$$(u^2 - v^2)^2 + (2uv)^2 = (u^2 + v^2)^2.$$

That is, for any choice of u and v, the integers $u^2 - v^2$, $2uv$, and $u^2 + v^2$ will satisfy the Pythagorean equation, implying that there are, indeed, infinitely many integral solutions.

Theorem 3.1. *Suppose x, y, z are the sides of a primitive Pythagorean triple. Then there exists integers u and v of opposite parity such that* $\gcd(u, v) = 1, u > v$ *and*

$$x = u^2 - v^2$$

$$y = 2uv$$

$$z = u^2 + v^2$$

(where x and y can be interchanged if necessary).

Proof. Suppose x, y, z are the sides of a primitive Pythagorean triple. That is, $\gcd(x, y, z) = 1$. We argue that x, y, and z are pairwise relatively prime, since any prime dividing any pair must divide the third by virtue of the equation $x^2 + y^2 = z^2$. Moreover, since x, y, and z are pairwise relatively prime, it follows that at most one of x, y and z is even. But since at least two of x, y, and z are odd, it follows that exactly one must be even.

We claim that z is odd. To see this, suppose on the contrary that z is even. It follows that x and y are both odd. Taking the equation modulo 4, we have

$$2 \equiv x^2 + y^2 = z^2 \equiv 0 \pmod 4$$

which is contradiction. Therefore z is odd and x and y have opposite parity. Without loss of generality, we may assume that x is odd. In particular, $z - x$ and $z + x$ are both even. That is, $z - x = 2m$ and $z + x = 2n$ for some $m, n \in \mathbb{Z}$. Adding these equations together,

we find that $2z = 2m + 2n$, or equivalently $z = m + n$. Likewise, if we subtract these equations, we find that $2x = 2m - 2n$, or equivalently $x = m - n$. It is not difficult to see that m and n are relatively prime. If $d|m$ and $d|n$, then $d|(m-n)$ and $d|(m+n)$. But since $\gcd(m - n, m + n) = \gcd(x, y) = 1$, this forces $d = 1$. Hence $\gcd(m, n) = 1$. Then

$$y^2 = z^2 - x^2 = (z - x)(z + x) = 4mn \implies \left(\frac{y}{2}\right)^2 = mn.$$

In particular, m and n are perfect squares. That is $m = u^2$ and $n = v^2$ for some $u, v \in \mathbb{Z}$. All together, we find that

$$\begin{aligned} x &= m - n = u^2 - v^2 \\ y &= 2uv \\ z &= m + n = u^2 + v^2. \end{aligned}$$

□

Remark. This proof is similar to the proof found in [1].

The following theorem is due to Fermat. We closely follow the proof found in [1], but we have added diagrams to help illustrate Fermat's descent argument.

Theorem 3.2 (Fermat)**.** *The equation $x^4 + y^4 = z^2$ has no positive integer solutions.*

Proof. Suppose on the contrary that $x, y, z \in \mathbb{N}$ such that $x^4 + y^4 = z^2$. Without loss of generality, we may assume that $\gcd(x, y, z) = 1$. Observe that $x^4 + y^4 = z^2$ can be expressed as $(x^2)^2 + (y^2)^2 = z^2$. That is, there exists a primitive Pythagorean triple with legs x^2 and y^2 and hypotenuse z (see Figure 11). Without loss of generality, assume that x^2 is odd. Then there must exist positive integers u and v of opposite parity such that

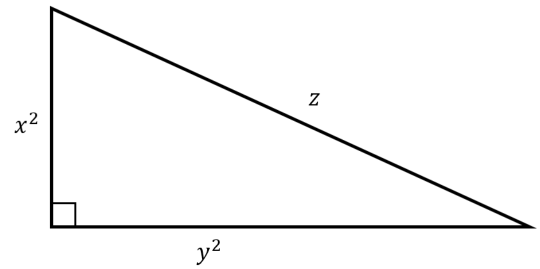

Figure 11: Pythagorean triple x^2, y^2, z

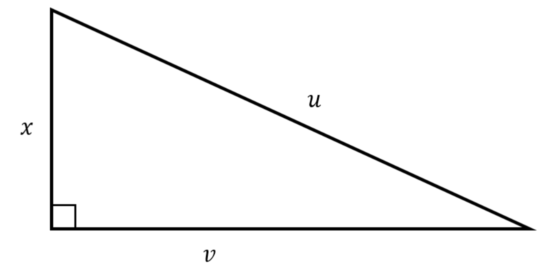

Figure 12: Pythagorean triple x, v, u

$\gcd(u, v) = 1$, $u > v$, and

$$x^2 = u^2 - v^2 \tag{1}$$

$$y^2 = 2uv \tag{2}$$

$$z = u^2 + v^2. \tag{3}$$

Equation (1) implies that there exists another primitive Pythagorean triple with legs x and v and hypotenuse u (see Figure 12). But since x is odd, we require v even. Hence there exist positive integers s and t of opposite parity such that $\gcd(s, t) = 1$, $s > t$, and,

$$x = s^2 - t^2 \tag{4}$$

$$v = 2st \tag{5}$$

$$u = s^2 + t^2. \tag{6}$$

Then

$$y^2 = 2uv = 4ust \implies \left(\frac{y}{2}\right)^2 = ust.$$

But since u, s, and t are pairwise relatively prime, it follows that $u = \overline{u}^2, s = \overline{s}^2$, and $t = \overline{t}^2$ for some $\overline{u}, \overline{s}, \overline{t} \in \mathbb{N}$. So

$$\overline{u}^2 = u = s^2 + t^2 = \overline{s}^4 + \overline{t}^4.$$

Thus, $\overline{u}, \overline{s}^2, \overline{t}^2$ is another solution with $\overline{u} \leq \overline{u}^2 = u \leq u^2 < u^2 + v^2 = z$. Geometrically, we have constructed another right triangle from our original solution with a smaller hypotenuse, as in Figure 13.

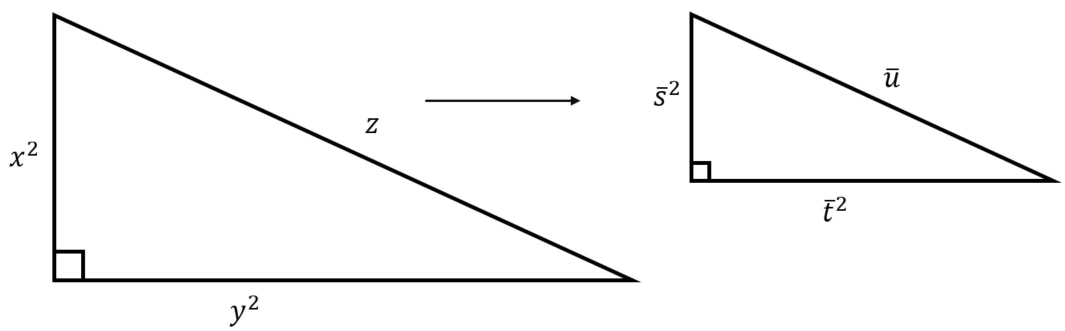

Figure 13: Decreased hypotenuse

We can continue this process indefinitely, obtaining infinitely many right triangles with decreasing hypotenuse lengths. This, however, is not possible since we are assuming that each hypotenuse is a positive integer, and the positive integers cannot be decreased indefinitely. Therefore there are no positive integer solutions of $x^4 + y^4 = z^2$. $\square$

The result above implies that $x^4 + y^4 = z^4$ has no positive integer solutions, for if it did, it would imply that x, y and z^2 would be a solution of $a^4 + b^4 = c^2$, which violates Theorem 3.2. Thus we have demonstrated the proof of Fermat's Last Theorem for the case $n = 4$. Since every positive integer $n \geq 3$ is divisible by either 4 or an odd prime p, we have reduced the problem to showing that $a^p + b^p = c^p$ has no positive integral solutions for odd primes p.

Section 3.2: The Modularity Theorem

Definition 3.3. The *modular group* Γ is defined as

$$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

The group Γ acts on $\mathcal{H} = \{a + bi : b > 0\}$ via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (z) = \frac{az + b}{cz + d}.$$

We can extend $\mathcal{H}$ by defining $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. Then Γ acts on $\overline{\mathcal{H}}$ just as before, with a few minor caveats. If $z = -d/c$, then the action defined above yields 0 in the denominator. To accommodate for this subtlety, we define the action to be ∞ whenever $z = -d/c$. On the other hand, if $z = \infty$, we define the action to be a/c if $c \neq 0$ and ∞ if $c = 0$. With these adjustments, the modular group acts on the entirety of $\overline{\mathcal{H}}$. For additional details regarding the modular group, see [2]. We now present the definition of a modular function as found in [5].

Definition 3.4. A meromorphic function $f : \overline{\mathcal{H}} \to \mathbb{C} \cup \{\infty\}$ is called a *modular function* if

$$f(Mz) = f(z) \text{ for all } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

Given these preliminary definitions, we now state the Modularity Theorem:

Theorem 3.5 (Wiles-Taylor)**.** *Suppose $E : y^2 = Ax^3 + Bx^2 + Cx + D$ is an elliptic defined over $\mathbb{Q}$. Then there are modular functions $f(z)$ and $g(z)$ for which*

$$E : g(z)^2 = Af(z)^3 + Bf(z)^2 + Cf(z) + D.$$

In the same way that the Weierstrass $\wp$-function can be used to parameterize an elliptic curve, the Theorem 3.5 implies that all rational elliptic curves can be parameterized using modular functions. Theorem 3.5 is equivalent to saying that elliptic curves are modular (see [9]).

Section 3.3: The Application to Fermat's Last Theorem

We aim to provide a very general idea of the proof. As we just discussed, we can focus on the equation $a^p + b^p = c^p$ for odd primes p. Now assume that there exists a solution of $a^p + b^p = c^p$. Next, we construct an elliptic curve using the assumed solution of Fermat's equation: $\mathcal{F} : y^2 = x(x - a^p)(x + b^p)$. This elliptic curve is called a Frey curve. In 1986, Ken Ribet proved that there are no modular functions which parameterize the curve $\mathcal{F}$. Subsequently, in 1995, Andrew Wiles and Richard Taylor proved Theorem 3.5 for a certain class of curves, including Frey curves. That is, it was shown that every rational elliptic curve can be parameterized by modular functions. Therefore the curve $\mathcal{F} : y^2 = x(x - a^p)(x + b^p)$ cannot possibly exist as violates Theorem 3.5, hence Fermat's Last Theorem holds. Further details regarding the history and Modularity Theorem can be found in [9]. The full proof of Fermat's Last Theorem due to Wiles and Taylor can be found in [11] and [12].

CHAPTER 4: CONCLUSION

In our investigation of elliptic curves we have covered a variety of topics ranging from the group law via the secant-tangent process to applications of elliptic curves in Fermat's Last Theorem. While most of the theorems we have presented are known, we have made every attempt possible to offer new insights, additional details, and, in some cases, new proofs entirely. The best example of such is our proof of associativity. We have used Maple to present an elementary proof of associativity which is unique to us. We hope this thesis will inspire some to explore the theory of elliptic curves and their numerous applications in greater detail.

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APPENDIX

The following is the printout of the Maple computations for $X_1 - X_2$ and $Y_1 - Y_2$ which is used in the proof of associativity.

$factor(X_1 - X_2)$

$$
\begin{aligned}
&((x_1^3 x_2 - x_1^3 x_3 - x_1 x_2^3 + x_1 x_3^3 + x_2^3 x_3 - x_2 x_3^3 + x_1 y_2^2 - x_1 y_3^2 - x_2 y_1^2 + x_2 y_3^2 + x_3 y_1^2 \qquad \textbf{(1)}\\
&\quad - x_3 y_2^2)\,(2 x_1^6 x_2^3 - 6 x_1^6 x_2^2 x_3 + 6 x_1^6 x_2 x_3^2 - 2 x_1^6 x_3^3 + 6 x_1^5 x_2^3 x_3 - 18 x_1^5 x_2^2 x_3^2\\
&\quad + 18 x_1^5 x_2 x_3^3 - 6 x_1^5 x_3^4 - 6 x_1^4 x_2^5 + 12 x_1^4 x_2^4 x_3 + 6 x_1^4 x_2^3 x_3^2 - 30 x_1^4 x_2^2 x_3^3\\
&\quad + 24 x_1^4 x_2 x_3^4 - 6 x_1^4 x_3^5 - 12 x_1^3 x_2^5 x_3 + 24 x_1^3 x_2^4 x_3^2 + 2 x_1^3 x_2^3 x_3^3 - 30 x_1^3 x_2^2 x_3^4\\
&\quad + 18 x_1^3 x_2 x_3^5 - 2 x_1^3 x_3^6 + 6 x_1^2 x_2^7 - 6 x_1^2 x_2^6 x_3 - 18 x_1^2 x_2^5 x_3^2 + 24 x_1^2 x_2^4 x_3^3\\
&\quad + 6 x_1^2 x_2^3 x_3^4 - 18 x_1^2 x_2^2 x_3^5 + 6 x_1^2 x_2 x_3^6 + 6 x_1 x_2^7 x_3 - 6 x_1 x_2^6 x_3^2 - 12 x_1 x_2^5 x_3^3\\
&\quad + 12 x_1 x_2^4 x_3^4 + 6 x_1 x_2^3 x_3^5 - 6 x_1 x_2^2 x_3^6 - 2 x_2^9 + 6 x_2^7 x_3^2 - 6 x_2^5 x_3^4 + 2 x_2^3 x_3^6\\
&\quad - 3 x_1^5 x_2 y_2^2 + 6 x_1^5 x_2 y_2 y_3 - 3 x_1^5 x_2 y_3^2 + 3 x_1^5 x_3 y_2^2 - 6 x_1^5 x_3 y_2 y_3 + 3 x_1^5 x_3 y_3^2\\
&\quad + 2 x_1^4 x_2^2 y_2^2 - 8 x_1^4 x_2^2 y_2 y_3 + 6 x_1^4 x_2^2 y_3^2 - 4 x_1^4 x_2 x_3 y_2^2 + 16 x_1^4 x_2 x_3 y_2 y_3\\
&\quad - 12 x_1^4 x_2 x_3 y_3^2 + 2 x_1^4 x_3^2 y_2^2 - 8 x_1^4 x_3^2 y_2 y_3 + 6 x_1^4 x_3^2 y_3^2 - 3 x_1^3 x_2^3 y_1^2 + 8 x_1^3 x_2^3 y_1 y_2\\
&\quad - 6 x_1^3 x_2^3 y_1 y_3 + 3 x_1^3 x_2^3 y_2^2 - 2 x_1^3 x_2^3 y_2 y_3 + 9 x_1^3 x_2^2 x_3 y_1^2 - 18 x_1^3 x_2^2 x_3 y_1 y_2\\
&\quad + 12 x_1^3 x_2^2 x_3 y_1 y_3 + 4 x_1^3 x_2^2 x_3 y_2^2 - 22 x_1^3 x_2^2 x_3 y_2 y_3 + 15 x_1^3 x_2^2 x_3 y_3^2 - 9 x_1^3 x_2 x_3^2 y_1^2\\
&\quad + 12 x_1^3 x_2 x_3^2 y_1 y_2 - 6 x_1^3 x_2 x_3^2 y_1 y_3 - 5 x_1^3 x_2 x_3^2 y_2^2 + 26 x_1^3 x_2 x_3^2 y_2 y_3 - 18 x_1^3 x_2 x_3^2 y_3^2\\
&\quad + 3 x_1^3 x_3^3 y_1^2 - 2 x_1^3 x_3^3 y_1 y_2 - 2 x_1^3 x_3^3 y_2^2 - 2 x_1^3 x_3^3 y_2 y_3 + 3 x_1^3 x_3^3 y_3^2 - 2 x_1^2 x_2^4 y_1 y_2\\
&\quad + 6 x_1^2 x_2^4 y_1 y_3 - 4 x_1^2 x_2^4 y_2^2 + 6 x_1^2 x_2^4 y_2 y_3 - 6 x_1^2 x_2^4 y_3^2 - 6 x_1^2 x_2^3 x_3 y_1^2\\
&\quad + 14 x_1^2 x_2^3 x_3 y_1 y_2 - 18 x_1^2 x_2^3 x_3 y_1 y_3 + x_1^2 x_2^3 x_3 y_2^2 + 12 x_1^2 x_2^3 x_3 y_2 y_3 - 3 x_1^2 x_2^3 x_3 y_3^2\\
&\quad + 18 x_1^2 x_2^2 x_3^2 y_1^2 - 30 x_1^2 x_2^2 x_3^2 y_1 y_2 + 18 x_1^2 x_2^2 x_3^2 y_1 y_3 + 6 x_1^2 x_2^2 x_3^2 y_2^2\\
&\quad - 30 x_1^2 x_2^2 x_3^2 y_2 y_3 + 18 x_1^2 x_2^2 x_3^2 y_3^2 - 18 x_1^2 x_2 x_3^3 y_1^2 + 26 x_1^2 x_2 x_3^3 y_1 y_2\\
&\quad - 6 x_1^2 x_2 x_3^3 y_1 y_3 - 5 x_1^2 x_2 x_3^3 y_2^2 + 12 x_1^2 x_2 x_3^3 y_2 y_3 - 9 x_1^2 x_2 x_3^3 y_3^2 + 6 x_1^2 x_3^4 y_1^2\\
&\quad - 8 x_1^2 x_3^4 y_1 y_2 + 2 x_1^2 x_3^4 y_2^2 + 3 x_1 x_2^5 y_1^2 - 8 x_1 x_2^5 y_1 y_2 + 6 x_1 x_2^5 y_1 y_3 - 4 x_1 x_2^5 y_2 y_3\\
&\quad + 3 x_1 x_2^5 y_3^2 - 6 x_1 x_2^4 x_3 y_1^2 + 8 x_1 x_2^4 x_3 y_1 y_2 - 4 x_1 x_2^4 x_3 y_2^2 + 8 x_1 x_2^4 x_3 y_2 y_3\\
&\quad - 6 x_1 x_2^4 x_3 y_3^2 - 3 x_1 x_2^3 x_3^2 y_1^2 + 12 x_1 x_2^3 x_3^2 y_1 y_2 - 18 x_1 x_2^3 x_3^2 y_1 y_3 + x_1 x_2^3 x_3^2 y_2^2\\
&\quad + 14 x_1 x_2^3 x_3^2 y_2 y_3 - 6 x_1 x_2^3 x_3^2 y_3^2 + 15 x_1 x_2^2 x_3^3 y_1^2 - 22 x_1 x_2^2 x_3^3 y_1 y_2\\
&\quad + 12 x_1 x_2^2 x_3^3 y_1 y_3 + 4 x_1 x_2^2 x_3^3 y_2^2 - 18 x_1 x_2^2 x_3^3 y_2 y_3 + 9 x_1 x_2^2 x_3^3 y_3^2 - 12 x_1 x_2 x_3^4 y_1^2\\
&\quad + 16 x_1 x_2 x_3^4 y_1 y_2 - 4 x_1 x_2 x_3^4 y_2^2 + 3 x_1 x_3^5 y_1^2 - 6 x_1 x_3^5 y_1 y_2 + 3 x_1 x_3^5 y_2^2 + 2 x_2^6 y_1 y_2\\
&\quad - 6 x_2^6 y_1 y_3 + 2 x_2^6 y_2^2 + 2 x_2^6 y_2 y_3 + 3 x_2^5 x_3 y_1^2 - 4 x_2^5 x_3 y_1 y_2 + 6 x_2^5 x_3 y_1 y_3\\
&\quad - 8 x_2^5 x_3 y_2 y_3 + 3 x_2^5 x_3 y_3^2 - 6 x_2^4 x_3^2 y_1^2 + 6 x_2^4 x_3^2 y_1 y_2 + 6 x_2^4 x_3^2 y_1 y_3 - 4 x_2^4 x_3^2 y_2^2\\
&\quad - 2 x_2^4 x_3^2 y_2 y_3 - 2 x_2^3 x_3^3 y_1 y_2 - 6 x_2^3 x_3^3 y_1 y_3 + 3 x_2^3 x_3^3 y_2^2 + 8 x_2^3 x_3^3 y_2 y_3 - 3 x_2^3 x_3^3 y_3^2\\
&\quad + 6 x_2^2 x_3^4 y_1^2 - 8 x_2^2 x_3^4 y_1 y_2 + 2 x_2^2 x_3^4 y_2^2 - 3 x_2 x_3^5 y_1^2 + 6 x_2 x_3^5 y_1 y_2 - 3 x_2 x_3^5 y_2^2
\end{aligned}
$$

$$-2x_1^3y_1y_2^3+6x_1^3y_1y_2^2y_3-6x_1^3y_1y_2y_3^2+2x_1^3y_1y_3^3+3x_1^3y_2^4-8x_1^3y_2^3y_3$$
$$+6x_1^3y_2^2y_3^2-x_1^3y_3^4+3x_1^2x_2y_1^2y_2^2-6x_1^2x_2y_1^2y_2y_3+3x_1^2x_2y_1^2y_3^2-4x_1^2x_2y_1y_2^3$$
$$+8x_1^2x_2y_1y_2^2y_3-4x_1^2x_2y_1y_2y_3^2-2x_1^2x_2y_2^4+4x_1^2x_2y_2^3y_3+x_1^2x_2y_2^2y_3^2$$
$$-6x_1^2x_2y_2y_3^3+3x_1^2x_2y_3^4-3x_1^2x_3y_1^2y_2^2+6x_1^2x_3y_1^2y_2y_3-3x_1^2x_3y_1^2y_3^2$$
$$+4x_1^2x_3y_1y_2^3-8x_1^2x_3y_1y_2^2y_3+4x_1^2x_3y_1y_2y_3^2-x_1^2x_3y_2^4+2x_1^2x_3y_2^3y_3$$
$$-x_1^2x_3y_2^2y_3^2-x_1x_2^2y_1^2y_2^2+4x_1x_2^2y_1^2y_2y_3-3x_1x_2^2y_1^2y_3^2+4x_1x_2^2y_1y_2^3$$
$$-18x_1x_2^2y_1y_2^2y_3+20x_1x_2^2y_1y_2y_3^2-6x_1x_2^2y_1y_3^3+8x_1x_2^2y_2^3y_3-17x_1x_2^2y_2^2y_3^2$$
$$+12x_1x_2^2y_2y_3^3-3x_1x_2^2y_3^4+2x_1x_2x_3y_1^2y_2^2-8x_1x_2x_3y_1^2y_2y_3+6x_1x_2x_3y_1^2y_3^2$$
$$+8x_1x_2x_3y_1y_2^2y_3-8x_1x_2x_3y_1y_2y_3^2-2x_1x_2x_3y_2^4+2x_1x_2x_3y_2^2y_3^2-x_1x_3^2y_1^2y_2^2$$
$$+4x_1x_3^2y_1^2y_2y_3-3x_1x_3^2y_1^2y_3^2+2x_1x_3^2y_1y_2^3-8x_1x_3^2y_1y_2^2y_3+6x_1x_3^2y_1y_2y_3^2$$
$$-x_1x_3^2y_2^4+4x_1x_3^2y_2^3y_3-3x_1x_3^2y_2^2y_3^2+x_2^3y_1^4-6x_2^3y_1^3y_2+4x_2^3y_1^3y_3$$
$$+10x_2^3y_1^2y_2^2-10x_2^3y_1^2y_2y_3-8x_2^3y_1y_2^3+16x_2^3y_1y_2^2y_3-10x_2^3y_1y_2y_3^2$$
$$+4x_2^3y_1y_3^3+2x_2^3y_2^4-8x_2^3y_2^3y_3+10x_2^3y_2^2y_3^2-6x_2^3y_2y_3^3+x_2^3y_3^4-3x_2^2x_3y_1^4$$
$$+12x_2^2x_3y_1^3y_2-6x_2^2x_3y_1^3y_3-17x_2^2x_3y_1^2y_2^2+20x_2^2x_3y_1^2y_2y_3-3x_2^2x_3y_1^2y_3^2$$
$$+8x_2^2x_3y_1y_2^3-18x_2^2x_3y_1y_2^2y_3+4x_2^2x_3y_1y_2y_3^2+4x_2^2x_3y_2^3y_3-x_2^2x_3y_2^2y_3^2$$
$$+3x_2x_3^2y_1^4-6x_2x_3^2y_1^3y_2+x_2x_3^2y_1^2y_2^2-4x_2x_3^2y_1^2y_2y_3+3x_2x_3^2y_1^2y_3^2$$
$$+4x_2x_3^2y_1y_2^3+8x_2x_3^2y_1y_2^2y_3-6x_2x_3^2y_1y_2y_3^2-2x_2x_3^2y_2^4-4x_2x_3^2y_2^3y_3$$
$$+3x_2x_3^2y_2^2y_3^2-x_3^3y_1^4+2x_3^3y_1^3y_3+6x_3^3y_1^2y_2^2-6x_3^3y_1^2y_2y_3-8x_3^3y_1y_2^3$$
$$+6x_3^3y_1y_2^2y_3+3x_3^3y_2^4-2x_3^3y_2^3y_3+2y_1^3y_2^3-6y_1^3y_2^2y_3+6y_1^3y_2y_3^2-2y_1^3y_3^3$$
$$-6y_1^2y_2^4+18y_1^2y_2^3y_3-18y_1^2y_2^2y_3^2+6y_1^2y_2y_3^3+6y_1y_2^5-18y_1y_2^4y_3+18y_1y_2^3y_3^2$$
$$-6y_1y_2^2y_3^3-2y_2^6+6y_2^5y_3-6y_2^4y_3^2+2y_2^3y_3^3)\Big)\Big/\Big(\big(x_1x_2^2-2x_1x_2x_3+x_1x_3^2$$
$$+x_2^3-x_2^2x_3-x_2x_3^2+x_3^3-y_2^2+2y_2y_3-y_3^2\big)^2\big(x_1^3-x_1^2x_2+x_1^2x_3-x_1x_2^2$$
$$-2x_1x_2x_3+x_2^3+x_2^2x_3-y_1^2+2y_1y_2-y_2^2\big)^2\Big)$$

$$factor(Y_1-Y_2)$$

$$-\Big(\big(x_1^3x_2-x_1^3x_3-x_1x_2^3+x_1x_3^3+x_2^3x_3-x_2x_3^3+x_1y_2^2-x_1y_3^2-x_2y_1^2+x_2y_3^2+x_3y_1^2 \tag{2}$$
$$-x_3y_2^2\big)\big(6x_1^9x_2^5y_3-6x_1^9x_2^4x_3y_2-24x_1^9x_2^4x_3y_3+24x_1^9x_2^3x_3^2y_2+36x_1^9x_2^3x_3^2y_3$$
$$-36x_1^9x_2^2x_3^3y_2-24x_1^9x_2^2x_3^3y_3+24x_1^9x_2x_3^4y_2+6x_1^9x_2x_3^4y_3-6x_1^9x_3^5y_2$$
$$-6x_1^8x_2^6y_1+6x_1^8x_2^6y_2-6x_1^8x_2^6y_3+30x_1^8x_2^5x_3y_1-24x_1^8x_2^5x_3y_2+48x_1^8x_2^5x_3y_3$$
$$-60x_1^8x_2^4x_3^2y_1+12x_1^8x_2^4x_3^2y_2-132x_1^8x_2^4x_3^2y_3+60x_1^8x_2^3x_3^3y_1+72x_1^8x_2^3x_3^3y_2$$

$$+168\,x_1^8x_2^3x_3^3y_3-30\,x_1^8x_2^2x_3^4y_1-138\,x_1^8x_2^2x_3^4y_2-102\,x_1^8x_2^2x_3^4y_3+6\,x_1^8x_2x_3^5y_1$$
$$+96\,x_1^8x_2x_3^5y_2+24\,x_1^8x_2x_3^5y_3-24\,x_1^8x_3^6y_2-24\,x_1^7x_2^7y_3-24\,x_1^7x_2^6x_3y_1$$
$$+48\,x_1^7x_2^6x_3y_2+48\,x_1^7x_2^6x_3y_3+120\,x_1^7x_2^5x_3^2y_1-168\,x_1^7x_2^5x_3^2y_2+84\,x_1^7x_2^5x_3^2y_3$$
$$-240\,x_1^7x_2^4x_3^3y_1+156\,x_1^7x_2^4x_3^3y_2-336\,x_1^7x_2^4x_3^3y_3+240\,x_1^7x_2^3x_3^4y_1$$
$$+96\,x_1^7x_2^3x_3^4y_2+384\,x_1^7x_2^3x_3^4y_3-120\,x_1^7x_2^2x_3^5y_1-264\,x_1^7x_2^2x_3^5y_2$$
$$-192\,x_1^7x_2^2x_3^5y_3+24\,x_1^7x_2x_3^6y_1+168\,x_1^7x_2x_3^6y_2+36\,x_1^7x_2x_3^6y_3-36\,x_1^7x_3^7y_2$$
$$+24\,x_1^6x_2^8y_1-24\,x_1^6x_2^8y_2+24\,x_1^6x_2^8y_3-96\,x_1^6x_2^7x_3y_1+72\,x_1^6x_2^7x_3y_2$$
$$-144\,x_1^6x_2^7x_3y_3+84\,x_1^6x_2^6x_3^2y_1+60\,x_1^6x_2^6x_3^2y_2+228\,x_1^6x_2^6x_3^2y_3+180\,x_1^6x_2^5x_3^3y_1$$
$$-408\,x_1^6x_2^5x_3^3y_2+72\,x_1^6x_2^5x_3^3y_3-480\,x_1^6x_2^4x_3^4y_1+408\,x_1^6x_2^4x_3^4y_2$$
$$-528\,x_1^6x_2^4x_3^4y_3+456\,x_1^6x_2^3x_3^5y_1+72\,x_1^6x_2^3x_3^5y_2+528\,x_1^6x_2^3x_3^5y_3$$
$$-204\,x_1^6x_2^2x_3^6y_1-324\,x_1^6x_2^2x_3^6y_2-204\,x_1^6x_2^2x_3^6y_3+36\,x_1^6x_2x_3^7y_1+168\,x_1^6x_2x_3^7y_2$$
$$+24\,x_1^6x_2x_3^7y_3-24\,x_1^6x_3^8y_2+36\,x_1^5x_2^9y_3+72\,x_1^5x_2^8x_3y_1-108\,x_1^5x_2^8x_3y_2$$
$$-288\,x_1^5x_2^7x_3^2y_1+288\,x_1^5x_2^7x_3^2y_2-324\,x_1^5x_2^7x_3^2y_3+336\,x_1^5x_2^6x_3^3y_1$$
$$-12\,x_1^5x_2^6x_3^3y_2+480\,x_1^5x_2^6x_3^3y_3+120\,x_1^5x_2^5x_3^4y_1-600\,x_1^5x_2^5x_3^4y_2$$
$$+66\,x_1^5x_2^5x_3^4y_3-600\,x_1^5x_2^4x_3^5y_1+534\,x_1^5x_2^4x_3^5y_2-600\,x_1^5x_2^4x_3^5y_3$$
$$+528\,x_1^5x_2^3x_3^6y_1+72\,x_1^5x_2^3x_3^6y_2+456\,x_1^5x_2^3x_3^6y_3-192\,x_1^5x_2^2x_3^7y_1$$
$$-264\,x_1^5x_2^2x_3^7y_2-120\,x_1^5x_2^2x_3^7y_3+24\,x_1^5x_2x_3^8y_1+96\,x_1^5x_2x_3^8y_2+6\,x_1^5x_2x_3^8y_3$$
$$-6\,x_1^5x_3^9y_2-36\,x_1^4x_2^{10}y_1+36\,x_1^4x_2^{10}y_2-36\,x_1^4x_2^{10}y_3+108\,x_1^4x_2^9x_3y_1$$
$$-72\,x_1^4x_2^9x_3y_2+144\,x_1^4x_2^9x_3y_3+36\,x_1^4x_2^8x_3^2y_1-180\,x_1^4x_2^8x_3^2y_2-36\,x_1^4x_2^8x_3^2y_3$$
$$-468\,x_1^4x_2^7x_3^3y_1+504\,x_1^4x_2^7x_3^3y_2-456\,x_1^4x_2^7x_3^3y_3+534\,x_1^4x_2^6x_3^4y_1$$
$$-78\,x_1^4x_2^6x_3^4y_2+534\,x_1^4x_2^6x_3^4y_3+66\,x_1^4x_2^5x_3^5y_1-600\,x_1^4x_2^5x_3^5y_2$$
$$+120\,x_1^4x_2^5x_3^5y_3-528\,x_1^4x_2^4x_3^6y_1+408\,x_1^4x_2^4x_3^6y_2-480\,x_1^4x_2^4x_3^6y_3$$
$$+384\,x_1^4x_2^3x_3^7y_1+96\,x_1^4x_2^3x_3^7y_2+240\,x_1^4x_2^3x_3^7y_3-102\,x_1^4x_2^2x_3^8y_1$$
$$-138\,x_1^4x_2^2x_3^8y_2-30\,x_1^4x_2^2x_3^8y_3+6\,x_1^4x_2x_3^9y_1+24\,x_1^4x_2x_3^9y_2-24\,x_1^3x_2^{11}y_3$$
$$-72\,x_1^3x_2^{10}x_3y_1+96\,x_1^3x_2^{10}x_3y_2-48\,x_1^3x_2^{10}x_3y_3+216\,x_1^3x_2^9x_3^2y_1-168\,x_1^3x_2^9x_3^2y_2$$
$$+252\,x_1^3x_2^9x_3^2y_3-48\,x_1^3x_2^8x_3^3y_1-204\,x_1^3x_2^8x_3^3y_2-48\,x_1^3x_2^8x_3^3y_3$$
$$-456\,x_1^3x_2^7x_3^4y_1+504\,x_1^3x_2^7x_3^4y_2-468\,x_1^3x_2^7x_3^4y_3+480\,x_1^3x_2^6x_3^5y_1$$
$$-12\,x_1^3x_2^6x_3^5y_2+336\,x_1^3x_2^6x_3^5y_3+72\,x_1^3x_2^5x_3^6y_1-408\,x_1^3x_2^5x_3^6y_2$$
$$+180\,x_1^3x_2^5x_3^6y_3-336\,x_1^3x_2^4x_3^7y_1+156\,x_1^3x_2^4x_3^7y_2-240\,x_1^3x_2^4x_3^7y_3$$

$$+ 168 x_1^3 x_2^3 x_3^8 y_1 + 72 x_1^3 x_2^3 x_3^8 y_2 + 60 x_1^3 x_2^3 x_3^8 y_3 - 24 x_1^3 x_2^2 x_3^9 y_1 - 36 x_1^3 x_2^2 x_3^9 y_2$$
$$+ 24 x_1^2 x_2^{12} y_1 - 24 x_1^2 x_2^{12} y_2 + 24 x_1^2 x_2^{12} y_3 - 48 x_1^2 x_2^{11} x_3 y_1 + 24 x_1^2 x_2^{11} x_3 y_2$$
$$- 48 x_1^2 x_2^{11} x_3 y_3 - 84 x_1^2 x_2^{10} x_3^2 y_1 + 132 x_1^2 x_2^{10} x_3^2 y_2 - 84 x_1^2 x_2^{10} x_3^2 y_3$$
$$+ 252 x_1^2 x_2^9 x_3^3 y_1 - 168 x_1^2 x_2^9 x_3^3 y_2 + 216 x_1^2 x_2^9 x_3^3 y_3 - 36 x_1^2 x_2^8 x_3^4 y_1$$
$$- 180 x_1^2 x_2^8 x_3^4 y_2 + 36 x_1^2 x_2^8 x_3^4 y_3 - 324 x_1^2 x_2^7 x_3^5 y_1 + 288 x_1^2 x_2^7 x_3^5 y_2$$
$$- 288 x_1^2 x_2^7 x_3^5 y_3 + 228 x_1^2 x_2^6 x_3^6 y_1 + 60 x_1^2 x_2^6 x_3^6 y_2 + 84 x_1^2 x_2^6 x_3^6 y_3 + 84 x_1^2 x_2^5 x_3^7 y_1$$
$$- 168 x_1^2 x_2^5 x_3^7 y_2 + 120 x_1^2 x_2^5 x_3^7 y_3 - 132 x_1^2 x_2^4 x_3^8 y_1 + 12 x_1^2 x_2^4 x_3^8 y_2$$
$$- 60 x_1^2 x_2^4 x_3^8 y_3 + 36 x_1^2 x_2^3 x_3^9 y_1 + 24 x_1^2 x_2^3 x_3^9 y_2 + 6 x_1 x_2^{13} y_3 + 24 x_1 x_2^{12} x_3 y_1$$
$$- 30 x_1 x_2^{12} x_3 y_2 + 24 x_1 x_2^{12} x_3 y_3 - 48 x_1 x_2^{11} x_3^2 y_1 + 24 x_1 x_2^{11} x_3^2 y_2 - 48 x_1 x_2^{11} x_3^2 y_3$$
$$- 48 x_1 x_2^{10} x_3^3 y_1 + 96 x_1 x_2^{10} x_3^3 y_2 - 72 x_1 x_2^{10} x_3^3 y_3 + 144 x_1 x_2^9 x_3^4 y_1 - 72 x_1 x_2^9 x_3^4 y_2$$
$$+ 108 x_1 x_2^9 x_3^4 y_3 - 108 x_1 x_2^8 x_3^5 y_2 + 72 x_1 x_2^8 x_3^5 y_3 - 144 x_1 x_2^7 x_3^6 y_1 + 72 x_1 x_2^7 x_3^6 y_2$$
$$- 96 x_1 x_2^7 x_3^6 y_3 + 48 x_1 x_2^6 x_3^7 y_1 + 48 x_1 x_2^6 x_3^7 y_2 - 24 x_1 x_2^6 x_3^7 y_3 + 48 x_1 x_2^5 x_3^8 y_1$$
$$- 24 x_1 x_2^5 x_3^8 y_2 + 30 x_1 x_2^5 x_3^8 y_3 - 24 x_1 x_2^4 x_3^9 y_1 - 6 x_1 x_2^4 x_3^9 y_2 - 6 x_2^{14} y_1 + 6 x_2^{14} y_2$$
$$- 6 x_2^{14} y_3 + 6 x_2^{13} x_3 y_1 + 24 x_2^{12} x_3^2 y_1 - 24 x_2^{12} x_3^2 y_2 + 24 x_2^{12} x_3^2 y_3 - 24 x_2^{11} x_3^3 y_1$$
$$- 36 x_2^{10} x_3^4 y_1 + 36 x_2^{10} x_3^4 y_2 - 36 x_2^{10} x_3^4 y_3 + 36 x_2^9 x_3^5 y_1 + 24 x_2^8 x_3^6 y_1 - 24 x_2^8 x_3^6 y_2$$
$$+ 24 x_2^8 x_3^6 y_3 - 24 x_2^7 x_3^7 y_1 - 6 x_2^6 x_3^8 y_1 + 6 x_2^6 x_3^8 y_2 - 6 x_2^6 x_3^8 y_3 + 6 x_2^5 x_3^9 y_1$$
$$+ 2 x_1^9 x_2^2 y_2^3 - 6 x_1^9 x_2^2 y_2^2 y_3 + 6 x_1^9 x_2^2 y_2 y_3^2 - 2 x_1^9 x_2^2 y_3^3 - 4 x_1^9 x_2 x_3 y_2^3$$
$$+ 12 x_1^9 x_2 x_3 y_2^2 y_3 - 12 x_1^9 x_2 x_3 y_2 y_3^2 + 4 x_1^9 x_2 x_3 y_3^3 + 2 x_1^9 x_3^2 y_2^3 - 6 x_1^9 x_3^2 y_2^2 y_3$$
$$+ 6 x_1^9 x_3^2 y_2 y_3^2 - 2 x_1^9 x_3^2 y_3^3 + 3 x_1^8 x_2^3 y_1 y_2^2 - 6 x_1^8 x_2^3 y_1 y_2 y_3 + 3 x_1^8 x_2^3 y_1 y_3^2$$
$$- 6 x_1^8 x_2^3 y_2^3 + 3 x_1^8 x_2^3 y_2^2 y_3 + 12 x_1^8 x_2^3 y_2 y_3^2 - 9 x_1^8 x_2^3 y_3^3 - 9 x_1^8 x_2^2 x_3 y_1 y_2^2$$
$$+ 18 x_1^8 x_2^2 x_3 y_1 y_2 y_3 - 9 x_1^8 x_2^2 x_3 y_1 y_3^2 + 33 x_1^8 x_2^2 x_3 y_2^3 - 54 x_1^8 x_2^2 x_3 y_2^2 y_3$$
$$+ 9 x_1^8 x_2^2 x_3 y_2 y_3^2 + 12 x_1^8 x_2^2 x_3 y_3^3 + 9 x_1^8 x_2 x_3^2 y_1 y_2^2 - 18 x_1^8 x_2 x_3^2 y_1 y_2 y_3$$
$$+ 9 x_1^8 x_2 x_3^2 y_1 y_3^2 - 48 x_1^8 x_2 x_3^2 y_2^3 + 99 x_1^8 x_2 x_3^2 y_2^2 y_3 - 54 x_1^8 x_2 x_3^2 y_2 y_3^2$$
$$+ 3 x_1^8 x_2 x_3^2 y_3^3 - 3 x_1^8 x_3^3 y_1 y_2^2 + 6 x_1^8 x_3^3 y_1 y_2 y_3 - 3 x_1^8 x_3^3 y_1 y_3^2 + 21 x_1^8 x_3^3 y_2^3$$
$$- 48 x_1^8 x_3^3 y_2^2 y_3 + 33 x_1^8 x_3^3 y_2 y_3^2 - 6 x_1^8 x_3^3 y_3^3 + 12 x_1^7 x_2^4 y_1 y_2^2 - 24 x_1^7 x_2^4 y_1 y_2 y_3$$
$$+ 12 x_1^7 x_2^4 y_1 y_3^2 - 19 x_1^7 x_2^4 y_2^3 + 63 x_1^7 x_2^4 y_2^2 y_3 - 81 x_1^7 x_2^4 y_2 y_3^2 + 37 x_1^7 x_2^4 y_3^3$$
$$- 24 x_1^7 x_2^3 x_3 y_1 y_2^2 + 48 x_1^7 x_2^3 x_3 y_1 y_2 y_3 - 24 x_1^7 x_2^3 x_3 y_1 y_3^2 + 4 x_1^7 x_2^3 x_3 y_2^3$$
$$- 54 x_1^7 x_2^3 x_3 y_2^2 y_3 + 144 x_1^7 x_2^3 x_3 y_2 y_3^2 - 94 x_1^7 x_2^3 x_3 y_3^3 + 90 x_1^7 x_2^2 x_3^2 y_2^3$$
$$- 180 x_1^7 x_2^2 x_3^2 y_2^2 y_3 + 18 x_1^7 x_2^2 x_3^2 y_2 y_3^2 + 72 x_1^7 x_2^2 x_3^2 y_3^3 + 24 x_1^7 x_2 x_3^3 y_1 y_2^2$$

$$- 48 x_1^7 x_2 x_3^3 y_1 y_2 y_3 + 24 x_1^7 x_2 x_3^3 y_1 y_3^2 - 116 x_1^7 x_2 x_3^3 y_2^3 + 270 x_1^7 x_2 x_3^3 y_2^2 y_3$$
$$- 144 x_1^7 x_2 x_3^3 y_2 y_3^2 - 10 x_1^7 x_2 x_3^3 y_3^3 - 12 x_1^7 x_3^4 y_1 y_2^2 + 24 x_1^7 x_3^4 y_1 y_2 y_3$$
$$- 12 x_1^7 x_3^4 y_1 y_3^2 + 41 x_1^7 x_3^4 y_2^3 - 99 x_1^7 x_3^4 y_2^2 y_3 + 63 x_1^7 x_3^4 y_2 y_3^2 - 5 x_1^7 x_3^4 y_3^3$$
$$- x_1^6 x_2^5 y_1^3 + 6 x_1^6 x_2^5 y_1^2 y_2 - 15 x_1^6 x_2^5 y_1^2 y_3 - 30 x_1^6 x_2^5 y_1 y_2^2 + 84 x_1^6 x_2^5 y_1 y_2 y_3$$
$$- 45 x_1^6 x_2^5 y_1 y_3^2 + 38 x_1^6 x_2^5 y_2^3 - 66 x_1^6 x_2^5 y_2^2 y_3 + 36 x_1^6 x_2^5 y_2 y_3^2 - 7 x_1^6 x_2^5 y_3^3$$
$$+ 5 x_1^6 x_2^4 x_3 y_1^3 - 9 x_1^6 x_2^4 x_3 y_1^2 y_2 + 54 x_1^6 x_2^4 x_3 y_1^2 y_3 + 75 x_1^6 x_2^4 x_3 y_1 y_2^2$$
$$- 276 x_1^6 x_2^4 x_3 y_1 y_2 y_3 + 156 x_1^6 x_2^4 x_3 y_1 y_3^2 - 141 x_1^6 x_2^4 x_3 y_2^3 + 330 x_1^6 x_2^4 x_3 y_2^2 y_3$$
$$- 330 x_1^6 x_2^4 x_3 y_2 y_3^2 + 136 x_1^6 x_2^4 x_3 y_3^3 - 10 x_1^6 x_2^3 x_3^2 y_1^3 - 24 x_1^6 x_2^3 x_3^2 y_1^2 y_2$$
$$- 66 x_1^6 x_2^3 x_3^2 y_1^2 y_3 - 9 x_1^6 x_2^3 x_3^2 y_1 y_2^2 + 282 x_1^6 x_2^3 x_3^2 y_1 y_2 y_3 - 183 x_1^6 x_2^3 x_3^2 y_1 y_3^2$$
$$+ 64 x_1^6 x_2^3 x_3^2 y_2^3 - 213 x_1^6 x_2^3 x_3^2 y_2^2 y_3 + 426 x_1^6 x_2^3 x_3^2 y_2 y_3^2 - 267 x_1^6 x_2^3 x_3^2 y_3^3$$
$$+ 10 x_1^6 x_2^2 x_3^3 y_1^3 + 66 x_1^6 x_2^2 x_3^3 y_1^2 y_2 + 24 x_1^6 x_2^2 x_3^3 y_1^2 y_3 - 123 x_1^6 x_2^2 x_3^3 y_1 y_2^2$$
$$- 30 x_1^6 x_2^2 x_3^3 y_1 y_2 y_3 + 63 x_1^6 x_2^2 x_3^3 y_1 y_3^2 + 175 x_1^6 x_2^2 x_3^3 y_2^3 - 384 x_1^6 x_2^2 x_3^3 y_2^2 y_3$$
$$+ 45 x_1^6 x_2^2 x_3^3 y_2 y_3^2 + 154 x_1^6 x_2^2 x_3^3 y_3^3 - 5 x_1^6 x_2 x_3^4 y_1^3 - 54 x_1^6 x_2 x_3^4 y_1^2 y_2$$
$$+ 9 x_1^6 x_2 x_3^4 y_1^2 y_3 + 123 x_1^6 x_2 x_3^4 y_1 y_2^2 - 102 x_1^6 x_2 x_3^4 y_1 y_2 y_3 + 24 x_1^6 x_2 x_3^4 y_1 y_3^2$$
$$- 168 x_1^6 x_2 x_3^4 y_2^3 + 417 x_1^6 x_2 x_3^4 y_2^2 y_3 - 228 x_1^6 x_2 x_3^4 y_2 y_3^2 - 16 x_1^6 x_2 x_3^4 y_3^3 + x_1^6 x_3^5 y_1^3$$
$$+ 15 x_1^6 x_3^5 y_1^2 y_2 - 6 x_1^6 x_3^5 y_1^2 y_3 - 36 x_1^6 x_3^5 y_1 y_2^2 + 42 x_1^6 x_3^5 y_1 y_2 y_3 - 15 x_1^6 x_3^5 y_1 y_3^2$$
$$+ 32 x_1^6 x_3^5 y_2^3 - 84 x_1^6 x_3^5 y_2^2 y_3 + 51 x_1^6 x_3^5 y_2 y_3^2 + 12 x_1^5 x_2^6 y_1^3 - 39 x_1^5 x_2^6 y_1^2 y_2$$
$$+ 21 x_1^5 x_2^6 y_1^2 y_3 + 18 x_1^5 x_2^6 y_1 y_2 y_3 + 6 x_1^5 x_2^6 y_1 y_3^2 + 32 x_1^5 x_2^6 y_2^3 - 132 x_1^5 x_2^6 y_2^2 y_3$$
$$+ 153 x_1^5 x_2^6 y_2 y_3^2 - 71 x_1^5 x_2^6 y_3^3 - 60 x_1^5 x_2^5 x_3 y_1^3 + 180 x_1^5 x_2^5 x_3 y_1^2 y_2$$
$$- 126 x_1^5 x_2^5 x_3 y_1^2 y_3 - 126 x_1^5 x_2^5 x_3 y_1 y_2^2 + 216 x_1^5 x_2^5 x_3 y_1 y_2 y_3 - 162 x_1^5 x_2^5 x_3 y_1 y_3^2$$
$$+ 114 x_1^5 x_2^5 x_3 y_2^3 - 144 x_1^5 x_2^5 x_3 y_2^2 y_3 + 90 x_1^5 x_2^5 x_3 y_2 y_3^2 + 18 x_1^5 x_2^5 x_3 y_3^3$$
$$+ 120 x_1^5 x_2^4 x_3^2 y_1^3 - 279 x_1^5 x_2^4 x_3^2 y_1^2 y_2 + 279 x_1^5 x_2^4 x_3^2 y_1^2 y_3 + 288 x_1^5 x_2^4 x_3^2 y_1 y_2^2$$
$$- 738 x_1^5 x_2^4 x_3^2 y_1 y_2 y_3 + 450 x_1^5 x_2^4 x_3^2 y_1 y_3^2 - 360 x_1^5 x_2^4 x_3^2 y_2^3 + 756 x_1^5 x_2^4 x_3^2 y_2^2 y_3$$
$$- 819 x_1^5 x_2^4 x_3^2 y_2 y_3^2 + 303 x_1^5 x_2^4 x_3^2 y_3^3 - 120 x_1^5 x_2^3 x_3^3 y_1^3 + 96 x_1^5 x_2^3 x_3^3 y_1^2 y_2$$
$$- 276 x_1^5 x_2^3 x_3^3 y_1^2 y_3 - 24 x_1^5 x_2^3 x_3^3 y_1 y_2^2 + 696 x_1^5 x_2^3 x_3^3 y_1 y_2 y_3 - 432 x_1^5 x_2^3 x_3^3 y_1 y_3^2$$
$$+ 122 x_1^5 x_2^3 x_3^3 y_2^3 - 282 x_1^5 x_2^3 x_3^3 y_2^2 y_3 + 606 x_1^5 x_2^3 x_3^3 y_2 y_3^2 - 386 x_1^5 x_2^3 x_3^3 y_3^3$$
$$+ 60 x_1^5 x_2^2 x_3^4 y_1^3 + 171 x_1^5 x_2^2 x_3^4 y_1^2 y_2 + 99 x_1^5 x_2^2 x_3^4 y_1^2 y_3 - 396 x_1^5 x_2^2 x_3^4 y_1 y_2^2$$
$$- 90 x_1^5 x_2^2 x_3^4 y_1 y_2 y_3 + 126 x_1^5 x_2^2 x_3^4 y_1 y_3^2 + 252 x_1^5 x_2^2 x_3^4 y_2^3 - 504 x_1^5 x_2^2 x_3^4 y_2^2 y_3$$
$$+ 135 x_1^5 x_2^2 x_3^4 y_2 y_3^2 + 147 x_1^5 x_2^2 x_3^4 y_3^3 - 12 x_1^5 x_2 x_3^5 y_1^3 - 180 x_1^5 x_2 x_3^5 y_1^2 y_2$$

$$+ 18 x_1^5 x_2 x_3^5 y_1^2 y_3 + 342 x_1^5 x_2 x_3^5 y_1 y_2^2 - 144 x_1^5 x_2 x_3^5 y_1 y_2 y_3 + 18 x_1^5 x_2 x_3^5 y_1 y_3^2$$
$$- 192 x_1^5 x_2 x_3^5 y_2^3 + 342 x_1^5 x_2 x_3^5 y_2^2 y_3 - 180 x_1^5 x_2 x_3^5 y_2 y_3^2 - 12 x_1^5 x_2 x_3^5 y_3^3$$
$$+ 51 x_1^5 x_3^6 y_1^2 y_2 - 15 x_1^5 x_3^6 y_1^2 y_3 - 84 x_1^5 x_3^6 y_1 y_2^2 + 42 x_1^5 x_3^6 y_1 y_2 y_3 - 6 x_1^5 x_3^6 y_1 y_3^2$$
$$+ 32 x_1^5 x_3^6 y_2^3 - 36 x_1^5 x_3^6 y_2^2 y_3 + 15 x_1^5 x_3^6 y_2 y_3^2 + x_1^5 x_3^6 y_3^3 + 8 x_1^4 x_2^7 y_1^3$$
$$- 30 x_1^4 x_2^7 y_1^2 y_2 + 24 x_1^4 x_2^7 y_1^2 y_3 + 87 x_1^4 x_2^7 y_1 y_2^2 - 162 x_1^4 x_2^7 y_1 y_2 y_3 + 75 x_1^4 x_2^7 y_1 y_3^2$$
$$- 82 x_1^4 x_2^7 y_2^3 + 159 x_1^4 x_2^7 y_2^2 y_3 - 126 x_1^4 x_2^7 y_2 y_3^2 + 47 x_1^4 x_2^7 y_3^3 + x_1^4 x_2^6 x_3 y_1^3$$
$$- 27 x_1^4 x_2^6 x_3 y_1^2 y_2 + 6 x_1^4 x_2^6 x_3 y_1^2 y_3 - 105 x_1^4 x_2^6 x_3 y_1 y_2^2 + 252 x_1^4 x_2^6 x_3 y_1 y_2 y_3$$
$$- 96 x_1^4 x_2^6 x_3 y_1 y_3^2 + 165 x_1^4 x_2^6 x_3 y_2^3 - 390 x_1^4 x_2^6 x_3 y_2^2 y_3 + 378 x_1^4 x_2^6 x_3 y_2 y_3^2$$
$$- 184 x_1^4 x_2^6 x_3 y_3^3 - 120 x_1^4 x_2^5 x_3^2 y_1^3 + 432 x_1^4 x_2^5 x_3^2 y_1^2 y_2 - 288 x_1^4 x_2^5 x_3^2 y_1^2 y_3$$
$$- 360 x_1^4 x_2^5 x_3^2 y_1 y_2^2 + 468 x_1^4 x_2^5 x_3^2 y_1 y_2 y_3 - 288 x_1^4 x_2^5 x_3^2 y_1 y_3^2 + 216 x_1^4 x_2^5 x_3^2 y_2^3$$
$$- 324 x_1^4 x_2^5 x_3^2 y_2^2 y_3 + 252 x_1^4 x_2^5 x_3^2 y_2 y_3^2 + 12 x_1^4 x_2^5 x_3^2 y_3^3 + 305 x_1^4 x_2^4 x_3^3 y_1^3$$
$$- 795 x_1^4 x_2^4 x_3^3 y_1^2 y_2 + 600 x_1^4 x_2^4 x_3^3 y_1^2 y_3 + 720 x_1^4 x_2^4 x_3^3 y_1 y_2^2 - 1266 x_1^4 x_2^4 x_3^3 y_1 y_2 y_3$$
$$+ 681 x_1^4 x_2^4 x_3^3 y_1 y_3^2 - 560 x_1^4 x_2^4 x_3^3 y_2^3 + 996 x_1^4 x_2^4 x_3^3 y_2^2 y_3 - 1071 x_1^4 x_2^4 x_3^3 y_2 y_3^2$$
$$+ 390 x_1^4 x_2^4 x_3^3 y_3^3 - 320 x_1^4 x_2^3 x_3^4 y_1^3 + 450 x_1^4 x_2^3 x_3^4 y_1^2 y_2 - 480 x_1^4 x_2^3 x_3^4 y_1^2 y_3$$
$$- 156 x_1^4 x_2^3 x_3^4 y_1 y_2^2 + 876 x_1^4 x_2^3 x_3^4 y_1 y_2 y_3 - 480 x_1^4 x_2^3 x_3^4 y_1 y_3^2 + 136 x_1^4 x_2^3 x_3^4 y_2^3$$
$$- 156 x_1^4 x_2^3 x_3^4 y_2^2 y_3 + 450 x_1^4 x_2^3 x_3^4 y_2 y_3^2 - 320 x_1^4 x_2^3 x_3^4 y_3^3 + 147 x_1^4 x_2^2 x_3^5 y_1^3$$
$$+ 135 x_1^4 x_2^2 x_3^5 y_1^2 y_2 + 126 x_1^4 x_2^2 x_3^5 y_1^2 y_3 - 504 x_1^4 x_2^2 x_3^5 y_1 y_2^2 - 90 x_1^4 x_2^2 x_3^5 y_1 y_2 y_3$$
$$+ 99 x_1^4 x_2^2 x_3^5 y_1 y_3^2 + 252 x_1^4 x_2^2 x_3^5 y_2^3 - 396 x_1^4 x_2^2 x_3^5 y_2^2 y_3 + 171 x_1^4 x_2^2 x_3^5 y_2 y_3^2$$
$$+ 60 x_1^4 x_2^2 x_3^5 y_3^3 - 16 x_1^4 x_2 x_3^6 y_1^3 - 228 x_1^4 x_2 x_3^6 y_1^2 y_2 + 24 x_1^4 x_2 x_3^6 y_1^2 y_3$$
$$+ 417 x_1^4 x_2 x_3^6 y_1 y_2^2 - 102 x_1^4 x_2 x_3^6 y_1 y_2 y_3 + 9 x_1^4 x_2 x_3^6 y_1 y_3^2 - 168 x_1^4 x_2 x_3^6 y_2^3$$
$$+ 123 x_1^4 x_2 x_3^6 y_2^2 y_3 - 54 x_1^4 x_2 x_3^6 y_2 y_3^2 - 5 x_1^4 x_2 x_3^6 y_3^3 - 5 x_1^4 x_3^7 y_1^3 + 63 x_1^4 x_3^7 y_1^2 y_2$$
$$- 12 x_1^4 x_3^7 y_1^2 y_3 - 99 x_1^4 x_3^7 y_1 y_2^2 + 24 x_1^4 x_3^7 y_1 y_2 y_3 + 41 x_1^4 x_3^7 y_2^3 - 12 x_1^4 x_3^7 y_2^2 y_3$$
$$- 24 x_1^3 x_2^8 y_1^3 + 78 x_1^3 x_2^8 y_1^2 y_2 - 42 x_1^3 x_2^8 y_1^2 y_3 - 36 x_1^3 x_2^8 y_1 y_2^2 + 36 x_1^3 x_2^8 y_1 y_2 y_3$$
$$- 48 x_1^3 x_2^8 y_1 y_3^2 - 15 x_1^3 x_2^8 y_2^3 + 99 x_1^3 x_2^8 y_2^2 y_3 - 87 x_1^3 x_2^8 y_2 y_3^2 + 39 x_1^3 x_2^8 y_3^3$$
$$+ 106 x_1^3 x_2^7 x_3 y_1^3 - 306 x_1^3 x_2^7 x_3 y_1^2 y_2 + 168 x_1^3 x_2^7 x_3 y_1^2 y_3 + 282 x_1^3 x_2^7 x_3 y_1 y_2^2$$
$$- 360 x_1^3 x_2^7 x_3 y_1 y_2 y_3 + 228 x_1^3 x_2^7 x_3 y_1 y_3^2 - 206 x_1^3 x_2^7 x_3 y_2^3 + 318 x_1^3 x_2^7 x_3 y_2^2 y_3$$
$$- 348 x_1^3 x_2^7 x_3 y_2 y_3^2 + 118 x_1^3 x_2^7 x_3 y_3^3 - 130 x_1^3 x_2^6 x_3^2 y_1^3 + 243 x_1^3 x_2^6 x_3^2 y_1^2 y_2$$
$$- 123 x_1^3 x_2^6 x_3^2 y_1^2 y_3 - 336 x_1^3 x_2^6 x_3^2 y_1 y_2^2 + 486 x_1^3 x_2^6 x_3^2 y_1 y_2 y_3 - 192 x_1^3 x_2^6 x_3^2 y_1 y_3^2$$
$$+ 307 x_1^3 x_2^6 x_3^2 y_2^3 - 549 x_1^3 x_2^6 x_3^2 y_2^2 y_3 + 552 x_1^3 x_2^6 x_3^2 y_2 y_3^2 - 258 x_1^3 x_2^6 x_3^2 y_3^3$$

$$
\begin{aligned}
&-94\,x_1^3x_2^5x_3^3y_1^3+516\,x_1^3x_2^5x_3^3y_1^2y_2-336\,x_1^3x_2^5x_3^3y_1^2y_3-462\,x_1^3x_2^5x_3^3y_1y_2^2\\
&+480\,x_1^3x_2^5x_3^3y_1y_2y_3-336\,x_1^3x_2^5x_3^3y_1y_3^2+272\,x_1^3x_2^5x_3^3y_2^3-462\,x_1^3x_2^5x_3^3y_2^2y_3\\
&+516\,x_1^3x_2^5x_3^3y_2y_3^2-94\,x_1^3x_2^5x_3^3y_3^3+390\,x_1^3x_2^4x_3^4y_1^3-1071\,x_1^3x_2^4x_3^4y_1^2y_2\\
&+681\,x_1^3x_2^4x_3^4y_1^2y_3+996\,x_1^3x_2^4x_3^4y_1y_2^2-1266\,x_1^3x_2^4x_3^4y_1y_2y_3+600\,x_1^3x_2^4x_3^4y_1y_3^2\\
&-560\,x_1^3x_2^4x_3^4y_2^3+720\,x_1^3x_2^4x_3^4y_2^2y_3-795\,x_1^3x_2^4x_3^4y_2y_3^2+305\,x_1^3x_2^4x_3^4y_3^3\\
&-386\,x_1^3x_2^3x_3^5y_1^3+606\,x_1^3x_2^3x_3^5y_1^2y_2-432\,x_1^3x_2^3x_3^5y_1^2y_3-282\,x_1^3x_2^3x_3^5y_1y_2^2\\
&+696\,x_1^3x_2^3x_3^5y_1y_2y_3-276\,x_1^3x_2^3x_3^5y_1y_3^2+122\,x_1^3x_2^3x_3^5y_2^3-24\,x_1^3x_2^3x_3^5y_2^2y_3\\
&+96\,x_1^3x_2^3x_3^5y_2y_3^2-120\,x_1^3x_2^3x_3^5y_3^3+154\,x_1^3x_2^2x_3^6y_1^3+45\,x_1^3x_2^2x_3^6y_1^2y_2\\
&+63\,x_1^3x_2^2x_3^6y_1^2y_3-384\,x_1^3x_2^2x_3^6y_1y_2^2-30\,x_1^3x_2^2x_3^6y_1y_2y_3+24\,x_1^3x_2^2x_3^6y_1y_3^2\\
&+175\,x_1^3x_2^2x_3^6y_2^3-123\,x_1^3x_2^2x_3^6y_2^2y_3+66\,x_1^3x_2^2x_3^6y_2y_3^2+10\,x_1^3x_2^2x_3^6y_3^3\\
&-10\,x_1^3x_2x_3^7y_1^3-144\,x_1^3x_2x_3^7y_1^2y_2+24\,x_1^3x_2x_3^7y_1^2y_3+270\,x_1^3x_2x_3^7y_1y_2^2\\
&-48\,x_1^3x_2x_3^7y_1y_2y_3-116\,x_1^3x_2x_3^7y_2^3+24\,x_1^3x_2x_3^7y_2^2y_3-6\,x_1^3x_3^8y_1^3\\
&+33\,x_1^3x_3^8y_1^2y_2-3\,x_1^3x_3^8y_1^2y_3-48\,x_1^3x_3^8y_1y_2^2+6\,x_1^3x_3^8y_1y_2y_3+21\,x_1^3x_3^8y_2^3\\
&-3\,x_1^3x_3^8y_2^2y_3-13\,x_1^2x_2^9y_1^3+42\,x_1^2x_2^9y_1^2y_2-3\,x_1^2x_2^9y_1^2y_3-96\,x_1^2x_2^9y_1y_2^2\\
&+96\,x_1^2x_2^9y_1y_2y_3-27\,x_1^2x_2^9y_1y_3^2+74\,x_1^2x_2^9y_2^3-132\,x_1^2x_2^9y_2^2y_3+96\,x_1^2x_2^9y_2y_3^2\\
&-37\,x_1^2x_2^9y_3^3-3\,x_1^2x_2^8x_3y_1^3+27\,x_1^2x_2^8x_3y_1^2y_2-90\,x_1^2x_2^8x_3y_1^2y_3\\
&+63\,x_1^2x_2^8x_3y_1y_2^2+36\,x_1^2x_2^8x_3y_1y_2y_3-72\,x_1^2x_2^8x_3y_1y_3^2-57\,x_1^2x_2^8x_3y_2^3\\
&+90\,x_1^2x_2^8x_3y_2^2y_3-18\,x_1^2x_2^8x_3y_2y_3^2+24\,x_1^2x_2^8x_3y_3^3+153\,x_1^2x_2^7x_3^2y_1^3\\
&-450\,x_1^2x_2^7x_3^2y_1^2y_2+297\,x_1^2x_2^7x_3^2y_1^2y_3+450\,x_1^2x_2^7x_3^2y_1y_2^2-612\,x_1^2x_2^7x_3^2y_1y_2y_3\\
&+297\,x_1^2x_2^7x_3^2y_1y_3^2-288\,x_1^2x_2^7x_3^2y_2^3+450\,x_1^2x_2^7x_3^2y_2^2y_3-450\,x_1^2x_2^7x_3^2y_2y_3^2\\
&+153\,x_1^2x_2^7x_3^2y_3^3-258\,x_1^2x_2^6x_3^3y_1^3+552\,x_1^2x_2^6x_3^3y_1^2y_2-192\,x_1^2x_2^6x_3^3y_1^2y_3\\
&-549\,x_1^2x_2^6x_3^3y_1y_2^2+486\,x_1^2x_2^6x_3^3y_1y_2y_3-123\,x_1^2x_2^6x_3^3y_1y_3^2+307\,x_1^2x_2^6x_3^3y_2^3\\
&-336\,x_1^2x_2^6x_3^3y_2^2y_3+243\,x_1^2x_2^6x_3^3y_2y_3^2-130\,x_1^2x_2^6x_3^3y_3^3+12\,x_1^2x_2^5x_3^4y_1^3\\
&+252\,x_1^2x_2^5x_3^4y_1^2y_2-288\,x_1^2x_2^5x_3^4y_1^2y_3-324\,x_1^2x_2^5x_3^4y_1y_2^2+468\,x_1^2x_2^5x_3^4y_1y_2y_3\\
&-288\,x_1^2x_2^5x_3^4y_1y_3^2+216\,x_1^2x_2^5x_3^4y_2^3-360\,x_1^2x_2^5x_3^4y_2^2y_3+432\,x_1^2x_2^5x_3^4y_2y_3^2\\
&-120\,x_1^2x_2^5x_3^4y_3^3+303\,x_1^2x_2^4x_3^5y_1^3-819\,x_1^2x_2^4x_3^5y_1^2y_2+450\,x_1^2x_2^4x_3^5y_1^2y_3\\
&+756\,x_1^2x_2^4x_3^5y_1y_2^2-738\,x_1^2x_2^4x_3^5y_1y_2y_3+279\,x_1^2x_2^4x_3^5y_1y_3^2-360\,x_1^2x_2^4x_3^5y_2^3\\
&+288\,x_1^2x_2^4x_3^5y_2^2y_3-279\,x_1^2x_2^4x_3^5y_2y_3^2+120\,x_1^2x_2^4x_3^5y_3^3-267\,x_1^2x_2^3x_3^6y_1^3\\
&+426\,x_1^2x_2^3x_3^6y_1^2y_2-183\,x_1^2x_2^3x_3^6y_1^2y_3-213\,x_1^2x_2^3x_3^6y_1y_2^2+282\,x_1^2x_2^3x_3^6y_1y_2y_3
\end{aligned}
$$

$$- 66 x_1^2 x_2^3 x_3^6 y_1 y_3^2 + 64 x_1^2 x_2^3 x_3^6 y_2^3 - 9 x_1^2 x_2^3 x_3^6 y_2^2 y_3 - 24 x_1^2 x_2^3 x_3^6 y_2 y_3^2$$
$$- 10 x_1^2 x_2^3 x_3^6 y_3^3 + 72 x_1^2 x_2^2 x_3^7 y_1^3 + 18 x_1^2 x_2^2 x_3^7 y_1^2 y_2 - 180 x_1^2 x_2^2 x_3^7 y_1 y_2^2$$
$$+ 90 x_1^2 x_2^2 x_3^7 y_2^3 + 3 x_1^2 x_2 x_3^8 y_1^3 - 54 x_1^2 x_2 x_3^8 y_1^2 y_2 + 9 x_1^2 x_2 x_3^8 y_1^2 y_3$$
$$+ 99 x_1^2 x_2 x_3^8 y_1 y_2^2 - 18 x_1^2 x_2 x_3^8 y_1 y_2 y_3 - 48 x_1^2 x_2 x_3^8 y_2^3 + 9 x_1^2 x_2 x_3^8 y_2^2 y_3$$
$$- 2 x_1^2 x_3^9 y_1^3 + 6 x_1^2 x_3^9 y_1^2 y_2 - 6 x_1^2 x_3^9 y_1 y_2^2 + 2 x_1^2 x_3^9 y_2^3 + 12 x_1 x_2^{10} y_1^3$$
$$- 39 x_1 x_2^{10} y_1^2 y_2 + 21 x_1 x_2^{10} y_1^2 y_3 + 24 x_1 x_2^{10} y_1 y_2^2 - 30 x_1 x_2^{10} y_1 y_2 y_3 + 30 x_1 x_2^{10} y_1 y_3^2$$
$$- 24 x_1 x_2^{10} y_2^2 y_3 + 9 x_1 x_2^{10} y_2 y_3^2 - 3 x_1 x_2^{10} y_3^3 - 46 x_1 x_2^9 x_3 y_1^3 + 126 x_1 x_2^9 x_3 y_1^2 y_2$$
$$- 42 x_1 x_2^9 x_3 y_1^2 y_3 - 132 x_1 x_2^9 x_3 y_1 y_2^2 + 96 x_1 x_2^9 x_3 y_1 y_2 y_3 - 42 x_1 x_2^9 x_3 y_1 y_3^2$$
$$+ 92 x_1 x_2^9 x_3 y_2^3 - 132 x_1 x_2^9 x_3 y_2^2 y_3 + 126 x_1 x_2^9 x_3 y_2 y_3^2 - 46 x_1 x_2^9 x_3 y_3^3$$
$$+ 24 x_1 x_2^8 x_3^2 y_1^3 - 18 x_1 x_2^8 x_3^2 y_1^2 y_2 - 72 x_1 x_2^8 x_3^2 y_1^2 y_3 + 90 x_1 x_2^8 x_3^2 y_1 y_2^2$$
$$+ 36 x_1 x_2^8 x_3^2 y_1 y_2 y_3 - 90 x_1 x_2^8 x_3^2 y_1 y_3^2 - 57 x_1 x_2^8 x_3^2 y_2^3 + 63 x_1 x_2^8 x_3^2 y_2^2 y_3$$
$$+ 27 x_1 x_2^8 x_3^2 y_2 y_3^2 - 3 x_1 x_2^8 x_3^2 y_3^3 + 118 x_1 x_2^7 x_3^3 y_1^3 - 348 x_1 x_2^7 x_3^3 y_1^2 y_2$$
$$+ 228 x_1 x_2^7 x_3^3 y_1^2 y_3 + 318 x_1 x_2^7 x_3^3 y_1 y_2^2 - 360 x_1 x_2^7 x_3^3 y_1 y_2 y_3 + 168 x_1 x_2^7 x_3^3 y_1 y_3^2$$
$$- 206 x_1 x_2^7 x_3^3 y_2^3 + 282 x_1 x_2^7 x_3^3 y_2^2 y_3 - 306 x_1 x_2^7 x_3^3 y_2 y_3^2 + 106 x_1 x_2^7 x_3^3 y_3^3$$
$$- 184 x_1 x_2^6 x_3^4 y_1^3 + 378 x_1 x_2^6 x_3^4 y_1^2 y_2 - 96 x_1 x_2^6 x_3^4 y_1^2 y_3 - 390 x_1 x_2^6 x_3^4 y_1 y_2^2$$
$$+ 252 x_1 x_2^6 x_3^4 y_1 y_2 y_3 + 6 x_1 x_2^6 x_3^4 y_1 y_3^2 + 165 x_1 x_2^6 x_3^4 y_2^3 - 105 x_1 x_2^6 x_3^4 y_2^2 y_3$$
$$- 27 x_1 x_2^6 x_3^4 y_2 y_3^2 + x_1 x_2^6 x_3^4 y_3^3 + 18 x_1 x_2^5 x_3^5 y_1^3 + 90 x_1 x_2^5 x_3^5 y_1^2 y_2$$
$$- 162 x_1 x_2^5 x_3^5 y_1^2 y_3 - 144 x_1 x_2^5 x_3^5 y_1 y_2^2 + 216 x_1 x_2^5 x_3^5 y_1 y_2 y_3 - 126 x_1 x_2^5 x_3^5 y_1 y_3^2$$
$$+ 114 x_1 x_2^5 x_3^5 y_2^3 - 126 x_1 x_2^5 x_3^5 y_2^2 y_3 + 180 x_1 x_2^5 x_3^5 y_2 y_3^2 - 60 x_1 x_2^5 x_3^5 y_3^3$$
$$+ 136 x_1 x_2^4 x_3^6 y_1^3 - 330 x_1 x_2^4 x_3^6 y_1^2 y_2 + 156 x_1 x_2^4 x_3^6 y_1^2 y_3 + 330 x_1 x_2^4 x_3^6 y_1 y_2^2$$
$$- 276 x_1 x_2^4 x_3^6 y_1 y_2 y_3 + 54 x_1 x_2^4 x_3^6 y_1 y_3^2 - 141 x_1 x_2^4 x_3^6 y_2^3 + 75 x_1 x_2^4 x_3^6 y_2^2 y_3$$
$$- 9 x_1 x_2^4 x_3^6 y_2 y_3^2 + 5 x_1 x_2^4 x_3^6 y_3^3 - 94 x_1 x_2^3 x_3^7 y_1^3 + 144 x_1 x_2^3 x_3^7 y_1^2 y_2$$
$$- 24 x_1 x_2^3 x_3^7 y_1^2 y_3 - 54 x_1 x_2^3 x_3^7 y_1 y_2^2 + 48 x_1 x_2^3 x_3^7 y_1 y_2 y_3 + 4 x_1 x_2^3 x_3^7 y_2^3$$
$$- 24 x_1 x_2^3 x_3^7 y_2^2 y_3 + 12 x_1 x_2^2 x_3^8 y_1^3 + 9 x_1 x_2^2 x_3^8 y_1^2 y_2 - 9 x_1 x_2^2 x_3^8 y_1^2 y_3$$
$$- 54 x_1 x_2^2 x_3^8 y_1 y_2^2 + 18 x_1 x_2^2 x_3^8 y_1 y_2 y_3 + 33 x_1 x_2^2 x_3^8 y_2^3 - 9 x_1 x_2^2 x_3^8 y_2^2 y_3$$
$$+ 4 x_1 x_2 x_3^9 y_1^3 - 12 x_1 x_2 x_3^9 y_1^2 y_2 + 12 x_1 x_2 x_3^9 y_1 y_2^2 - 4 x_1 x_2 x_3^9 y_2^3 + 6 x_2^{11} y_1^3$$
$$- 18 x_2^{11} y_1^2 y_2 - 6 x_2^{11} y_1^2 y_3 + 36 x_2^{11} y_1 y_2^2 - 12 x_2^{11} y_1 y_2 y_3 - 6 x_2^{11} y_1 y_3^2 - 24 x_2^{11} y_2^3$$
$$+ 36 x_2^{11} y_2^2 y_3 - 18 x_2^{11} y_2 y_3^2 + 6 x_2^{11} y_3^3 - 3 x_2^{10} x_3 y_1^3 + 9 x_2^{10} x_3 y_1^2 y_2 + 30 x_2^{10} x_3 y_1^2 y_3$$
$$- 24 x_2^{10} x_3 y_1 y_2^2 - 30 x_2^{10} x_3 y_1 y_2 y_3 + 21 x_2^{10} x_3 y_1 y_3^2 + 24 x_2^{10} x_3 y_2^2 y_3 - 39 x_2^{10} x_3 y_2 y_3^2$$

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\begin{aligned}
&+ 12 x_2^{10} x_3 y_3^3 - 37 x_2^9 x_3^2 y_1^3 + 96 x_2^9 x_3^2 y_1^2 y_2 - 27 x_2^9 x_3^2 y_1^2 y_3 - 132 x_2^9 x_3^2 y_1 y_2^2 \\
&+ 96 x_2^9 x_3^2 y_1 y_2 y_3 - 3 x_2^9 x_3^2 y_1 y_3^2 + 74 x_2^9 x_3^2 y_2^3 - 96 x_2^9 x_3^2 y_2^2 y_3 + 42 x_2^9 x_3^2 y_2 y_3^2 \\
&- 13 x_2^9 x_3^2 y_3^3 + 39 x_2^8 x_3^3 y_1^3 - 87 x_2^8 x_3^3 y_1^2 y_2 - 48 x_2^8 x_3^3 y_1^2 y_3 + 99 x_2^8 x_3^3 y_1 y_2^2 \\
&+ 36 x_2^8 x_3^3 y_1 y_2 y_3 - 42 x_2^8 x_3^3 y_1 y_3^2 - 15 x_2^8 x_3^3 y_2^3 - 36 x_2^8 x_3^3 y_2^2 y_3 + 78 x_2^8 x_3^3 y_2 y_3^2 \\
&- 24 x_2^8 x_3^3 y_3^3 + 47 x_2^7 x_3^4 y_1^3 - 126 x_2^7 x_3^4 y_1^2 y_2 + 75 x_2^7 x_3^4 y_1^2 y_3 + 159 x_2^7 x_3^4 y_1 y_2^2 \\
&- 162 x_2^7 x_3^4 y_1 y_2 y_3 + 24 x_2^7 x_3^4 y_1 y_3^2 - 82 x_2^7 x_3^4 y_2^3 + 87 x_2^7 x_3^4 y_2^2 y_3 - 30 x_2^7 x_3^4 y_2 y_3^2 \\
&+ 8 x_2^7 x_3^4 y_3^3 - 71 x_2^6 x_3^5 y_1^3 + 153 x_2^6 x_3^5 y_1^2 y_2 + 6 x_2^6 x_3^5 y_1^2 y_3 - 132 x_2^6 x_3^5 y_1 y_2^2 \\
&+ 18 x_2^6 x_3^5 y_1 y_2 y_3 + 21 x_2^6 x_3^5 y_1 y_3^2 + 32 x_2^6 x_3^5 y_2^3 - 39 x_2^6 x_3^5 y_2 y_3^2 + 12 x_2^6 x_3^5 y_3^3 \\
&- 7 x_2^5 x_3^6 y_1^3 + 36 x_2^5 x_3^6 y_1^2 y_2 - 45 x_2^5 x_3^6 y_1^2 y_3 - 66 x_2^5 x_3^6 y_1 y_2^2 + 84 x_2^5 x_3^6 y_1 y_2 y_3 \\
&- 15 x_2^5 x_3^6 y_1 y_3^2 + 38 x_2^5 x_3^6 y_2^3 - 30 x_2^5 x_3^6 y_2^2 y_3 + 6 x_2^5 x_3^6 y_2 y_3^2 - x_2^5 x_3^6 y_3^3 \\
&+ 37 x_2^4 x_3^7 y_1^3 - 81 x_2^4 x_3^7 y_1^2 y_2 + 12 x_2^4 x_3^7 y_1^2 y_3 + 63 x_2^4 x_3^7 y_1 y_2^2 - 24 x_2^4 x_3^7 y_1 y_2 y_3 \\
&- 19 x_2^4 x_3^7 y_2^3 + 12 x_2^4 x_3^7 y_2^2 y_3 - 9 x_2^3 x_3^8 y_1^3 + 12 x_2^3 x_3^8 y_1^2 y_2 + 3 x_2^3 x_3^8 y_1^2 y_3 \\
&+ 3 x_2^3 x_3^8 y_1 y_2^2 - 6 x_2^3 x_3^8 y_1 y_2 y_3 - 6 x_2^3 x_3^8 y_2^3 + 3 x_2^3 x_3^8 y_2^2 y_3 - 2 x_2^2 x_3^9 y_1^3 \\
&+ 6 x_2^2 x_3^9 y_1^2 y_2 - 6 x_2^2 x_3^9 y_1 y_2^2 + 2 x_2^2 x_3^9 y_2^3 - 3 x_1^8 y_2^5 + 15 x_1^8 y_2^4 y_3 - 30 x_1^8 y_2^3 y_3^2 \\
&+ 30 x_1^8 y_2^2 y_3^3 - 15 x_1^8 y_2 y_3^4 + 3 x_1^8 y_3^5 - 6 x_1^7 x_2 y_1 y_2^4 + 24 x_1^7 x_2 y_1 y_2^3 y_3 \\
&- 36 x_1^7 x_2 y_1 y_2^2 y_3^2 + 24 x_1^7 x_2 y_1 y_2 y_3^3 - 6 x_1^7 x_2 y_1 y_3^4 + 13 x_1^7 x_2 y_2^5 - 57 x_1^7 x_2 y_2^4 y_3 \\
&+ 98 x_1^7 x_2 y_2^3 y_3^2 - 82 x_1^7 x_2 y_2^2 y_3^3 + 33 x_1^7 x_2 y_2 y_3^4 - 5 x_1^7 x_2 y_3^5 + 6 x_1^7 x_3 y_1 y_2^4 \\
&- 24 x_1^7 x_3 y_1 y_2^3 y_3 + 36 x_1^7 x_3 y_1 y_2^2 y_3^2 - 24 x_1^7 x_3 y_1 y_2 y_3^3 + 6 x_1^7 x_3 y_1 y_3^4 - 13 x_1^7 x_3 y_2^5 \\
&+ 57 x_1^7 x_3 y_2^4 y_3 - 98 x_1^7 x_3 y_2^3 y_3^2 + 82 x_1^7 x_3 y_2^2 y_3^3 - 33 x_1^7 x_3 y_2 y_3^4 + 5 x_1^7 x_3 y_3^5 \\
&- 7 x_1^6 x_2^2 y_1^2 y_2^3 + 21 x_1^6 x_2^2 y_1^2 y_2^2 y_3 - 21 x_1^6 x_2^2 y_1^2 y_2 y_3^2 + 7 x_1^6 x_2^2 y_1^2 y_3^3 \\
&+ 15 x_1^6 x_2^2 y_1 y_2^4 - 60 x_1^6 x_2^2 y_1 y_2^3 y_3 + 90 x_1^6 x_2^2 y_1 y_2^2 y_3^2 - 60 x_1^6 x_2^2 y_1 y_2 y_3^3 \\
&+ 15 x_1^6 x_2^2 y_1 y_3^4 - 3 x_1^6 x_2^2 y_2^5 + 18 x_1^6 x_2^2 y_2^4 y_3 - 23 x_1^6 x_2^2 y_2^3 y_3^2 - 9 x_1^6 x_2^2 y_2^2 y_3^3 \\
&+ 30 x_1^6 x_2^2 y_2 y_3^4 - 13 x_1^6 x_2^2 y_3^5 + 14 x_1^6 x_2 x_3 y_1^2 y_2^3 - 42 x_1^6 x_2 x_3 y_1^2 y_2^2 y_3 \\
&+ 42 x_1^6 x_2 x_3 y_1^2 y_2 y_3^2 - 14 x_1^6 x_2 x_3 y_1^2 y_3^3 - 30 x_1^6 x_2 x_3 y_1 y_2^4 + 120 x_1^6 x_2 x_3 y_1 y_2^3 y_3 \\
&- 180 x_1^6 x_2 x_3 y_1 y_2^2 y_3^2 + 120 x_1^6 x_2 x_3 y_1 y_2 y_3^3 - 30 x_1^6 x_2 x_3 y_1 y_3^4 + 32 x_1^6 x_2 x_3 y_2^5 \\
&- 174 x_1^6 x_2 x_3 y_2^4 y_3 + 338 x_1^6 x_2 x_3 y_2^3 y_3^2 - 290 x_1^6 x_2 x_3 y_2^2 y_3^3 + 102 x_1^6 x_2 x_3 y_2 y_3^4 \\
&- 8 x_1^6 x_2 x_3 y_3^5 - 7 x_1^6 x_3^2 y_1^2 y_2^3 + 21 x_1^6 x_3^2 y_1^2 y_2^2 y_3 - 21 x_1^6 x_3^2 y_1^2 y_2 y_3^2 \\
&+ 7 x_1^6 x_3^2 y_1^2 y_3^3 + 15 x_1^6 x_3^2 y_1 y_2^4 - 60 x_1^6 x_3^2 y_1 y_2^3 y_3 + 90 x_1^6 x_3^2 y_1 y_2^2 y_3^2 \\
&- 60 x_1^6 x_3^2 y_1 y_2 y_3^3 + 15 x_1^6 x_3^2 y_1 y_3^4 - 8 x_1^6 x_3^2 y_2^5 + 51 x_1^6 x_3^2 y_2^4 y_3 - 105 x_1^6 x_3^2 y_2^3 y_3^2
\end{aligned}
$$

$$+ 89 x_1^6 x_3^2 y_2^2 y_3^3 - 27 x_1^6 x_3^2 y_2 y_3^4 - 6 x_1^5 x_2^3 y_1^3 y_2^2 + 12 x_1^5 x_2^3 y_1^3 y_2 y_3 - 6 x_1^5 x_2^3 y_1^3 y_3^2$$
$$+ 24 x_1^5 x_2^3 y_1^2 y_2^3 - 42 x_1^5 x_2^3 y_1^2 y_2^2 y_3 + 12 x_1^5 x_2^3 y_1^2 y_2 y_3^2 + 6 x_1^5 x_2^3 y_1^2 y_3^3$$
$$- 6 x_1^5 x_2^3 y_1 y_2^4 - 24 x_1^5 x_2^3 y_1 y_2^3 y_3 + 78 x_1^5 x_2^3 y_1 y_2^2 y_3^2 - 60 x_1^5 x_2^3 y_1 y_2 y_3^3$$
$$+ 12 x_1^5 x_2^3 y_1 y_3^4 - 37 x_1^5 x_2^3 y_2^5 + 141 x_1^5 x_2^3 y_2^4 y_3 - 230 x_1^5 x_2^3 y_2^3 y_3^2 + 220 x_1^5 x_2^3 y_2^2 y_3^3$$
$$- 129 x_1^5 x_2^3 y_2 y_3^4 + 35 x_1^5 x_2^3 y_3^5 + 18 x_1^5 x_2^2 x_3 y_1^3 y_2^2 - 36 x_1^5 x_2^2 x_3 y_1^3 y_2 y_3$$
$$+ 18 x_1^5 x_2^2 x_3 y_1^3 y_3^2 - 102 x_1^5 x_2^2 x_3 y_1^2 y_2^3 + 216 x_1^5 x_2^2 x_3 y_1^2 y_2^2 y_3$$
$$- 126 x_1^5 x_2^2 x_3 y_1^2 y_2 y_3^2 + 12 x_1^5 x_2^2 x_3 y_1^2 y_3^3 + 123 x_1^5 x_2^2 x_3 y_1 y_2^4 - 300 x_1^5 x_2^2 x_3 y_1 y_2^3 y_3$$
$$+ 252 x_1^5 x_2^2 x_3 y_1 y_2^2 y_3^2 - 96 x_1^5 x_2^2 x_3 y_1 y_2 y_3^3 + 21 x_1^5 x_2^2 x_3 y_1 y_3^4 - 24 x_1^5 x_2^2 x_3 y_2^5$$
$$+ 141 x_1^5 x_2^2 x_3 y_2^4 y_3 - 270 x_1^5 x_2^2 x_3 y_2^3 y_3^2 + 186 x_1^5 x_2^2 x_3 y_2^2 y_3^3 - 6 x_1^5 x_2^2 x_3 y_2 y_3^4$$
$$- 27 x_1^5 x_2^2 x_3 y_3^5 - 18 x_1^5 x_2 x_3^2 y_1^3 y_2^2 + 36 x_1^5 x_2 x_3^2 y_1^3 y_2 y_3 - 18 x_1^5 x_2 x_3^2 y_1^3 y_3^2$$
$$+ 132 x_1^5 x_2 x_3^2 y_1^2 y_2^3 - 306 x_1^5 x_2 x_3^2 y_1^2 y_2^2 y_3 + 216 x_1^5 x_2 x_3^2 y_1^2 y_2 y_3^2$$
$$- 42 x_1^5 x_2 x_3^2 y_1^2 y_3^3 - 192 x_1^5 x_2 x_3^2 y_1 y_2^4 + 528 x_1^5 x_2 x_3^2 y_1 y_2^3 y_3 - 522 x_1^5 x_2 x_3^2 y_1 y_2^2 y_3^2$$
$$+ 228 x_1^5 x_2 x_3^2 y_1 y_2 y_3^3 - 42 x_1^5 x_2 x_3^2 y_1 y_3^4 + 72 x_1^5 x_2 x_3^2 y_2^5 - 300 x_1^5 x_2 x_3^2 y_2^4 y_3$$
$$+ 480 x_1^5 x_2 x_3^2 y_2^3 y_3^2 - 342 x_1^5 x_2 x_3^2 y_2^2 y_3^3 + 84 x_1^5 x_2 x_3^2 y_2 y_3^4 + 6 x_1^5 x_2 x_3^2 y_3^5$$
$$+ 6 x_1^5 x_3^3 y_1^3 y_2^2 - 12 x_1^5 x_3^3 y_1^3 y_2 y_3 + 6 x_1^5 x_3^3 y_1^3 y_3^2 - 54 x_1^5 x_3^3 y_1^2 y_2^3$$
$$+ 132 x_1^5 x_3^3 y_1^2 y_2^2 y_3 - 102 x_1^5 x_3^3 y_1^2 y_2 y_3^2 + 24 x_1^5 x_3^3 y_1^2 y_3^3 + 75 x_1^5 x_3^3 y_1 y_2^4$$
$$- 204 x_1^5 x_3^3 y_1 y_2^3 y_3 + 192 x_1^5 x_3^3 y_1 y_2^2 y_3^2 - 72 x_1^5 x_3^3 y_1 y_2 y_3^3 + 9 x_1^5 x_3^3 y_1 y_3^4$$
$$- 23 x_1^5 x_3^3 y_2^5 + 78 x_1^5 x_3^3 y_2^4 y_3 - 100 x_1^5 x_3^3 y_2^3 y_3^2 + 56 x_1^5 x_3^3 y_2^2 y_3^3 - 9 x_1^5 x_3^3 y_2 y_3^4$$
$$- 2 x_1^5 x_3^3 y_3^5 - 23 x_1^4 x_2^4 y_1^3 y_2^2 + 48 x_1^4 x_2^4 y_1^3 y_2 y_3 - 25 x_1^4 x_2^4 y_1^3 y_3^2 + 82 x_1^4 x_2^4 y_1^2 y_2^3$$
$$- 207 x_1^4 x_2^4 y_1^2 y_2^2 y_3 + 174 x_1^4 x_2^4 y_1^2 y_2 y_3^2 - 49 x_1^4 x_2^4 y_1^2 y_3^3 - 96 x_1^4 x_2^4 y_1 y_2^4$$
$$+ 300 x_1^4 x_2^4 y_1 y_2^3 y_3 - 393 x_1^4 x_2^4 y_1 y_2^2 y_3^2 + 252 x_1^4 x_2^4 y_1 y_2 y_3^3 - 63 x_1^4 x_2^4 y_1 y_3^4$$
$$+ 52 x_1^4 x_2^4 y_2^5 - 156 x_1^4 x_2^4 y_2^4 y_3 + 214 x_1^4 x_2^4 y_2^3 y_3^2 - 173 x_1^4 x_2^4 y_2^2 y_3^3 + 78 x_1^4 x_2^4 y_2 y_3^4$$
$$- 15 x_1^4 x_2^4 y_3^5 + 44 x_1^4 x_2^3 x_3 y_1^3 y_2^2 - 96 x_1^4 x_2^3 x_3 y_1^3 y_2 y_3 + 52 x_1^4 x_2^3 x_3 y_1^3 y_3^2$$
$$- 106 x_1^4 x_2^3 x_3 y_1^2 y_2^3 + 282 x_1^4 x_2^3 x_3 y_1^2 y_2^2 y_3 - 270 x_1^4 x_2^3 x_3 y_1^2 y_2 y_3^2$$
$$+ 94 x_1^4 x_2^3 x_3 y_1^2 y_3^3 + 93 x_1^4 x_2^3 x_3 y_1 y_2^4 - 276 x_1^4 x_2^3 x_3 y_1 y_2^3 y_3 + 426 x_1^4 x_2^3 x_3 y_1 y_2^2 y_3^2$$
$$- 324 x_1^4 x_2^3 x_3 y_1 y_2 y_3^3 + 81 x_1^4 x_2^3 x_3 y_1 y_3^4 - 101 x_1^4 x_2^3 x_3 y_2^5 + 270 x_1^4 x_2^3 x_3 y_2^4 y_3$$
$$- 408 x_1^4 x_2^3 x_3 y_2^3 y_3^2 + 450 x_1^4 x_2^3 x_3 y_2^2 y_3^3 - 291 x_1^4 x_2^3 x_3 y_2 y_3^4 + 80 x_1^4 x_2^3 x_3 y_3^5$$
$$+ 6 x_1^4 x_2^2 x_3^2 y_1^3 y_2^2 - 6 x_1^4 x_2^2 x_3^2 y_1^3 y_3^2 - 144 x_1^4 x_2^2 x_3^2 y_1^2 y_2^3 + 306 x_1^4 x_2^2 x_3^2 y_1^2 y_2^2 y_3$$
$$- 144 x_1^4 x_2^2 x_3^2 y_1^2 y_2 y_3^2 - 18 x_1^4 x_2^2 x_3^2 y_1^2 y_3^3 + 213 x_1^4 x_2^2 x_3^2 y_1 y_2^4$$

$$- 588\, x_1^4 x_2^2 x_3^2 y_1 y_2^3 y_3 + 432\, x_1^4 x_2^2 x_3^2 y_1 y_2^2 y_3^2 - 60\, x_1^4 x_2^2 x_3^2 y_1 y_2 y_3^3$$
$$+ 3\, x_1^4 x_2^2 x_3^2 y_1 y_3^4 - 45\, x_1^4 x_2^2 x_3^2 y_2^5 + 312\, x_1^4 x_2^2 x_3^2 y_2^4 y_3 - 522\, x_1^4 x_2^2 x_3^2 y_2^3 y_3^2$$
$$+ 318\, x_1^4 x_2^2 x_3^2 y_2^2 y_3^3 - 33\, x_1^4 x_2^2 x_3^2 y_2 y_3^4 - 30\, x_1^4 x_2^2 x_3^2 y_3^5 - 52\, x_1^4 x_2 x_3^3 y_1^3 y_2^2$$
$$+ 96\, x_1^4 x_2 x_3^3 y_1^3 y_2 y_3 - 44\, x_1^4 x_2 x_3^3 y_1^3 y_3^2 + 278\, x_1^4 x_2 x_3^3 y_1^2 y_2^3 - 630\, x_1^4 x_2 x_3^3 y_1^2 y_2^2 y_3$$
$$+ 402\, x_1^4 x_2 x_3^3 y_1^2 y_2 y_3^2 - 50\, x_1^4 x_2 x_3^3 y_1^2 y_3^3 - 345\, x_1^4 x_2 x_3^3 y_1 y_2^4 + 900\, x_1^4 x_2 x_3^3 y_1 y_2^3 y_3$$
$$- 714\, x_1^4 x_2 x_3^3 y_1 y_2^2 y_3^2 + 180\, x_1^4 x_2 x_3^3 y_1 y_2 y_3^3 - 21\, x_1^4 x_2 x_3^3 y_1 y_3^4 + 99\, x_1^4 x_2 x_3^3 y_2^5$$
$$- 336\, x_1^4 x_2 x_3^3 y_2^4 y_3 + 376\, x_1^4 x_2 x_3^3 y_2^3 y_3^2 - 170\, x_1^4 x_2 x_3^3 y_2^2 y_3^3 + 21\, x_1^4 x_2 x_3^3 y_2 y_3^4$$
$$+ 10\, x_1^4 x_2 x_3^3 y_3^5 + 25\, x_1^4 x_3^4 y_1^3 y_2^2 - 48\, x_1^4 x_3^4 y_1^3 y_2 y_3 + 23\, x_1^4 x_3^4 y_1^3 y_3^2$$
$$- 110\, x_1^4 x_3^4 y_1^2 y_2^3 + 249\, x_1^4 x_3^4 y_1^2 y_2^2 y_3 - 162\, x_1^4 x_3^4 y_1^2 y_2 y_3^2 + 23\, x_1^4 x_3^4 y_1^2 y_3^3$$
$$+ 135\, x_1^4 x_3^4 y_1 y_2^4 - 336\, x_1^4 x_3^4 y_1 y_2^3 y_3 + 249\, x_1^4 x_3^4 y_1 y_2^2 y_3^2 - 48\, x_1^4 x_3^4 y_1 y_2 y_3^3$$
$$- 50\, x_1^4 x_3^4 y_2^5 + 135\, x_1^4 x_3^4 y_2^4 y_3 - 110\, x_1^4 x_3^4 y_2^3 y_3^2 + 25\, x_1^4 x_3^4 y_2^2 y_3^3 + 2\, x_1^3 x_2^5 y_1^5$$
$$- 15\, x_1^3 x_2^5 y_1^4 y_2 + 15\, x_1^3 x_2^5 y_1^4 y_3 + 54\, x_1^3 x_2^5 y_1^3 y_2^2 - 108\, x_1^3 x_2^5 y_1^3 y_2 y_3$$
$$+ 46\, x_1^3 x_2^5 y_1^3 y_3^2 - 100\, x_1^3 x_2^5 y_1^2 y_2^3 + 258\, x_1^3 x_2^5 y_1^2 y_2^2 y_3 - 186\, x_1^3 x_2^5 y_1^2 y_2 y_3^2$$
$$+ 32\, x_1^3 x_2^5 y_1^2 y_3^3 + 42\, x_1^3 x_2^5 y_1 y_2^4 - 132\, x_1^3 x_2^5 y_1 y_2^3 y_3 + 174\, x_1^3 x_2^5 y_1 y_2^2 y_3^2$$
$$- 120\, x_1^3 x_2^5 y_1 y_2 y_3^3 + 42\, x_1^3 x_2^5 y_1 y_3^4 + 30\, x_1^3 x_2^5 y_2^5 - 108\, x_1^3 x_2^5 y_2^4 y_3$$
$$+ 136\, x_1^3 x_2^5 y_2^3 y_3^2 - 102\, x_1^3 x_2^5 y_2^2 y_3^3 + 63\, x_1^3 x_2^5 y_2 y_3^4 - 23\, x_1^3 x_2^5 y_3^5 - 10\, x_1^3 x_2^4 x_3 y_1^5$$
$$+ 51\, x_1^3 x_2^4 x_3 y_1^4 y_2 - 51\, x_1^3 x_2^4 x_3 y_1^4 y_3 - 160\, x_1^3 x_2^4 x_3 y_1^3 y_2^2 + 360\, x_1^3 x_2^4 x_3 y_1^3 y_2 y_3$$
$$- 160\, x_1^3 x_2^4 x_3 y_1^3 y_3^2 + 328\, x_1^3 x_2^4 x_3 y_1^2 y_2^3 - 948\, x_1^3 x_2^4 x_3 y_1^2 y_2^2 y_3$$
$$+ 774\, x_1^3 x_2^4 x_3 y_1^2 y_2 y_3^2 - 174\, x_1^3 x_2^4 x_3 y_1^2 y_3^3 - 282\, x_1^3 x_2^4 x_3 y_1 y_2^4$$
$$+ 900\, x_1^3 x_2^4 x_3 y_1 y_2^3 y_3 - 1116\, x_1^3 x_2^4 x_3 y_1 y_2^2 y_3^2 + 612\, x_1^3 x_2^4 x_3 y_1 y_2 y_3^3$$
$$- 144\, x_1^3 x_2^4 x_3 y_1 y_3^4 + 128\, x_1^3 x_2^4 x_3 y_2^5 - 426\, x_1^3 x_2^4 x_3 y_2^4 y_3 + 732\, x_1^3 x_2^4 x_3 y_2^3 y_3^2$$
$$- 688\, x_1^3 x_2^4 x_3 y_2^2 y_3^3 + 339\, x_1^3 x_2^4 x_3 y_2 y_3^4 - 65\, x_1^3 x_2^4 x_3 y_3^5 + 20\, x_1^3 x_2^3 x_3^2 y_1^5$$
$$- 54\, x_1^3 x_2^3 x_3^2 y_1^4 y_2 + 54\, x_1^3 x_2^3 x_3^2 y_1^4 y_3 + 100\, x_1^3 x_2^3 x_3^2 y_1^3 y_2^2 - 360\, x_1^3 x_2^3 x_3^2 y_1^3 y_2 y_3$$
$$+ 180\, x_1^3 x_2^3 x_3^2 y_1^3 y_3^2 - 180\, x_1^3 x_2^3 x_3^2 y_1^2 y_2^3 + 792\, x_1^3 x_2^3 x_3^2 y_1^2 y_2^2 y_3$$
$$- 792\, x_1^3 x_2^3 x_3^2 y_1^2 y_2 y_3^2 + 220\, x_1^3 x_2^3 x_3^2 y_1^2 y_3^3 + 204\, x_1^3 x_2^3 x_3^2 y_1 y_2^4$$
$$- 696\, x_1^3 x_2^3 x_3^2 y_1 y_2^3 y_3 + 1020\, x_1^3 x_2^3 x_3^2 y_1 y_2^2 y_3^2 - 576\, x_1^3 x_2^3 x_3^2 y_1 y_2 y_3^3$$
$$+ 108\, x_1^3 x_2^3 x_3^2 y_1 y_3^4 - 150\, x_1^3 x_2^3 x_3^2 y_2^5 + 270\, x_1^3 x_2^3 x_3^2 y_2^4 y_3 - 288\, x_1^3 x_2^3 x_3^2 y_2^3 y_3^2$$
$$+ 236\, x_1^3 x_2^3 x_3^2 y_2^2 y_3^3 - 168\, x_1^3 x_2^3 x_3^2 y_2 y_3^4 + 60\, x_1^3 x_2^3 x_3^2 y_3^5 - 20\, x_1^3 x_2^2 x_3^3 y_1^5$$
$$+ 6\, x_1^3 x_2^2 x_3^3 y_1^4 y_2 - 6\, x_1^3 x_2^2 x_3^3 y_1^4 y_3 + 120\, x_1^3 x_2^2 x_3^3 y_1^3 y_2^2 - 40\, x_1^3 x_2^2 x_3^3 y_1^3 y_3^2$$

$$-324\,x_1^3x_2^2x_3^3y_1^2y_2^3+420\,x_1^3x_2^2x_3^3y_1^2y_2^2y_3-96\,x_1^3x_2^2x_3^3y_1^2y_2y_3^2$$
$$-40\,x_1^3x_2^2x_3^3y_1^2y_3^3+294\,x_1^3x_2^2x_3^3y_1y_2^4-768\,x_1^3x_2^2x_3^3y_1y_2^3y_3$$
$$+420\,x_1^3x_2^2x_3^3y_1y_2^2y_3^2-6\,x_1^3x_2^2x_3^3y_1y_3^4-36\,x_1^3x_2^2x_3^3y_2^5+294\,x_1^3x_2^2x_3^3y_2^4y_3$$
$$-324\,x_1^3x_2^2x_3^3y_2^3y_3^2+120\,x_1^3x_2^2x_3^3y_2^2y_3^3+6\,x_1^3x_2^2x_3^3y_2y_3^4-20\,x_1^3x_2^2x_3^3y_3^5$$
$$+10\,x_1^3x_2x_3^4y_1^5+21\,x_1^3x_2x_3^4y_1^4y_2-21\,x_1^3x_2x_3^4y_1^4y_3-170\,x_1^3x_2x_3^4y_1^3y_2^2$$
$$+180\,x_1^3x_2x_3^4y_1^3y_2y_3-50\,x_1^3x_2x_3^4y_1^3y_3^2+376\,x_1^3x_2x_3^4y_1^2y_2^3-714\,x_1^3x_2x_3^4y_1^2y_2^2y_3$$
$$+402\,x_1^3x_2x_3^4y_1^2y_2y_3^2-44\,x_1^3x_2x_3^4y_1^2y_3^3-336\,x_1^3x_2x_3^4y_1y_2^4+900\,x_1^3x_2x_3^4y_1y_2^3y_3$$
$$-630\,x_1^3x_2x_3^4y_1y_2^2y_3^2+96\,x_1^3x_2x_3^4y_1y_2y_3^3+99\,x_1^3x_2x_3^4y_2^5-345\,x_1^3x_2x_3^4y_2^4y_3$$
$$+278\,x_1^3x_2x_3^4y_2^3y_3^2-52\,x_1^3x_2x_3^4y_2^2y_3^3-2\,x_1^3x_3^5y_1^5-9\,x_1^3x_3^5y_1^4y_2+9\,x_1^3x_3^5y_1^4y_3$$
$$+56\,x_1^3x_3^5y_1^3y_2^2-72\,x_1^3x_3^5y_1^3y_2y_3+24\,x_1^3x_3^5y_1^3y_3^2-100\,x_1^3x_3^5y_1^2y_2^3$$
$$+192\,x_1^3x_3^5y_1^2y_2^2y_3-102\,x_1^3x_3^5y_1^2y_2y_3^2+6\,x_1^3x_3^5y_1^2y_3^3+78\,x_1^3x_3^5y_1y_2^4$$
$$-204\,x_1^3x_3^5y_1y_2^3y_3+132\,x_1^3x_3^5y_1y_2^2y_3^2-12\,x_1^3x_3^5y_1y_2y_3^3-23\,x_1^3x_3^5y_2^5$$
$$+75\,x_1^3x_3^5y_2^4y_3-54\,x_1^3x_3^5y_2^3y_3^2+6\,x_1^3x_3^5y_2^2y_3^3-6\,x_1^2x_2^6y_1^5+33\,x_1^2x_2^6y_1^4y_2$$
$$-15\,x_1^2x_2^6y_1^4y_3-28\,x_1^2x_2^6y_1^3y_2^2-12\,x_1^2x_2^6y_1^3y_2y_3+16\,x_1^2x_2^6y_1^3y_3^2$$
$$-63\,x_1^2x_2^6y_1^2y_2^3+177\,x_1^2x_2^6y_1^2y_2^2y_3-123\,x_1^2x_2^6y_1^2y_2y_3^2+33\,x_1^2x_2^6y_1^2y_3^3$$
$$+123\,x_1^2x_2^6y_1y_2^4-312\,x_1^2x_2^6y_1y_2^3y_3+294\,x_1^2x_2^6y_1y_2^2y_3^2-156\,x_1^2x_2^6y_1y_2y_3^3$$
$$+33\,x_1^2x_2^6y_1y_3^4-76\,x_1^2x_2^6y_2^5+195\,x_1^2x_2^6y_2^4y_3-209\,x_1^2x_2^6y_2^3y_3^2+161\,x_1^2x_2^6y_2^2y_3^3$$
$$-90\,x_1^2x_2^6y_2y_3^4+25\,x_1^2x_2^6y_3^5+30\,x_1^2x_2^5x_3y_1^5-156\,x_1^2x_2^5x_3y_1^4y_2$$
$$+78\,x_1^2x_2^5x_3y_1^4y_3+258\,x_1^2x_2^5x_3y_1^3y_2^2-228\,x_1^2x_2^5x_3y_1^3y_2y_3+54\,x_1^2x_2^5x_3y_1^3y_3^2$$
$$-186\,x_1^2x_2^5x_3y_1^2y_2^3+324\,x_1^2x_2^5x_3y_1^2y_2^2y_3-270\,x_1^2x_2^5x_3y_1^2y_2y_3^2$$
$$+48\,x_1^2x_2^5x_3y_1^2y_3^3+24\,x_1^2x_2^5x_3y_1y_2^4-192\,x_1^2x_2^5x_3y_1y_2^3y_3+414\,x_1^2x_2^5x_3y_1y_2^2y_3^2$$
$$-228\,x_1^2x_2^5x_3y_1y_2y_3^3+60\,x_1^2x_2^5x_3y_1y_3^4+42\,x_1^2x_2^5x_3y_2^5-30\,x_1^2x_2^5x_3y_2^4y_3$$
$$-126\,x_1^2x_2^5x_3y_2^3y_3^2+132\,x_1^2x_2^5x_3y_2^2y_3^3-48\,x_1^2x_2^5x_3y_2y_3^4-60\,x_1^2x_2^4x_3^2y_1^5$$
$$+267\,x_1^2x_2^4x_3^2y_1^4y_2-147\,x_1^2x_2^4x_3^2y_1^4y_3-531\,x_1^2x_2^4x_3^2y_1^3y_2^2$$
$$+708\,x_1^2x_2^4x_3^2y_1^3y_2y_3-237\,x_1^2x_2^4x_3^2y_1^3y_3^2+684\,x_1^2x_2^4x_3^2y_1^2y_2^3$$
$$-1521\,x_1^2x_2^4x_3^2y_1^2y_2^2y_3+1134\,x_1^2x_2^4x_3^2y_1^2y_2y_3^2-237\,x_1^2x_2^4x_3^2y_1^2y_3^3$$
$$-480\,x_1^2x_2^4x_3^2y_1y_2^4+1320\,x_1^2x_2^4x_3^2y_1y_2^3y_3-1521\,x_1^2x_2^4x_3^2y_1y_2^2y_3^2$$
$$+708\,x_1^2x_2^4x_3^2y_1y_2y_3^3-147\,x_1^2x_2^4x_3^2y_1y_3^4+180\,x_1^2x_2^4x_3^2y_2^5-480\,x_1^2x_2^4x_3^2y_2^4y_3$$
$$+684\,x_1^2x_2^4x_3^2y_2^3y_3^2-531\,x_1^2x_2^4x_3^2y_2^2y_3^3+267\,x_1^2x_2^4x_3^2y_2y_3^4-60\,x_1^2x_2^4x_3^2y_3^5$$

$$+ 60 x_1^2 x_2^3 x_3^3 y_1^5 - 168 x_1^2 x_2^3 x_3^3 y_1^4 y_2 + 108 x_1^2 x_2^3 x_3^3 y_1^4 y_3 + 236 x_1^2 x_2^3 x_3^3 y_1^3 y_2^2$$
$$- 576 x_1^2 x_2^3 x_3^3 y_1^3 y_2 y_3 + 220 x_1^2 x_2^3 x_3^3 y_1^3 y_3^2 - 288 x_1^2 x_2^3 x_3^3 y_1^2 y_2^3$$
$$+ 1020 x_1^2 x_2^3 x_3^3 y_1^2 y_2^2 y_3 - 792 x_1^2 x_2^3 x_3^3 y_1^2 y_2 y_3^2 + 180 x_1^2 x_2^3 x_3^3 y_1^2 y_3^3$$
$$+ 270 x_1^2 x_2^3 x_3^3 y_1 y_2^4 - 696 x_1^2 x_2^3 x_3^3 y_1 y_2^3 y_3 + 792 x_1^2 x_2^3 x_3^3 y_1 y_2^2 y_3^2$$
$$- 360 x_1^2 x_2^3 x_3^3 y_1 y_2 y_3^3 + 54 x_1^2 x_2^3 x_3^3 y_1 y_3^4 - 150 x_1^2 x_2^3 x_3^3 y_2^5 + 204 x_1^2 x_2^3 x_3^3 y_2^4 y_3$$
$$- 180 x_1^2 x_2^3 x_3^3 y_2^3 y_3^2 + 100 x_1^2 x_2^3 x_3^3 y_2^2 y_3^3 - 54 x_1^2 x_2^3 x_3^3 y_2 y_3^4 + 20 x_1^2 x_2^3 x_3^3 y_3^5$$
$$- 30 x_1^2 x_2^2 x_3^4 y_1^5 - 33 x_1^2 x_2^2 x_3^4 y_1^4 y_2 + 3 x_1^2 x_2^2 x_3^4 y_1^4 y_3 + 318 x_1^2 x_2^2 x_3^4 y_1^3 y_2^2$$
$$- 60 x_1^2 x_2^2 x_3^4 y_1^3 y_2 y_3 - 18 x_1^2 x_2^2 x_3^4 y_1^3 y_3^2 - 522 x_1^2 x_2^2 x_3^4 y_1^2 y_2^3$$
$$+ 432 x_1^2 x_2^2 x_3^4 y_1^2 y_2^2 y_3 - 144 x_1^2 x_2^2 x_3^4 y_1^2 y_2 y_3^2 - 6 x_1^2 x_2^2 x_3^4 y_1^2 y_3^3$$
$$+ 312 x_1^2 x_2^2 x_3^4 y_1 y_2^4 - 588 x_1^2 x_2^2 x_3^4 y_1 y_2^3 y_3 + 306 x_1^2 x_2^2 x_3^4 y_1 y_2^2 y_3^2 - 45 x_1^2 x_2^2 x_3^4 y_2^5$$
$$+ 213 x_1^2 x_2^2 x_3^4 y_2^4 y_3 - 144 x_1^2 x_2^2 x_3^4 y_2^3 y_3^2 + 6 x_1^2 x_2^2 x_3^4 y_2^2 y_3^3 + 6 x_1^2 x_2 x_3^5 y_1^5$$
$$+ 84 x_1^2 x_2 x_3^5 y_1^4 y_2 - 42 x_1^2 x_2 x_3^5 y_1^4 y_3 - 342 x_1^2 x_2 x_3^5 y_1^3 y_2^2 + 228 x_1^2 x_2 x_3^5 y_1^3 y_2 y_3$$
$$- 42 x_1^2 x_2 x_3^5 y_1^3 y_3^2 + 480 x_1^2 x_2 x_3^5 y_1^2 y_2^3 - 522 x_1^2 x_2 x_3^5 y_1^2 y_2^2 y_3$$
$$+ 216 x_1^2 x_2 x_3^5 y_1^2 y_2 y_3^2 - 18 x_1^2 x_2 x_3^5 y_1^2 y_3^3 - 300 x_1^2 x_2 x_3^5 y_1 y_2^4 + 528 x_1^2 x_2 x_3^5 y_1 y_2^3 y_3$$
$$- 306 x_1^2 x_2 x_3^5 y_1 y_2^2 y_3^2 + 36 x_1^2 x_2 x_3^5 y_1 y_2 y_3^3 + 72 x_1^2 x_2 x_3^5 y_2^5 - 192 x_1^2 x_2 x_3^5 y_2^4 y_3$$
$$+ 132 x_1^2 x_2 x_3^5 y_2^3 y_3^2 - 18 x_1^2 x_2 x_3^5 y_2^2 y_3^3 - 27 x_1^2 x_3^6 y_1^4 y_2 + 15 x_1^2 x_3^6 y_1^4 y_3$$
$$+ 89 x_1^2 x_3^6 y_1^3 y_2^2 - 60 x_1^2 x_3^6 y_1^3 y_2 y_3 + 7 x_1^2 x_3^6 y_1^3 y_3^2 - 105 x_1^2 x_3^6 y_1^2 y_2^3$$
$$+ 90 x_1^2 x_3^6 y_1^2 y_2^2 y_3 - 21 x_1^2 x_3^6 y_1^2 y_2 y_3^2 + 51 x_1^2 x_3^6 y_1 y_2^4 - 60 x_1^2 x_3^6 y_1 y_2^3 y_3$$
$$+ 21 x_1^2 x_3^6 y_1 y_2^2 y_3^2 - 8 x_1^2 x_3^6 y_2^5 + 15 x_1^2 x_3^6 y_2^4 y_3 - 7 x_1^2 x_3^6 y_2^3 y_3^2 - 8 x_1 x_2^7 y_1^5$$
$$+ 45 x_1 x_2^7 y_1^4 y_2 - 15 x_1 x_2^7 y_1^4 y_3 - 108 x_1 x_2^7 y_1^3 y_2^2 + 96 x_1 x_2^7 y_1^3 y_2 y_3 - 40 x_1 x_2^7 y_1^3 y_3^2$$
$$+ 136 x_1 x_2^7 y_1^2 y_2^3 - 216 x_1 x_2^7 y_1^2 y_2^2 y_3 + 174 x_1 x_2^7 y_1^2 y_2 y_3^2 - 38 x_1 x_2^7 y_1^2 y_3^3$$
$$- 60 x_1 x_2^7 y_1 y_2^4 + 132 x_1 x_2^7 y_1 y_2^3 y_3 - 216 x_1 x_2^7 y_1 y_2^2 y_3^2 + 156 x_1 x_2^7 y_1 y_2 y_3^3$$
$$- 48 x_1 x_2^7 y_1 y_3^4 + 24 x_1 x_2^7 y_2^4 y_3 - 4 x_1 x_2^7 y_2^3 y_3^2 - 36 x_1 x_2^7 y_2^2 y_3^3 + 33 x_1 x_2^7 y_2 y_3^4$$
$$- 7 x_1 x_2^7 y_3^5 + 23 x_1 x_2^6 x_3 y_1^5 - 108 x_1 x_2^6 x_3 y_1^4 y_2 + 21 x_1 x_2^6 x_3 y_1^4 y_3 + 236 x_1 x_2^6 x_3 y_1^3 y_2^2$$
$$- 156 x_1 x_2^6 x_3 y_1^3 y_2 y_3 + 74 x_1 x_2^6 x_3 y_1^3 y_3^2 - 304 x_1 x_2^6 x_3 y_1^2 y_2^3 + 384 x_1 x_2^6 x_3 y_1^2 y_2^2 y_3$$
$$- 324 x_1 x_2^6 x_3 y_1^2 y_2 y_3^2 + 74 x_1 x_2^6 x_3 y_1^2 y_3^3 + 222 x_1 x_2^6 x_3 y_1 y_2^4 - 360 x_1 x_2^6 x_3 y_1 y_2^3 y_3$$
$$+ 384 x_1 x_2^6 x_3 y_1 y_2^2 y_3^2 - 156 x_1 x_2^6 x_3 y_1 y_2 y_3^3 + 21 x_1 x_2^6 x_3 y_1 y_3^4 - 100 x_1 x_2^6 x_3 y_2^5$$
$$+ 222 x_1 x_2^6 x_3 y_2^4 y_3 - 304 x_1 x_2^6 x_3 y_2^3 y_3^2 + 236 x_1 x_2^6 x_3 y_2^2 y_3^3 - 108 x_1 x_2^6 x_3 y_2 y_3^4$$
$$+ 23 x_1 x_2^6 x_3 y_3^5 - 48 x_1 x_2^5 x_3^2 y_1^4 y_2 + 60 x_1 x_2^5 x_3^2 y_1^4 y_3 + 132 x_1 x_2^5 x_3^2 y_1^3 y_2^2$$

$$- 228 x_1 x_2^5 x_3^2 y_1^3 y_2 y_3 + 48 x_1 x_2^5 x_3^2 y_1^3 y_3^2 - 126 x_1 x_2^5 x_3^2 y_1^2 y_2^3 + 414 x_1 x_2^5 x_3^2 y_1^2 y_2^2 y_3$$
$$- 270 x_1 x_2^5 x_3^2 y_1^2 y_2 y_3^2 + 54 x_1 x_2^5 x_3^2 y_1^2 y_3^3 - 30 x_1 x_2^5 x_3^2 y_1 y_2^4 - 192 x_1 x_2^5 x_3^2 y_1 y_2^3 y_3$$
$$+ 324 x_1 x_2^5 x_3^2 y_1 y_2^2 y_3^2 - 228 x_1 x_2^5 x_3^2 y_1 y_2 y_3^3 + 78 x_1 x_2^5 x_3^2 y_1 y_3^4 + 42 x_1 x_2^5 x_3^2 y_2^5$$
$$+ 24 x_1 x_2^5 x_3^2 y_2^4 y_3 - 186 x_1 x_2^5 x_3^2 y_2^3 y_3^2 + 258 x_1 x_2^5 x_3^2 y_2^2 y_3^3 - 156 x_1 x_2^5 x_3^2 y_2 y_3^4$$
$$+ 30 x_1 x_2^5 x_3^2 y_3^5 - 65 x_1 x_2^4 x_3^3 y_1^5 + 339 x_1 x_2^4 x_3^3 y_1^4 y_2 - 144 x_1 x_2^4 x_3^3 y_1^4 y_3$$
$$- 688 x_1 x_2^4 x_3^3 y_1^3 y_2^2 + 612 x_1 x_2^4 x_3^3 y_1^3 y_2 y_3 - 174 x_1 x_2^4 x_3^3 y_1^3 y_3^2 + 732 x_1 x_2^4 x_3^3 y_1^2 y_2^3$$
$$- 1116 x_1 x_2^4 x_3^3 y_1^2 y_2^2 y_3 + 774 x_1 x_2^4 x_3^3 y_1^2 y_2 y_3^2 - 160 x_1 x_2^4 x_3^3 y_1^2 y_3^3$$
$$- 426 x_1 x_2^4 x_3^3 y_1 y_2^4 + 900 x_1 x_2^4 x_3^3 y_1 y_2^3 y_3 - 948 x_1 x_2^4 x_3^3 y_1 y_2^2 y_3^2$$
$$+ 360 x_1 x_2^4 x_3^3 y_1 y_2 y_3^3 - 51 x_1 x_2^4 x_3^3 y_1 y_3^4 + 128 x_1 x_2^4 x_3^3 y_2^5 - 282 x_1 x_2^4 x_3^3 y_2^4 y_3$$
$$+ 328 x_1 x_2^4 x_3^3 y_2^3 y_3^2 - 160 x_1 x_2^4 x_3^3 y_2^2 y_3^3 + 51 x_1 x_2^4 x_3^3 y_2 y_3^4 - 10 x_1 x_2^4 x_3^3 y_3^5$$
$$+ 80 x_1 x_2^3 x_3^4 y_1^5 - 291 x_1 x_2^3 x_3^4 y_1^4 y_2 + 81 x_1 x_2^3 x_3^4 y_1^4 y_3 + 450 x_1 x_2^3 x_3^4 y_1^3 y_2^2$$
$$- 324 x_1 x_2^3 x_3^4 y_1^3 y_2 y_3 + 94 x_1 x_2^3 x_3^4 y_1^3 y_3^2 - 408 x_1 x_2^3 x_3^4 y_1^2 y_2^3 + 426 x_1 x_2^3 x_3^4 y_1^2 y_2^2 y_3$$
$$- 270 x_1 x_2^3 x_3^4 y_1^2 y_2 y_3^2 + 52 x_1 x_2^3 x_3^4 y_1^2 y_3^3 + 270 x_1 x_2^3 x_3^4 y_1 y_2^4 - 276 x_1 x_2^3 x_3^4 y_1 y_2^3 y_3$$
$$+ 282 x_1 x_2^3 x_3^4 y_1 y_2^2 y_3^2 - 96 x_1 x_2^3 x_3^4 y_1 y_2 y_3^3 - 101 x_1 x_2^3 x_3^4 y_2^5 + 93 x_1 x_2^3 x_3^4 y_2^4 y_3$$
$$- 106 x_1 x_2^3 x_3^4 y_2^3 y_3^2 + 44 x_1 x_2^3 x_3^4 y_2^2 y_3^3 - 27 x_1 x_2^2 x_3^5 y_1^5 - 6 x_1 x_2^2 x_3^5 y_1^4 y_2$$
$$+ 21 x_1 x_2^2 x_3^5 y_1^4 y_3 + 186 x_1 x_2^2 x_3^5 y_1^3 y_2^2 - 96 x_1 x_2^2 x_3^5 y_1^3 y_2 y_3 + 12 x_1 x_2^2 x_3^5 y_1^3 y_3^2$$
$$- 270 x_1 x_2^2 x_3^5 y_1^2 y_2^3 + 252 x_1 x_2^2 x_3^5 y_1^2 y_2^2 y_3 - 126 x_1 x_2^2 x_3^5 y_1^2 y_2 y_3^2$$
$$+ 18 x_1 x_2^2 x_3^5 y_1^2 y_3^3 + 141 x_1 x_2^2 x_3^5 y_1 y_2^4 - 300 x_1 x_2^2 x_3^5 y_1 y_2^3 y_3 + 216 x_1 x_2^2 x_3^5 y_1 y_2^2 y_3^2$$
$$- 36 x_1 x_2^2 x_3^5 y_1 y_2 y_3^3 - 24 x_1 x_2^2 x_3^5 y_2^5 + 123 x_1 x_2^2 x_3^5 y_2^4 y_3 - 102 x_1 x_2^2 x_3^5 y_2^3 y_3^2$$
$$+ 18 x_1 x_2^2 x_3^5 y_2^2 y_3^3 - 8 x_1 x_2 x_3^6 y_1^5 + 102 x_1 x_2 x_3^6 y_1^4 y_2 - 30 x_1 x_2 x_3^6 y_1^4 y_3$$
$$- 290 x_1 x_2 x_3^6 y_1^3 y_2^2 + 120 x_1 x_2 x_3^6 y_1^3 y_2 y_3 - 14 x_1 x_2 x_3^6 y_1^3 y_3^2 + 338 x_1 x_2 x_3^6 y_1^2 y_2^3$$
$$- 180 x_1 x_2 x_3^6 y_1^2 y_2^2 y_3 + 42 x_1 x_2 x_3^6 y_1^2 y_2 y_3^2 - 174 x_1 x_2 x_3^6 y_1 y_2^4 + 120 x_1 x_2 x_3^6 y_1 y_2^3 y_3$$
$$- 42 x_1 x_2 x_3^6 y_1 y_2^2 y_3^2 + 32 x_1 x_2 x_3^6 y_2^5 - 30 x_1 x_2 x_3^6 y_2^4 y_3 + 14 x_1 x_2 x_3^6 y_2^3 y_3^2$$
$$+ 5 x_1 x_3^7 y_1^5 - 33 x_1 x_3^7 y_1^4 y_2 + 6 x_1 x_3^7 y_1^4 y_3 + 82 x_1 x_3^7 y_1^3 y_2^2 - 24 x_1 x_3^7 y_1^3 y_2 y_3$$
$$- 98 x_1 x_3^7 y_1^2 y_2^3 + 36 x_1 x_3^7 y_1^2 y_2^2 y_3 + 57 x_1 x_3^7 y_1 y_2^4 - 24 x_1 x_3^7 y_1 y_2^3 y_3 - 13 x_1 x_3^7 y_2^5$$
$$+ 6 x_1 x_3^7 y_2^4 y_3 - 3 x_2^8 y_1^4 y_2 + 15 x_2^8 y_1^4 y_3 - 9 x_2^8 y_1^3 y_2^2 - 36 x_2^8 y_1^3 y_2 y_3 + 9 x_2^8 y_1^3 y_3^2$$
$$+ 48 x_2^8 y_1^2 y_2^3 + 9 x_2^8 y_1^2 y_2^2 y_3 - 30 x_2^8 y_1^2 y_2 y_3^2 + 9 x_2^8 y_1^2 y_3^3 - 72 x_2^8 y_1 y_2^4$$
$$+ 72 x_2^8 y_1 y_2^3 y_3 + 9 x_2^8 y_1 y_2^2 y_3^2 - 36 x_2^8 y_1 y_2 y_3^3 + 15 x_2^8 y_1 y_3^4 + 36 x_2^8 y_2^5 - 72 x_2^8 y_2^4 y_3$$
$$+ 48 x_2^8 y_2^3 y_3^2 - 9 x_2^8 y_2^2 y_3^3 - 3 x_2^8 y_2 y_3^4 - 7 x_2^7 x_3 y_1^5 + 33 x_2^7 x_3 y_1^4 y_2 - 48 x_2^7 x_3 y_1^4 y_3$$

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\begin{aligned}
&-36\,x_2^7 x_3 y_1^3 y_2^2+156\,x_2^7 x_3 y_1^3 y_2 y_3-38\,x_2^7 x_3 y_1^3 y_3^2-4\,x_2^7 x_3 y_1^2 y_2^3\\
&-216\,x_2^7 x_3 y_1^2 y_2^2 y_3+174\,x_2^7 x_3 y_1^2 y_2 y_3^2-40\,x_2^7 x_3 y_1^2 y_3^3+24\,x_2^7 x_3 y_1 y_2^4\\
&+132\,x_2^7 x_3 y_1 y_2^3 y_3-216\,x_2^7 x_3 y_1 y_2^2 y_3^2+96\,x_2^7 x_3 y_1 y_2 y_3^3-15\,x_2^7 x_3 y_1 y_3^4\\
&-60\,x_2^7 x_3 y_2^4 y_3+136\,x_2^7 x_3 y_2^3 y_3^2-108\,x_2^7 x_3 y_2^2 y_3^3+45\,x_2^7 x_3 y_2 y_3^4-8\,x_2^7 x_3 y_3^5\\
&+25\,x_2^6 x_3^2 y_1^5-90\,x_2^6 x_3^2 y_1^4 y_2+33\,x_2^6 x_3^2 y_1^4 y_3+161\,x_2^6 x_3^2 y_1^3 y_2^2-156\,x_2^6 x_3^2 y_1^3 y_2 y_3\\
&+33\,x_2^6 x_3^2 y_1^3 y_3^2-209\,x_2^6 x_3^2 y_1^2 y_2^3+294\,x_2^6 x_3^2 y_1^2 y_2^2 y_3-123\,x_2^6 x_3^2 y_1^2 y_2 y_3^2\\
&+16\,x_2^6 x_3^2 y_1^2 y_3^3+195\,x_2^6 x_3^2 y_1 y_2^4-312\,x_2^6 x_3^2 y_1 y_2^3 y_3+177\,x_2^6 x_3^2 y_1 y_2^2 y_3^2\\
&-12\,x_2^6 x_3^2 y_1 y_2 y_3^3-15\,x_2^6 x_3^2 y_1 y_3^4-76\,x_2^6 x_3^2 y_2^5+123\,x_2^6 x_3^2 y_2^4 y_3-63\,x_2^6 x_3^2 y_2^3 y_3^2\\
&-28\,x_2^6 x_3^2 y_2^2 y_3^3+33\,x_2^6 x_3^2 y_2 y_3^4-6\,x_2^6 x_3^2 y_3^5-23\,x_2^5 x_3^3 y_1^5+63\,x_2^5 x_3^3 y_1^4 y_2\\
&+42\,x_2^5 x_3^3 y_1^4 y_3-102\,x_2^5 x_3^3 y_1^3 y_2^2-120\,x_2^5 x_3^3 y_1^3 y_2 y_3+32\,x_2^5 x_3^3 y_1^3 y_3^2\\
&+136\,x_2^5 x_3^3 y_1^2 y_2^3+174\,x_2^5 x_3^3 y_1^2 y_2^2 y_3-186\,x_2^5 x_3^3 y_1^2 y_2 y_3^2+46\,x_2^5 x_3^3 y_1^2 y_3^3\\
&-108\,x_2^5 x_3^3 y_1 y_2^4-132\,x_2^5 x_3^3 y_1 y_2^3 y_3+258\,x_2^5 x_3^3 y_1 y_2^2 y_3^2-108\,x_2^5 x_3^3 y_1 y_2 y_3^3\\
&+15\,x_2^5 x_3^3 y_1 y_3^4+30\,x_2^5 x_3^3 y_2^5+42\,x_2^5 x_3^3 y_2^4 y_3-100\,x_2^5 x_3^3 y_2^3 y_3^2+54\,x_2^5 x_3^3 y_2^2 y_3^3\\
&-15\,x_2^5 x_3^3 y_2 y_3^4+2\,x_2^5 x_3^3 y_3^5-15\,x_2^4 x_3^4 y_1^5+78\,x_2^4 x_3^4 y_1^4 y_2-63\,x_2^4 x_3^4 y_1^4 y_3\\
&-173\,x_2^4 x_3^4 y_1^3 y_2^2+252\,x_2^4 x_3^4 y_1^3 y_2 y_3-49\,x_2^4 x_3^4 y_1^3 y_3^2+214\,x_2^4 x_3^4 y_1^2 y_2^3\\
&-393\,x_2^4 x_3^4 y_1^2 y_2^2 y_3+174\,x_2^4 x_3^4 y_1^2 y_2 y_3^2-25\,x_2^4 x_3^4 y_1^2 y_3^3-156\,x_2^4 x_3^4 y_1 y_2^4\\
&+300\,x_2^4 x_3^4 y_1 y_2^3 y_3-207\,x_2^4 x_3^4 y_1 y_2^2 y_3^2+48\,x_2^4 x_3^4 y_1 y_2 y_3^3+52\,x_2^4 x_3^4 y_2^5\\
&-96\,x_2^4 x_3^4 y_2^4 y_3+82\,x_2^4 x_3^4 y_2^3 y_3^2-23\,x_2^4 x_3^4 y_2^2 y_3^3+35\,x_2^3 x_3^5 y_1^5-129\,x_2^3 x_3^5 y_1^4 y_2\\
&+12\,x_2^3 x_3^5 y_1^4 y_3+220\,x_2^3 x_3^5 y_1^3 y_2^2-60\,x_2^3 x_3^5 y_1^3 y_2 y_3+6\,x_2^3 x_3^5 y_1^3 y_3^2\\
&-230\,x_2^3 x_3^5 y_1^2 y_2^3+78\,x_2^3 x_3^5 y_1^2 y_2^2 y_3+12\,x_2^3 x_3^5 y_1^2 y_2 y_3^2-6\,x_2^3 x_3^5 y_1^2 y_3^3\\
&+141\,x_2^3 x_3^5 y_1 y_2^4-24\,x_2^3 x_3^5 y_1 y_2^3 y_3-42\,x_2^3 x_3^5 y_1 y_2^2 y_3^2+12\,x_2^3 x_3^5 y_1 y_2 y_3^3\\
&-37\,x_2^3 x_3^5 y_2^5-6\,x_2^3 x_3^5 y_2^4 y_3+24\,x_2^3 x_3^5 y_2^3 y_3^2-6\,x_2^3 x_3^5 y_2^2 y_3^3-13\,x_2^2 x_3^6 y_1^5\\
&+30\,x_2^2 x_3^6 y_1^4 y_2+15\,x_2^2 x_3^6 y_1^4 y_3-9\,x_2^2 x_3^6 y_1^3 y_2^2-60\,x_2^2 x_3^6 y_1^3 y_2 y_3+7\,x_2^2 x_3^6 y_1^3 y_3^2\\
&-23\,x_2^2 x_3^6 y_1^2 y_2^3+90\,x_2^2 x_3^6 y_1^2 y_2^2 y_3-21\,x_2^2 x_3^6 y_1^2 y_2 y_3^2+18\,x_2^2 x_3^6 y_1 y_2^4\\
&-60\,x_2^2 x_3^6 y_1 y_2^3 y_3+21\,x_2^2 x_3^6 y_1 y_2^2 y_3^2-3\,x_2^2 x_3^6 y_2^5+15\,x_2^2 x_3^6 y_2^4 y_3-7\,x_2^2 x_3^6 y_2^3 y_3^2\\
&-5\,x_2 x_3^7 y_1^5+33\,x_2 x_3^7 y_1^4 y_2-6\,x_2 x_3^7 y_1^4 y_3-82\,x_2 x_3^7 y_1^3 y_2^2+24\,x_2 x_3^7 y_1^3 y_2 y_3\\
&+98\,x_2 x_3^7 y_1^2 y_2^3-36\,x_2 x_3^7 y_1^2 y_2^2 y_3-57\,x_2 x_3^7 y_1 y_2^4+24\,x_2 x_3^7 y_1 y_2^3 y_3+13\,x_2 x_3^7 y_2^5\\
&-6\,x_2 x_3^7 y_2^4 y_3+3\,x_3^8 y_1^5-15\,x_3^8 y_1^4 y_2+30\,x_3^8 y_1^3 y_2^2-30\,x_3^8 y_1^2 y_2^3+15\,x_3^8 y_1 y_2^4\\
&-3\,x_3^8 y_2^5+6\,x_1^5 y_1^2 y_2^5-30\,x_1^5 y_1^2 y_2^4 y_3+60\,x_1^5 y_1^2 y_2^3 y_3^2-60\,x_1^5 y_1^2 y_2^2 y_3^3
\end{aligned}
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$$+ 30 x_1^5 y_1^2 y_2 y_3^4 - 6 x_1^5 y_1^2 y_3^5 - 9 x_1^5 y_1 y_2^6 + 48 x_1^5 y_1 y_2^5 y_3 - 105 x_1^5 y_1 y_2^4 y_3^2$$
$$+ 120 x_1^5 y_1 y_2^3 y_3^3 - 75 x_1^5 y_1 y_2^2 y_3^4 + 24 x_1^5 y_1 y_2 y_3^5 - 3 x_1^5 y_1 y_3^6 + 2 x_1^5 y_2^7$$
$$- 15 x_1^5 y_2^6 y_3 + 44 x_1^5 y_2^5 y_3^2 - 65 x_1^5 y_2^4 y_3^3 + 50 x_1^5 y_2^3 y_3^4 - 17 x_1^5 y_2^2 y_3^5 + x_1^5 y_3^7$$
$$+ 12 x_1^4 x_2 y_1^3 y_2^4 - 48 x_1^4 x_2 y_1^3 y_2^3 y_3 + 72 x_1^4 x_2 y_1^3 y_2^2 y_3^2 - 48 x_1^4 x_2 y_1^3 y_2 y_3^3$$
$$+ 12 x_1^4 x_2 y_1^3 y_3^4 - 43 x_1^4 x_2 y_1^2 y_2^5 + 183 x_1^4 x_2 y_1^2 y_2^4 y_3 - 302 x_1^4 x_2 y_1^2 y_2^3 y_3^2$$
$$+ 238 x_1^4 x_2 y_1^2 y_2^2 y_3^3 - 87 x_1^4 x_2 y_1^2 y_2 y_3^4 + 11 x_1^4 x_2 y_1^2 y_3^5 + 42 x_1^4 x_2 y_1 y_2^6$$
$$- 192 x_1^4 x_2 y_1 y_2^5 y_3 + 354 x_1^4 x_2 y_1 y_2^4 y_3^2 - 336 x_1^4 x_2 y_1 y_2^3 y_3^3 + 174 x_1^4 x_2 y_1 y_2^2 y_3^4$$
$$- 48 x_1^4 x_2 y_1 y_2 y_3^5 + 6 x_1^4 x_2 y_1 y_3^6 - 6 x_1^4 x_2 y_2^7 + 42 x_1^4 x_2 y_2^6 y_3 - 119 x_1^4 x_2 y_2^5 y_3^2$$
$$+ 171 x_1^4 x_2 y_2^4 y_3^3 - 124 x_1^4 x_2 y_2^3 y_3^4 + 32 x_1^4 x_2 y_2^2 y_3^5 + 9 x_1^4 x_2 y_2 y_3^6 - 5 x_1^4 x_2 y_3^7$$
$$- 12 x_1^4 x_3 y_1^3 y_2^4 + 48 x_1^4 x_3 y_1^3 y_2^3 y_3 - 72 x_1^4 x_3 y_1^3 y_2^2 y_3^2 + 48 x_1^4 x_3 y_1^3 y_2 y_3^3$$
$$- 12 x_1^4 x_3 y_1^3 y_3^4 + 43 x_1^4 x_3 y_1^2 y_2^5 - 183 x_1^4 x_3 y_1^2 y_2^4 y_3 + 302 x_1^4 x_3 y_1^2 y_2^3 y_3^2$$
$$- 238 x_1^4 x_3 y_1^2 y_2^2 y_3^3 + 87 x_1^4 x_3 y_1^2 y_2 y_3^4 - 11 x_1^4 x_3 y_1^2 y_3^5 - 48 x_1^4 x_3 y_1 y_2^6$$
$$+ 216 x_1^4 x_3 y_1 y_2^5 y_3 - 384 x_1^4 x_3 y_1 y_2^4 y_3^2 + 336 x_1^4 x_3 y_1 y_2^3 y_3^3 - 144 x_1^4 x_3 y_1 y_2^2 y_3^4$$
$$+ 24 x_1^4 x_3 y_1 y_2 y_3^5 + 17 x_1^4 x_3 y_2^7 - 81 x_1^4 x_3 y_2^6 y_3 + 154 x_1^4 x_3 y_2^5 y_3^2 - 146 x_1^4 x_3 y_2^4 y_3^3$$
$$+ 69 x_1^4 x_3 y_2^3 y_3^4 - 13 x_1^4 x_3 y_2^2 y_3^5 + 8 x_1^3 x_2^2 y_1^4 y_2^3 - 24 x_1^3 x_2^2 y_1^4 y_2^2 y_3$$
$$+ 24 x_1^3 x_2^2 y_1^4 y_2 y_3^2 - 8 x_1^3 x_2^2 y_1^4 y_3^3 - 30 x_1^3 x_2^2 y_1^3 y_2^4 + 108 x_1^3 x_2^2 y_1^3 y_2^3 y_3$$
$$- 144 x_1^3 x_2^2 y_1^3 y_2^2 y_3^2 + 84 x_1^3 x_2^2 y_1^3 y_2 y_3^3 - 18 x_1^3 x_2^2 y_1^3 y_3^4 + 33 x_1^3 x_2^2 y_1^2 y_2^5$$
$$- 141 x_1^3 x_2^2 y_1^2 y_2^4 y_3 + 218 x_1^3 x_2^2 y_1^2 y_2^3 y_3^2 - 138 x_1^3 x_2^2 y_1^2 y_2^2 y_3^3 + 21 x_1^3 x_2^2 y_1^2 y_2 y_3^4$$
$$+ 7 x_1^3 x_2^2 y_1^2 y_3^5 - 6 x_1^3 x_2^2 y_1 y_2^6 + 54 x_1^3 x_2^2 y_1 y_2^5 y_3 - 114 x_1^3 x_2^2 y_1 y_2^4 y_3^2$$
$$+ 72 x_1^3 x_2^2 y_1 y_2^3 y_3^3 + 18 x_1^3 x_2^2 y_1 y_2^2 y_3^4 - 30 x_1^3 x_2^2 y_1 y_2 y_3^5 + 6 x_1^3 x_2^2 y_1 y_3^6$$
$$- 15 x_1^3 x_2^2 y_2^7 + 33 x_1^3 x_2^2 y_2^6 y_3 + 6 x_1^3 x_2^2 y_2^5 y_3^2 - 60 x_1^3 x_2^2 y_2^4 y_3^3 + 29 x_1^3 x_2^2 y_2^3 y_3^4$$
$$+ 33 x_1^3 x_2^2 y_2^2 y_3^5 - 36 x_1^3 x_2^2 y_2 y_3^6 + 10 x_1^3 x_2^2 y_3^7 - 16 x_1^3 x_2 x_3 y_1^4 y_2^3$$
$$+ 48 x_1^3 x_2 x_3 y_1^4 y_2^2 y_3 - 48 x_1^3 x_2 x_3 y_1^4 y_2 y_3^2 + 16 x_1^3 x_2 x_3 y_1^4 y_3^3 + 60 x_1^3 x_2 x_3 y_1^3 y_2^4$$
$$- 216 x_1^3 x_2 x_3 y_1^3 y_2^3 y_3 + 288 x_1^3 x_2 x_3 y_1^3 y_2^2 y_3^2 - 168 x_1^3 x_2 x_3 y_1^3 y_2 y_3^3$$
$$+ 36 x_1^3 x_2 x_3 y_1^3 y_3^4 - 98 x_1^3 x_2 x_3 y_1^2 y_2^5 + 444 x_1^3 x_2 x_3 y_1^2 y_2^4 y_3 - 764 x_1^3 x_2 x_3 y_1^2 y_2^3 y_3^2$$
$$+ 608 x_1^3 x_2 x_3 y_1^2 y_2^2 y_3^3 - 210 x_1^3 x_2 x_3 y_1^2 y_2 y_3^4 + 20 x_1^3 x_2 x_3 y_1^2 y_3^5 + 72 x_1^3 x_2 x_3 y_1 y_2^6$$
$$- 408 x_1^3 x_2 x_3 y_1 y_2^5 y_3 + 840 x_1^3 x_2 x_3 y_1 y_2^4 y_3^2 - 792 x_1^3 x_2 x_3 y_1 y_2^3 y_3^3$$
$$+ 336 x_1^3 x_2 x_3 y_1 y_2^2 y_3^4 - 48 x_1^3 x_2 x_3 y_1 y_2 y_3^5 - 18 x_1^3 x_2 x_3 y_2^7 + 132 x_1^3 x_2 x_3 y_2^6 y_3$$
$$- 316 x_1^3 x_2 x_3 y_2^5 y_3^2 + 336 x_1^3 x_2 x_3 y_2^4 y_3^3 - 162 x_1^3 x_2 x_3 y_2^3 y_3^4 + 28 x_1^3 x_2 x_3 y_2^2 y_3^5$$

$$+ 8 x_1^3 x_3^2 y_1^4 y_2^3 - 24 x_1^3 x_3^2 y_1^4 y_2^2 y_3 + 24 x_1^3 x_3^2 y_1^4 y_2 y_3^2 - 8 x_1^3 x_3^2 y_1^4 y_3^3$$
$$- 30 x_1^3 x_3^2 y_1^3 y_2^4 + 108 x_1^3 x_3^2 y_1^3 y_2^3 y_3 - 144 x_1^3 x_3^2 y_1^3 y_2^2 y_3^2 + 84 x_1^3 x_3^2 y_1^3 y_2 y_3^3$$
$$- 18 x_1^3 x_3^2 y_1^3 y_3^4 + 41 x_1^3 x_3^2 y_1^2 y_2^5 - 183 x_1^3 x_3^2 y_1^2 y_2^4 y_3 + 306 x_1^3 x_3^2 y_1^2 y_2^3 y_3^2$$
$$- 230 x_1^3 x_3^2 y_1^2 y_2^2 y_3^3 + 69 x_1^3 x_3^2 y_1^2 y_2 y_3^4 - 3 x_1^3 x_3^2 y_1^2 y_3^5 - 24 x_1^3 x_3^2 y_1 y_2^6$$
$$+ 138 x_1^3 x_3^2 y_1 y_2^5 y_3 - 276 x_1^3 x_3^2 y_1 y_2^4 y_3^2 + 240 x_1^3 x_3^2 y_1 y_2^3 y_3^3 - 84 x_1^3 x_3^2 y_1 y_2^2 y_3^4$$
$$+ 6 x_1^3 x_3^2 y_1 y_2 y_3^5 + 5 x_1^3 x_3^2 y_2^7 - 39 x_1^3 x_3^2 y_2^6 y_3 + 90 x_1^3 x_3^2 y_2^5 y_3^2 - 86 x_1^3 x_3^2 y_2^4 y_3^3$$
$$+ 33 x_1^3 x_3^2 y_2^3 y_3^4 - 3 x_1^3 x_3^2 y_2^2 y_3^5 + 3 x_1^2 x_2^3 y_1^5 y_2^2 - 6 x_1^2 x_2^3 y_1^5 y_2 y_3 + 3 x_1^2 x_2^3 y_1^5 y_3^2$$
$$- 18 x_1^2 x_2^3 y_1^4 y_2^3 + 39 x_1^2 x_2^3 y_1^4 y_2^2 y_3 - 24 x_1^2 x_2^3 y_1^4 y_2 y_3^2 + 3 x_1^2 x_2^3 y_1^4 y_3^3$$
$$+ 17 x_1^2 x_2^3 y_1^3 y_2^4 - 12 x_1^2 x_2^3 y_1^3 y_2^3 y_3 - 40 x_1^2 x_2^3 y_1^3 y_2^2 y_3^2 + 48 x_1^2 x_2^3 y_1^3 y_2 y_3^3$$
$$- 13 x_1^2 x_2^3 y_1^3 y_3^4 + 36 x_1^2 x_2^3 y_1^2 y_2^5 - 189 x_1^2 x_2^3 y_1^2 y_2^4 y_3 + 360 x_1^2 x_2^3 y_1^2 y_2^3 y_3^2$$
$$- 324 x_1^2 x_2^3 y_1^2 y_2^2 y_3^3 + 144 x_1^2 x_2^3 y_1^2 y_2 y_3^4 - 27 x_1^2 x_2^3 y_1^2 y_3^5 - 54 x_1^2 x_2^3 y_1 y_2^6$$
$$+ 228 x_1^2 x_2^3 y_1 y_2^5 y_3 - 423 x_1^2 x_2^3 y_1 y_2^4 y_3^2 + 468 x_1^2 x_2^3 y_1 y_2^3 y_3^3 - 333 x_1^2 x_2^3 y_1 y_2^2 y_3^4$$
$$+ 138 x_1^2 x_2^3 y_1 y_2 y_3^5 - 24 x_1^2 x_2^3 y_1 y_3^6 + 26 x_1^2 x_2^3 y_2^7 - 90 x_1^2 x_2^3 y_2^6 y_3 + 134 x_1^2 x_2^3 y_2^5 y_3^2$$
$$- 145 x_1^2 x_2^3 y_2^4 y_3^3 + 152 x_1^2 x_2^3 y_2^3 y_3^4 - 121 x_1^2 x_2^3 y_2^2 y_3^5 + 54 x_1^2 x_2^3 y_2 y_3^6$$
$$- 10 x_1^2 x_2^3 y_3^7 - 9 x_1^2 x_2^2 x_3 y_1^5 y_2^2 + 18 x_1^2 x_2^2 x_3 y_1^5 y_2 y_3 - 9 x_1^2 x_2^2 x_3 y_1^5 y_3^2$$
$$+ 69 x_1^2 x_2^2 x_3 y_1^4 y_2^3 - 162 x_1^2 x_2^2 x_3 y_1^4 y_2^2 y_3 + 117 x_1^2 x_2^2 x_3 y_1^4 y_2 y_3^2 - 24 x_1^2 x_2^2 x_3 y_1^4 y_3^3$$
$$- 156 x_1^2 x_2^2 x_3 y_1^3 y_2^4 + 408 x_1^2 x_2^2 x_3 y_1^3 y_2^3 y_3 - 366 x_1^2 x_2^2 x_3 y_1^3 y_2^2 y_3^2$$
$$+ 132 x_1^2 x_2^2 x_3 y_1^3 y_2 y_3^3 - 18 x_1^2 x_2^2 x_3 y_1^3 y_3^4 + 144 x_1^2 x_2^2 x_3 y_1^2 y_2^5$$
$$- 450 x_1^2 x_2^2 x_3 y_1^2 y_2^4 y_3 + 534 x_1^2 x_2^2 x_3 y_1^2 y_2^3 y_3^2 - 288 x_1^2 x_2^2 x_3 y_1^2 y_2^2 y_3^3$$
$$+ 54 x_1^2 x_2^2 x_3 y_1^2 y_2 y_3^4 + 6 x_1^2 x_2^2 x_3 y_1^2 y_3^5 - 39 x_1^2 x_2^2 x_3 y_1 y_2^6 + 198 x_1^2 x_2^2 x_3 y_1 y_2^5 y_3$$
$$- 369 x_1^2 x_2^2 x_3 y_1 y_2^4 y_3^2 + 300 x_1^2 x_2^2 x_3 y_1 y_2^3 y_3^3 - 90 x_1^2 x_2^2 x_3 y_1 y_2^2 y_3^4 - 9 x_1^2 x_2^2 x_3 y_2^7$$
$$- 12 x_1^2 x_2^2 x_3 y_2^6 y_3 + 93 x_1^2 x_2^2 x_3 y_2^5 y_3^2 - 120 x_1^2 x_2^2 x_3 y_2^4 y_3^3 + 54 x_1^2 x_2^2 x_3 y_2^3 y_3^4$$
$$- 6 x_1^2 x_2^2 x_3 y_2^2 y_3^5 + 9 x_1^2 x_2 x_3^2 y_1^5 y_2^2 - 18 x_1^2 x_2 x_3^2 y_1^5 y_2 y_3 + 9 x_1^2 x_2 x_3^2 y_1^5 y_3^2$$
$$- 84 x_1^2 x_2 x_3^2 y_1^4 y_2^3 + 207 x_1^2 x_2 x_3^2 y_1^4 y_2^2 y_3 - 162 x_1^2 x_2 x_3^2 y_1^4 y_2 y_3^2 + 39 x_1^2 x_2 x_3^2 y_1^4 y_3^3$$
$$+ 225 x_1^2 x_2 x_3^2 y_1^3 y_2^4 - 636 x_1^2 x_2 x_3^2 y_1^3 y_2^3 y_3 + 636 x_1^2 x_2 x_3^2 y_1^3 y_2^2 y_3^2$$
$$- 264 x_1^2 x_2 x_3^2 y_1^3 y_2 y_3^3 + 39 x_1^2 x_2 x_3^2 y_1^3 y_3^4 - 258 x_1^2 x_2 x_3^2 y_1^2 y_2^5$$
$$+ 855 x_1^2 x_2 x_3^2 y_1^2 y_2^4 y_3 - 1080 x_1^2 x_2 x_3^2 y_1^2 y_2^3 y_3^2 + 636 x_1^2 x_2 x_3^2 y_1^2 y_2^2 y_3^3$$
$$- 162 x_1^2 x_2 x_3^2 y_1^2 y_2 y_3^4 + 9 x_1^2 x_2 x_3^2 y_1^2 y_3^5 + 132 x_1^2 x_2 x_3^2 y_1 y_2^6 - 540 x_1^2 x_2 x_3^2 y_1 y_2^5 y_3$$
$$+ 855 x_1^2 x_2 x_3^2 y_1 y_2^4 y_3^2 - 636 x_1^2 x_2 x_3^2 y_1 y_2^3 y_3^3 + 207 x_1^2 x_2 x_3^2 y_1 y_2^2 y_3^4$$

$$-18 x_1^2 x_2 x_3^2 y_1 y_2 y_3^5 - 24 x_1^2 x_2 x_3^2 y_2^7 + 132 x_1^2 x_2 x_3^2 y_2^6 y_3 - 258 x_1^2 x_2 x_3^2 y_2^5 y_3^2$$
$$+ 225 x_1^2 x_2 x_3^2 y_2^4 y_3^3 - 84 x_1^2 x_2 x_3^2 y_2^3 y_3^4 + 9 x_1^2 x_2 x_3^2 y_2^2 y_3^5 - 3 x_1^2 x_3^3 y_1^5 y_2^2$$
$$+ 6 x_1^2 x_3^3 y_1^5 y_2 y_3 - 3 x_1^2 x_3^3 y_1^5 y_3^2 + 33 x_1^2 x_3^3 y_1^4 y_2^3 - 84 x_1^2 x_3^3 y_1^4 y_2^2 y_3$$
$$+ 69 x_1^2 x_3^3 y_1^4 y_2 y_3^2 - 18 x_1^2 x_3^3 y_1^4 y_3^3 - 86 x_1^2 x_3^3 y_1^3 y_2^4 + 240 x_1^2 x_3^3 y_1^3 y_2^3 y_3$$
$$- 230 x_1^2 x_3^3 y_1^3 y_2^2 y_3^2 + 84 x_1^2 x_3^3 y_1^3 y_2 y_3^3 - 8 x_1^2 x_3^3 y_1^3 y_3^4 + 90 x_1^2 x_3^3 y_1^2 y_2^5$$
$$- 276 x_1^2 x_3^3 y_1^2 y_2^4 y_3 + 306 x_1^2 x_3^3 y_1^2 y_2^3 y_3^2 - 144 x_1^2 x_3^3 y_1^2 y_2^2 y_3^3 + 24 x_1^2 x_3^3 y_1^2 y_2 y_3^4$$
$$- 39 x_1^2 x_3^3 y_1 y_2^6 + 138 x_1^2 x_3^3 y_1 y_2^5 y_3 - 183 x_1^2 x_3^3 y_1 y_2^4 y_3^2 + 108 x_1^2 x_3^3 y_1 y_2^3 y_3^3$$
$$- 24 x_1^2 x_3^3 y_1 y_2^2 y_3^4 + 5 x_1^2 x_3^3 y_2^7 - 24 x_1^2 x_3^3 y_2^6 y_3 + 41 x_1^2 x_3^3 y_2^5 y_3^2 - 30 x_1^2 x_3^3 y_2^4 y_3^3$$
$$+ 8 x_1^2 x_3^3 y_2^3 y_3^4 + 11 x_1 x_2^4 y_1^5 y_2^2 - 24 x_1 x_2^4 y_1^5 y_2 y_3 + 13 x_1 x_2^4 y_1^5 y_3^2 - 63 x_1 x_2^4 y_1^4 y_2^3$$
$$+ 150 x_1 x_2^4 y_1^4 y_2^2 y_3 - 105 x_1 x_2^4 y_1^4 y_2 y_3^2 + 18 x_1 x_2^4 y_1^4 y_3^3 + 138 x_1 x_2^4 y_1^3 y_2^4$$
$$- 372 x_1 x_2^4 y_1^3 y_2^3 y_3 + 358 x_1 x_2^4 y_1^3 y_2^2 y_3^2 - 156 x_1 x_2^4 y_1^3 y_2 y_3^3 + 32 x_1 x_2^4 y_1^3 y_3^4$$
$$- 139 x_1 x_2^4 y_1^2 y_2^5 + 435 x_1 x_2^4 y_1^2 y_2^4 y_3 - 572 x_1 x_2^4 y_1^2 y_2^3 y_3^2 + 410 x_1 x_2^4 y_1^2 y_2^2 y_3^3$$
$$- 153 x_1 x_2^4 y_1^2 y_2 y_3^4 + 19 x_1 x_2^4 y_1^2 y_3^5 + 48 x_1 x_2^4 y_1 y_2^6 - 174 x_1 x_2^4 y_1 y_2^5 y_3$$
$$+ 336 x_1 x_2^4 y_1 y_2^4 y_3^2 - 408 x_1 x_2^4 y_1 y_2^3 y_3^3 + 291 x_1 x_2^4 y_1 y_2^2 y_3^4 - 114 x_1 x_2^4 y_1 y_2 y_3^5$$
$$+ 21 x_1 x_2^4 y_1 y_3^6 - 35 x_1 x_2^4 y_2^5 y_3^2 + 111 x_1 x_2^4 y_2^4 y_3^3 - 145 x_1 x_2^4 y_2^3 y_3^4 + 100 x_1 x_2^4 y_2^2 y_3^5$$
$$- 36 x_1 x_2^4 y_2 y_3^6 + 5 x_1 x_2^4 y_3^7 - 20 x_1 x_2^3 x_3 y_1^5 y_2^2 + 48 x_1 x_2^3 x_3 y_1^5 y_2 y_3$$
$$- 28 x_1 x_2^3 x_3 y_1^5 y_3^2 + 102 x_1 x_2^3 x_3 y_1^4 y_2^3 - 252 x_1 x_2^3 x_3 y_1^4 y_2^2 y_3 + 174 x_1 x_2^3 x_3 y_1^4 y_2 y_3^2$$
$$- 24 x_1 x_2^3 x_3 y_1^4 y_3^3 - 208 x_1 x_2^3 x_3 y_1^3 y_2^4 + 528 x_1 x_2^3 x_3 y_1^3 y_2^3 y_3 - 440 x_1 x_2^3 x_3 y_1^3 y_2^2 y_3^2$$
$$+ 144 x_1 x_2^3 x_3 y_1^3 y_2 y_3^3 - 24 x_1 x_2^3 x_3 y_1^3 y_3^4 + 226 x_1 x_2^3 x_3 y_1^2 y_2^5 - 612 x_1 x_2^3 x_3 y_1^2 y_2^4 y_3$$
$$+ 680 x_1 x_2^3 x_3 y_1^2 y_2^3 y_3^2 - 440 x_1 x_2^3 x_3 y_1^2 y_2^2 y_3^3 + 174 x_1 x_2^3 x_3 y_1^2 y_2 y_3^4$$
$$- 28 x_1 x_2^3 x_3 y_1^2 y_3^5 - 144 x_1 x_2^3 x_3 y_1 y_2^6 + 432 x_1 x_2^3 x_3 y_1 y_2^5 y_3 - 612 x_1 x_2^3 x_3 y_1 y_2^4 y_3^2$$
$$+ 528 x_1 x_2^3 x_3 y_1 y_2^3 y_3^3 - 252 x_1 x_2^3 x_3 y_1 y_2^2 y_3^4 + 48 x_1 x_2^3 x_3 y_1 y_2 y_3^5 + 44 x_1 x_2^3 x_3 y_2^7$$
$$- 144 x_1 x_2^3 x_3 y_2^6 y_3 + 226 x_1 x_2^3 x_3 y_2^5 y_3^2 - 208 x_1 x_2^3 x_3 y_2^4 y_3^3 + 102 x_1 x_2^3 x_3 y_2^3 y_3^4$$
$$- 20 x_1 x_2^3 x_3 y_2^2 y_3^5 - 6 x_1 x_2^2 x_3^2 y_1^5 y_2^2 + 6 x_1 x_2^2 x_3^2 y_1^5 y_3^2 + 54 x_1 x_2^2 x_3^2 y_1^4 y_2^3$$
$$- 90 x_1 x_2^2 x_3^2 y_1^4 y_2^2 y_3 + 54 x_1 x_2^2 x_3^2 y_1^4 y_2 y_3^2 - 18 x_1 x_2^2 x_3^2 y_1^4 y_3^3 - 120 x_1 x_2^2 x_3^2 y_1^3 y_2^4$$
$$+ 300 x_1 x_2^2 x_3^2 y_1^3 y_2^3 y_3 - 288 x_1 x_2^2 x_3^2 y_1^3 y_2^2 y_3^2 + 132 x_1 x_2^2 x_3^2 y_1^3 y_2 y_3^3$$
$$- 24 x_1 x_2^2 x_3^2 y_1^3 y_3^4 + 93 x_1 x_2^2 x_3^2 y_1^2 y_2^5 - 369 x_1 x_2^2 x_3^2 y_1^2 y_2^4 y_3$$
$$+ 534 x_1 x_2^2 x_3^2 y_1^2 y_2^3 y_3^2 - 366 x_1 x_2^2 x_3^2 y_1^2 y_2^2 y_3^3 + 117 x_1 x_2^2 x_3^2 y_1^2 y_2 y_3^4$$
$$- 9 x_1 x_2^2 x_3^2 y_1^2 y_3^5 - 12 x_1 x_2^2 x_3^2 y_1 y_2^6 + 198 x_1 x_2^2 x_3^2 y_1 y_2^5 y_3 - 450 x_1 x_2^2 x_3^2 y_1 y_2^4 y_3^2$$

$$+ 408 x_1 x_2^2 x_3^2 y_1 y_2^3 y_3^3 - 162 x_1 x_2^2 x_3^2 y_1 y_2^2 y_3^4 + 18 x_1 x_2^2 x_3^2 y_1 y_2 y_3^5 - 9 x_1 x_2^2 x_3^2 y_2^7$$
$$- 39 x_1 x_2^2 x_3^2 y_2^6 y_3 + 144 x_1 x_2^2 x_3^2 y_2^5 y_3^2 - 156 x_1 x_2^2 x_3^2 y_2^4 y_3^3 + 69 x_1 x_2^2 x_3^2 y_2^3 y_3^4$$
$$- 9 x_1 x_2^2 x_3^2 y_2^2 y_3^5 + 28 x_1 x_2 x_3^3 y_1^5 y_2^2 - 48 x_1 x_2 x_3^3 y_1^5 y_2 y_3 + 20 x_1 x_2 x_3^3 y_1^5 y_3^2$$
$$- 162 x_1 x_2 x_3^3 y_1^4 y_2^3 + 336 x_1 x_2 x_3^3 y_1^4 y_2^2 y_3 - 210 x_1 x_2 x_3^3 y_1^4 y_2 y_3^2 + 36 x_1 x_2 x_3^3 y_1^4 y_3^3$$
$$+ 336 x_1 x_2 x_3^3 y_1^3 y_2^4 - 792 x_1 x_2 x_3^3 y_1^3 y_2^3 y_3 + 608 x_1 x_2 x_3^3 y_1^3 y_2^2 y_3^2$$
$$- 168 x_1 x_2 x_3^3 y_1^3 y_2 y_3^3 + 16 x_1 x_2 x_3^3 y_1^3 y_3^4 - 316 x_1 x_2 x_3^3 y_1^2 y_2^5 + 840 x_1 x_2 x_3^3 y_1^2 y_2^4 y_3$$
$$- 764 x_1 x_2 x_3^3 y_1^2 y_2^3 y_3^2 + 288 x_1 x_2 x_3^3 y_1^2 y_2^2 y_3^3 - 48 x_1 x_2 x_3^3 y_1^2 y_2 y_3^4$$
$$+ 132 x_1 x_2 x_3^3 y_1 y_2^6 - 408 x_1 x_2 x_3^3 y_1 y_2^5 y_3 + 444 x_1 x_2 x_3^3 y_1 y_2^4 y_3^2 - 216 x_1 x_2 x_3^3 y_1 y_2^3 y_3^3$$
$$+ 48 x_1 x_2 x_3^3 y_1 y_2^2 y_3^4 - 18 x_1 x_2 x_3^3 y_2^7 + 72 x_1 x_2 x_3^3 y_2^6 y_3 - 98 x_1 x_2 x_3^3 y_2^5 y_3^2$$
$$+ 60 x_1 x_2 x_3^3 y_2^4 y_3^3 - 16 x_1 x_2 x_3^3 y_2^3 y_3^4 - 13 x_1 x_3^4 y_1^5 y_2^2 + 24 x_1 x_3^4 y_1^5 y_2 y_3$$
$$- 11 x_1 x_3^4 y_1^5 y_3^2 + 69 x_1 x_3^4 y_1^4 y_2^3 - 144 x_1 x_3^4 y_1^4 y_2^2 y_3 + 87 x_1 x_3^4 y_1^4 y_2 y_3^2$$
$$- 12 x_1 x_3^4 y_1^4 y_3^3 - 146 x_1 x_3^4 y_1^3 y_2^4 + 336 x_1 x_3^4 y_1^3 y_2^3 y_3 - 238 x_1 x_3^4 y_1^3 y_2^2 y_3^2$$
$$+ 48 x_1 x_3^4 y_1^3 y_2 y_3^3 + 154 x_1 x_3^4 y_1^2 y_2^5 - 384 x_1 x_3^4 y_1^2 y_2^4 y_3 + 302 x_1 x_3^4 y_1^2 y_2^3 y_3^2$$
$$- 72 x_1 x_3^4 y_1^2 y_2^2 y_3^3 - 81 x_1 x_3^4 y_1 y_2^6 + 216 x_1 x_3^4 y_1 y_2^5 y_3 - 183 x_1 x_3^4 y_1 y_2^4 y_3^2$$
$$+ 48 x_1 x_3^4 y_1 y_2^3 y_3^3 + 17 x_1 x_3^4 y_2^7 - 48 x_1 x_3^4 y_2^6 y_3 + 43 x_1 x_3^4 y_2^5 y_3^2 - 12 x_1 x_3^4 y_2^4 y_3^3$$
$$- x_2^5 y_1^7 + 9 x_2^5 y_1^6 y_2 - 6 x_2^5 y_1^6 y_3 - 27 x_2^5 y_1^5 y_2^2 + 30 x_2^5 y_1^5 y_2 y_3 - 4 x_2^5 y_1^5 y_3^2$$
$$+ 38 x_2^5 y_1^4 y_2^3 - 75 x_2^5 y_1^4 y_2^2 y_3 + 45 x_2^5 y_1^4 y_2 y_3^2 - 13 x_2^5 y_1^4 y_3^3 - 12 x_2^5 y_1^3 y_2^4$$
$$+ 84 x_2^5 y_1^3 y_2^3 y_3 - 126 x_2^5 y_1^3 y_2^2 y_3^2 + 72 x_2^5 y_1^3 y_2 y_3^3 - 13 x_2^5 y_1^3 y_3^4 - 42 x_2^5 y_1^2 y_2^5$$
$$+ 12 x_2^5 y_1^2 y_2^4 y_3 + 116 x_2^5 y_1^2 y_2^3 y_3^2 - 126 x_2^5 y_1^2 y_2^2 y_3^3 + 45 x_2^5 y_1^2 y_2 y_3^4 - 4 x_2^5 y_1^2 y_3^5$$
$$+ 60 x_2^5 y_1 y_2^6 - 108 x_2^5 y_1 y_2^5 y_3 + 12 x_2^5 y_1 y_2^4 y_3^2 + 84 x_2^5 y_1 y_2^3 y_3^3 - 75 x_2^5 y_1 y_2^2 y_3^4$$
$$+ 30 x_2^5 y_1 y_2 y_3^5 - 6 x_2^5 y_1 y_3^6 - 24 x_2^5 y_2^7 + 60 x_2^5 y_2^6 y_3 - 42 x_2^5 y_2^5 y_3^2 - 12 x_2^5 y_2^4 y_3^3$$
$$+ 38 x_2^5 y_2^3 y_3^4 - 27 x_2^5 y_2^2 y_3^5 + 9 x_2^5 y_2 y_3^6 - x_2^5 y_3^7 + 5 x_2^4 x_3 y_1^7 - 36 x_2^4 x_3 y_1^6 y_2$$
$$+ 21 x_2^4 x_3 y_1^6 y_3 + 100 x_2^4 x_3 y_1^5 y_2^2 - 114 x_2^4 x_3 y_1^5 y_2 y_3 + 19 x_2^4 x_3 y_1^5 y_3^2$$
$$- 145 x_2^4 x_3 y_1^4 y_2^3 + 291 x_2^4 x_3 y_1^4 y_2^2 y_3 - 153 x_2^4 x_3 y_1^4 y_2 y_3^2 + 32 x_2^4 x_3 y_1^4 y_3^3$$
$$+ 111 x_2^4 x_3 y_1^3 y_2^4 - 408 x_2^4 x_3 y_1^3 y_2^3 y_3 + 410 x_2^4 x_3 y_1^3 y_2^2 y_3^2 - 156 x_2^4 x_3 y_1^3 y_2 y_3^3$$
$$+ 18 x_2^4 x_3 y_1^3 y_3^4 - 35 x_2^4 x_3 y_1^2 y_2^5 + 336 x_2^4 x_3 y_1^2 y_2^4 y_3 - 572 x_2^4 x_3 y_1^2 y_2^3 y_3^2$$
$$+ 358 x_2^4 x_3 y_1^2 y_2^2 y_3^3 - 105 x_2^4 x_3 y_1^2 y_2 y_3^4 + 13 x_2^4 x_3 y_1^2 y_3^5 - 174 x_2^4 x_3 y_1 y_2^5 y_3$$
$$+ 435 x_2^4 x_3 y_1 y_2^4 y_3^2 - 372 x_2^4 x_3 y_1 y_2^3 y_3^3 + 150 x_2^4 x_3 y_1 y_2^2 y_3^4 - 24 x_2^4 x_3 y_1 y_2 y_3^5$$
$$+ 48 x_2^4 x_3 y_2^6 y_3 - 139 x_2^4 x_3 y_2^5 y_3^2 + 138 x_2^4 x_3 y_2^4 y_3^3 - 63 x_2^4 x_3 y_2^3 y_3^4 + 11 x_2^4 x_3 y_2^2 y_3^5$$

$$- 10 x_2^3 x_3^2 y_1^7 + 54 x_2^3 x_3^2 y_1^6 y_2 - 24 x_2^3 x_3^2 y_1^6 y_3 - 121 x_2^3 x_3^2 y_1^5 y_2^2 + 138 x_2^3 x_3^2 y_1^5 y_2 y_3$$
$$- 27 x_2^3 x_3^2 y_1^5 y_3^2 + 152 x_2^3 x_3^2 y_1^4 y_2^3 - 333 x_2^3 x_3^2 y_1^4 y_2^2 y_3 + 144 x_2^3 x_3^2 y_1^4 y_2 y_3^2$$
$$- 13 x_2^3 x_3^2 y_1^4 y_3^3 - 145 x_2^3 x_3^2 y_1^3 y_2^4 + 468 x_2^3 x_3^2 y_1^3 y_2^3 y_3 - 324 x_2^3 x_3^2 y_1^3 y_2^2 y_3^2$$
$$+ 48 x_2^3 x_3^2 y_1^3 y_2 y_3^3 + 3 x_2^3 x_3^2 y_1^3 y_3^4 + 134 x_2^3 x_3^2 y_1^2 y_2^5 - 423 x_2^3 x_3^2 y_1^2 y_2^4 y_3$$
$$+ 360 x_2^3 x_3^2 y_1^2 y_2^3 y_3^2 - 40 x_2^3 x_3^2 y_1^2 y_2^2 y_3^3 - 24 x_2^3 x_3^2 y_1^2 y_2 y_3^4 + 3 x_2^3 x_3^2 y_1^2 y_3^5$$
$$- 90 x_2^3 x_3^2 y_1 y_2^6 + 228 x_2^3 x_3^2 y_1 y_2^5 y_3 - 189 x_2^3 x_3^2 y_1 y_2^4 y_3^2 - 12 x_2^3 x_3^2 y_1 y_2^3 y_3^3$$
$$+ 39 x_2^3 x_3^2 y_1 y_2^2 y_3^4 - 6 x_2^3 x_3^2 y_1 y_2 y_3^5 + 26 x_2^3 x_3^2 y_2^7 - 54 x_2^3 x_3^2 y_2^6 y_3 + 36 x_2^3 x_3^2 y_2^5 y_3^2$$
$$+ 17 x_2^3 x_3^2 y_2^4 y_3^3 - 18 x_2^3 x_3^2 y_2^3 y_3^4 + 3 x_2^3 x_3^2 y_2^2 y_3^5 + 10 x_2^2 x_3^3 y_1^7 - 36 x_2^2 x_3^3 y_1^6 y_2$$
$$+ 6 x_2^2 x_3^3 y_1^6 y_3 + 33 x_2^2 x_3^3 y_1^5 y_2^2 - 30 x_2^2 x_3^3 y_1^5 y_2 y_3 + 7 x_2^2 x_3^3 y_1^5 y_3^2 + 29 x_2^2 x_3^3 y_1^4 y_2^3$$
$$+ 18 x_2^2 x_3^3 y_1^4 y_2^2 y_3 + 21 x_2^2 x_3^3 y_1^4 y_2 y_3^2 - 18 x_2^2 x_3^3 y_1^4 y_3^3 - 60 x_2^2 x_3^3 y_1^3 y_2^4$$
$$+ 72 x_2^2 x_3^3 y_1^3 y_2^3 y_3 - 138 x_2^2 x_3^3 y_1^3 y_2^2 y_3^2 + 84 x_2^2 x_3^3 y_1^3 y_2 y_3^3 - 8 x_2^2 x_3^3 y_1^3 y_3^4$$
$$+ 6 x_2^2 x_3^3 y_1^2 y_2^5 - 114 x_2^2 x_3^3 y_1^2 y_2^4 y_3 + 218 x_2^2 x_3^3 y_1^2 y_2^3 y_3^2 - 144 x_2^2 x_3^3 y_1^2 y_2^2 y_3^3$$
$$+ 24 x_2^2 x_3^3 y_1^2 y_2 y_3^4 + 33 x_2^2 x_3^3 y_1 y_2^6 + 54 x_2^2 x_3^3 y_1 y_2^5 y_3 - 141 x_2^2 x_3^3 y_1 y_2^4 y_3^2$$
$$+ 108 x_2^2 x_3^3 y_1 y_2^3 y_3^3 - 24 x_2^2 x_3^3 y_1 y_2^2 y_3^4 - 15 x_2^2 x_3^3 y_2^7 - 6 x_2^2 x_3^3 y_2^6 y_3$$
$$+ 33 x_2^2 x_3^3 y_2^5 y_3^2 - 30 x_2^2 x_3^3 y_2^4 y_3^3 + 8 x_2^2 x_3^3 y_2^3 y_3^4 - 5 x_2 x_3^4 y_1^7 + 9 x_2 x_3^4 y_1^6 y_2$$
$$+ 6 x_2 x_3^4 y_1^6 y_3 + 32 x_2 x_3^4 y_1^5 y_2^2 - 48 x_2 x_3^4 y_1^5 y_2 y_3 + 11 x_2 x_3^4 y_1^5 y_3^2 - 124 x_2 x_3^4 y_1^4 y_2^3$$
$$+ 174 x_2 x_3^4 y_1^4 y_2^2 y_3 - 87 x_2 x_3^4 y_1^4 y_2 y_3^2 + 12 x_2 x_3^4 y_1^4 y_3^3 + 171 x_2 x_3^4 y_1^3 y_2^4$$
$$- 336 x_2 x_3^4 y_1^3 y_2^3 y_3 + 238 x_2 x_3^4 y_1^3 y_2^2 y_3^2 - 48 x_2 x_3^4 y_1^3 y_2 y_3^3 - 119 x_2 x_3^4 y_1^2 y_2^5$$
$$+ 354 x_2 x_3^4 y_1^2 y_2^4 y_3 - 302 x_2 x_3^4 y_1^2 y_2^3 y_3^2 + 72 x_2 x_3^4 y_1^2 y_2^2 y_3^3 + 42 x_2 x_3^4 y_1 y_2^6$$
$$- 192 x_2 x_3^4 y_1 y_2^5 y_3 + 183 x_2 x_3^4 y_1 y_2^4 y_3^2 - 48 x_2 x_3^4 y_1 y_2^3 y_3^3 - 6 x_2 x_3^4 y_2^7$$
$$+ 42 x_2 x_3^4 y_2^6 y_3 - 43 x_2 x_3^4 y_2^5 y_3^2 + 12 x_2 x_3^4 y_2^4 y_3^3 + x_3^5 y_1^7 - 3 x_3^5 y_1^6 y_3 - 17 x_3^5 y_1^5 y_2^2$$
$$+ 24 x_3^5 y_1^5 y_2 y_3 - 6 x_3^5 y_1^5 y_3^2 + 50 x_3^5 y_1^4 y_2^3 - 75 x_3^5 y_1^4 y_2^2 y_3 + 30 x_3^5 y_1^4 y_2 y_3^2$$
$$- 65 x_3^5 y_1^3 y_2^4 + 120 x_3^5 y_1^3 y_2^3 y_3 - 60 x_3^5 y_1^3 y_2^2 y_3^2 + 44 x_3^5 y_1^2 y_2^5 - 105 x_3^5 y_1^2 y_2^4 y_3$$
$$+ 60 x_3^5 y_1^2 y_2^3 y_3^2 - 15 x_3^5 y_1 y_2^6 + 48 x_3^5 y_1 y_2^5 y_3 - 30 x_3^5 y_1 y_2^4 y_3^2 + 2 x_3^5 y_2^7 - 9 x_3^5 y_2^6 y_3$$
$$+ 6 x_3^5 y_2^5 y_3^2 - 3 x_1^2 y_1^4 y_2^5 + 15 x_1^2 y_1^4 y_2^4 y_3 - 30 x_1^2 y_1^4 y_2^3 y_3^2 + 30 x_1^2 y_1^4 y_2^2 y_3^3$$
$$- 15 x_1^2 y_1^4 y_2 y_3^4 + 3 x_1^2 y_1^4 y_3^5 + 9 x_1^2 y_1^3 y_2^6 - 48 x_1^2 y_1^3 y_2^5 y_3 + 105 x_1^2 y_1^3 y_2^4 y_3^2$$
$$- 120 x_1^2 y_1^3 y_2^3 y_3^3 + 75 x_1^2 y_1^3 y_2^2 y_3^4 - 24 x_1^2 y_1^3 y_2 y_3^5 + 3 x_1^2 y_1^3 y_3^6 - 9 x_1^2 y_1^2 y_2^7$$
$$+ 54 x_1^2 y_1^2 y_2^6 y_3 - 135 x_1^2 y_1^2 y_2^5 y_3^2 + 180 x_1^2 y_1^2 y_2^4 y_3^3 - 135 x_1^2 y_1^2 y_2^3 y_3^4$$
$$+ 54 x_1^2 y_1^2 y_2^2 y_3^5 - 9 x_1^2 y_1^2 y_2 y_3^6 + 3 x_1^2 y_1 y_2^8 - 24 x_1^2 y_1 y_2^7 y_3 + 75 x_1^2 y_1 y_2^6 y_3^2$$

$$- 120\, x_1^2 y_1 y_2^5 y_3^3 + 105\, x_1^2 y_1 y_2^4 y_3^4 - 48\, x_1^2 y_1 y_2^3 y_3^5 + 9\, x_1^2 y_1 y_2^2 y_3^6 + 3\, x_1^2 y_2^8 y_3$$
$$- 15\, x_1^2 y_2^7 y_3^2 + 30\, x_1^2 y_2^6 y_3^3 - 30\, x_1^2 y_2^5 y_3^4 + 15\, x_1^2 y_2^4 y_3^5 - 3\, x_1^2 y_2^3 y_3^6 - 6\, x_1 x_2 y_1^5 y_2^4$$
$$+ 24\, x_1 x_2 y_1^5 y_2^3 y_3 - 36\, x_1 x_2 y_1^5 y_2^2 y_3^2 + 24\, x_1 x_2 y_1^5 y_2 y_3^3 - 6\, x_1 x_2 y_1^5 y_3^4 + 30\, x_1 x_2 y_1^4 y_2^5$$
$$- 126\, x_1 x_2 y_1^4 y_2^4 y_3 + 204\, x_1 x_2 y_1^4 y_2^3 y_3^2 - 156\, x_1 x_2 y_1^4 y_2^2 y_3^3 + 54\, x_1 x_2 y_1^4 y_2 y_3^4$$
$$- 6\, x_1 x_2 y_1^4 y_3^5 - 54\, x_1 x_2 y_1^3 y_2^6 + 240\, x_1 x_2 y_1^3 y_2^5 y_3 - 426\, x_1 x_2 y_1^3 y_2^4 y_3^2$$
$$+ 384\, x_1 x_2 y_1^3 y_2^3 y_3^3 - 186\, x_1 x_2 y_1^3 y_2^2 y_3^4 + 48\, x_1 x_2 y_1^3 y_2 y_3^5 - 6\, x_1 x_2 y_1^3 y_3^6$$
$$+ 42\, x_1 x_2 y_1^2 y_2^7 - 204\, x_1 x_2 y_1^2 y_2^6 y_3 + 414\, x_1 x_2 y_1^2 y_2^5 y_3^2 - 456\, x_1 x_2 y_1^2 y_2^4 y_3^3$$
$$+ 294\, x_1 x_2 y_1^2 y_2^3 y_3^4 - 108\, x_1 x_2 y_1^2 y_2^2 y_3^5 + 18\, x_1 x_2 y_1^2 y_2 y_3^6 - 12\, x_1 x_2 y_1 y_2^8$$
$$+ 72\, x_1 x_2 y_1 y_2^7 y_3 - 186\, x_1 x_2 y_1 y_2^6 y_3^2 + 264\, x_1 x_2 y_1 y_2^5 y_3^3 - 216\, x_1 x_2 y_1 y_2^4 y_3^4$$
$$+ 96\, x_1 x_2 y_1 y_2^3 y_3^5 - 18\, x_1 x_2 y_1 y_2^2 y_3^6 - 6\, x_1 x_2 y_2^8 y_3 + 30\, x_1 x_2 y_2^7 y_3^2 - 60\, x_1 x_2 y_2^6 y_3^3$$
$$+ 60\, x_1 x_2 y_2^5 y_3^4 - 30\, x_1 x_2 y_2^4 y_3^5 + 6\, x_1 x_2 y_2^3 y_3^6 + 6\, x_1 x_3 y_1^5 y_2^4 - 24\, x_1 x_3 y_1^5 y_2^3 y_3$$
$$+ 36\, x_1 x_3 y_1^5 y_2^2 y_3^2 - 24\, x_1 x_3 y_1^5 y_2 y_3^3 + 6\, x_1 x_3 y_1^5 y_3^4 - 30\, x_1 x_3 y_1^4 y_2^5$$
$$+ 126\, x_1 x_3 y_1^4 y_2^4 y_3 - 204\, x_1 x_3 y_1^4 y_2^3 y_3^2 + 156\, x_1 x_3 y_1^4 y_2^2 y_3^3 - 54\, x_1 x_3 y_1^4 y_2 y_3^4$$
$$+ 6\, x_1 x_3 y_1^4 y_3^5 + 60\, x_1 x_3 y_1^3 y_2^6 - 264\, x_1 x_3 y_1^3 y_2^5 y_3 + 456\, x_1 x_3 y_1^3 y_2^4 y_3^2$$
$$- 384\, x_1 x_3 y_1^3 y_2^3 y_3^3 + 156\, x_1 x_3 y_1^3 y_2^2 y_3^4 - 24\, x_1 x_3 y_1^3 y_2 y_3^5 - 60\, x_1 x_3 y_1^2 y_2^7$$
$$+ 276\, x_1 x_3 y_1^2 y_2^6 y_3 - 504\, x_1 x_3 y_1^2 y_2^5 y_3^2 + 456\, x_1 x_3 y_1^2 y_2^4 y_3^3 - 204\, x_1 x_3 y_1^2 y_2^3 y_3^4$$
$$+ 36\, x_1 x_3 y_1^2 y_2^2 y_3^5 + 30\, x_1 x_3 y_1 y_2^8 - 144\, x_1 x_3 y_1 y_2^7 y_3 + 276\, x_1 x_3 y_1 y_2^6 y_3^2$$
$$- 264\, x_1 x_3 y_1 y_2^5 y_3^3 + 126\, x_1 x_3 y_1 y_2^4 y_3^4 - 24\, x_1 x_3 y_1 y_2^3 y_3^5 - 6\, x_1 x_3 y_2^9 + 30\, x_1 x_3 y_2^8 y_3$$
$$- 60\, x_1 x_3 y_2^7 y_3^2 + 60\, x_1 x_3 y_2^6 y_3^3 - 30\, x_1 x_3 y_2^5 y_3^4 + 6\, x_1 x_3 y_2^4 y_3^5 - 3\, x_2^2 y_1^6 y_2^3$$
$$+ 9\, x_2^2 y_1^6 y_2^2 y_3 - 9\, x_2^2 y_1^6 y_2 y_3^2 + 3\, x_2^2 y_1^6 y_3^3 + 15\, x_2^2 y_1^5 y_2^4 - 48\, x_2^2 y_1^5 y_2^3 y_3$$
$$+ 54\, x_2^2 y_1^5 y_2^2 y_3^2 - 24\, x_2^2 y_1^5 y_2 y_3^3 + 3\, x_2^2 y_1^5 y_3^4 - 27\, x_2^2 y_1^4 y_2^5 + 96\, x_2^2 y_1^4 y_2^4 y_3$$
$$- 129\, x_2^2 y_1^4 y_2^3 y_3^2 + 81\, x_2^2 y_1^4 y_2^2 y_3^3 - 24\, x_2^2 y_1^4 y_2 y_3^4 + 3\, x_2^2 y_1^4 y_3^5 + 15\, x_2^2 y_1^3 y_2^6$$
$$- 72\, x_2^2 y_1^3 y_2^5 y_3 + 141\, x_2^2 y_1^3 y_2^4 y_3^2 - 144\, x_2^2 y_1^3 y_2^3 y_3^3 + 81\, x_2^2 y_1^3 y_2^2 y_3^4$$
$$- 24\, x_2^2 y_1^3 y_2 y_3^5 + 3\, x_2^2 y_1^3 y_3^6 + 12\, x_2^2 y_1^2 y_2^7 - 15\, x_2^2 y_1^2 y_2^6 y_3 - 54\, x_2^2 y_1^2 y_2^5 y_3^2$$
$$+ 141\, x_2^2 y_1^2 y_2^4 y_3^3 - 129\, x_2^2 y_1^2 y_2^3 y_3^4 + 54\, x_2^2 y_1^2 y_2^2 y_3^5 - 9\, x_2^2 y_1^2 y_2 y_3^6 - 18\, x_2^2 y_1 y_2^8$$
$$+ 48\, x_2^2 y_1 y_2^7 y_3 - 15\, x_2^2 y_1 y_2^6 y_3^2 - 72\, x_2^2 y_1 y_2^5 y_3^3 + 96\, x_2^2 y_1 y_2^4 y_3^4 - 48\, x_2^2 y_1 y_2^3 y_3^5$$
$$+ 9\, x_2^2 y_1 y_2^2 y_3^6 + 6\, x_2^2 y_2^9 - 18\, x_2^2 y_2^8 y_3 + 12\, x_2^2 y_2^7 y_3^2 + 15\, x_2^2 y_2^6 y_3^3 - 27\, x_2^2 y_2^5 y_3^4$$
$$+ 15\, x_2^2 y_2^4 y_3^5 - 3\, x_2^2 y_2^3 y_3^6 + 6\, x_2 x_3 y_1^6 y_2^3 - 18\, x_2 x_3 y_1^6 y_2^2 y_3 + 18\, x_2 x_3 y_1^6 y_2 y_3^2$$
$$- 6\, x_2 x_3 y_1^6 y_3^3 - 30\, x_2 x_3 y_1^5 y_2^4 + 96\, x_2 x_3 y_1^5 y_2^3 y_3 - 108\, x_2 x_3 y_1^5 y_2^2 y_3^2$$

$$+48\,x_2 x_3 y_1^5 y_2 y_3^3 - 6\,x_2 x_3 y_1^5 y_3^4 + 60\,x_2 x_3 y_1^4 y_2^5 - 216\,x_2 x_3 y_1^4 y_2^4 y_3$$
$$+294\,x_2 x_3 y_1^4 y_2^3 y_3^2 - 186\,x_2 x_3 y_1^4 y_2^2 y_3^3 + 54\,x_2 x_3 y_1^4 y_2 y_3^4 - 6\,x_2 x_3 y_1^4 y_3^5$$
$$-60\,x_2 x_3 y_1^3 y_2^6 + 264\,x_2 x_3 y_1^3 y_2^5 y_3 - 456\,x_2 x_3 y_1^3 y_2^4 y_3^2 + 384\,x_2 x_3 y_1^3 y_2^3 y_3^3$$
$$-156\,x_2 x_3 y_1^3 y_2^2 y_3^4 + 24\,x_2 x_3 y_1^3 y_2 y_3^5 + 30\,x_2 x_3 y_1^2 y_2^7 - 186\,x_2 x_3 y_1^2 y_2^6 y_3$$
$$+414\,x_2 x_3 y_1^2 y_2^5 y_3^2 - 426\,x_2 x_3 y_1^2 y_2^4 y_3^3 + 204\,x_2 x_3 y_1^2 y_2^3 y_3^4 - 36\,x_2 x_3 y_1^2 y_2^2 y_3^5$$
$$-6\,x_2 x_3 y_1 y_2^8 + 72\,x_2 x_3 y_1 y_2^7 y_3 - 204\,x_2 x_3 y_1 y_2^6 y_3^2 + 240\,x_2 x_3 y_1 y_2^5 y_3^3$$
$$-126\,x_2 x_3 y_1 y_2^4 y_3^4 + 24\,x_2 x_3 y_1 y_2^3 y_3^5 - 12\,x_2 x_3 y_2^8 y_3 + 42\,x_2 x_3 y_2^7 y_3^2 - 54\,x_2 x_3 y_2^6 y_3^3$$
$$+30\,x_2 x_3 y_2^5 y_3^4 - 6\,x_2 x_3 y_2^4 y_3^5 - 3\,x_3^2 y_1^6 y_2^3 + 9\,x_3^2 y_1^6 y_2^2 y_3 - 9\,x_3^2 y_1^6 y_2 y_3^2$$
$$+3\,x_3^2 y_1^6 y_3^3 + 15\,x_3^2 y_1^5 y_2^4 - 48\,x_3^2 y_1^5 y_2^3 y_3 + 54\,x_3^2 y_1^5 y_2^2 y_3^2 - 24\,x_3^2 y_1^5 y_2 y_3^3$$
$$+3\,x_3^2 y_1^5 y_3^4 - 30\,x_3^2 y_1^4 y_2^5 + 105\,x_3^2 y_1^4 y_2^4 y_3 - 135\,x_3^2 y_1^4 y_2^3 y_3^2 + 75\,x_3^2 y_1^4 y_2^2 y_3^3$$
$$-15\,x_3^2 y_1^4 y_2 y_3^4 + 30\,x_3^2 y_1^3 y_2^6 - 120\,x_3^2 y_1^3 y_2^5 y_3 + 180\,x_3^2 y_1^3 y_2^4 y_3^2 - 120\,x_3^2 y_1^3 y_2^3 y_3^3$$
$$+30\,x_3^2 y_1^3 y_2^2 y_3^4 - 15\,x_3^2 y_1^2 y_2^7 + 75\,x_3^2 y_1^2 y_2^6 y_3 - 135\,x_3^2 y_1^2 y_2^5 y_3^2 + 105\,x_3^2 y_1^2 y_2^4 y_3^3$$
$$-30\,x_3^2 y_1^2 y_2^3 y_3^4 + 3\,x_3^2 y_1 y_2^8 - 24\,x_3^2 y_1 y_2^7 y_3 + 54\,x_3^2 y_1 y_2^6 y_3^2 - 48\,x_3^2 y_1 y_2^5 y_3^3$$
$$+15\,x_3^2 y_1 y_2^4 y_3^4 + 3\,x_3^2 y_2^8 y_3 - 9\,x_3^2 y_2^7 y_3^2 + 9\,x_3^2 y_2^6 y_3^3 - 3\,x_3^2 y_2^5 y_3^4)\big)\Big/\Big(\big(x_1 x_2^2$$
$$-2\,x_1 x_2 x_3 + x_1 x_3^2 + x_2^3 - x_2^2 x_3 - x_2 x_3^2 + x_3^3 - y_2^2 + 2\,y_2 y_3 - y_3^2\big)^3\,\big(x_1^3 - x_1^2 x_2$$
$$+x_1^2 x_3 - x_1 x_2^2 - 2\,x_1 x_2 x_3 + x_2^3 + x_2^2 x_3 - y_1^2 + 2\,y_1 y_2 - y_2^2\big)^3\Big)$$

$$\mathit{expand}\big(x_1^3 x_2 - x_1^3 x_3 - x_1 x_2^3 + x_1 x_3^3 + x_2^3 x_3 - x_2 x_3^3 + x_1\,(x_2^3 + A\,x_2 + B) - x_1\,(x_3^3 + A\,x_3 + B)$$
$$- x_2\,(x_1^3 + A\,x_1 + B) + x_2\,(x_3^3 + A\,x_3 + B) + x_3\,(x_1^3 + A\,x_1 + B) - x_3\,(x_2^3 + A\,x_2 + B)\big)$$

$$0 \tag{3}$$