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On the Hamiltonicity of Subgroup Lattices

Nicholas Charles Fleece
Missouri State University, Fleece965@live.missouristate.edu

ON THE HAMILTONICITY OF SUBGROUP LATTICES

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

In this paper we discuss the Hamiltonicity of the subgroup lattices of different classes of groups. We provide sufficient conditions for the Hamiltonicity of the subgroup lattices of cube-free abelian groups. We also prove the non-Hamiltonicity of the subgroup lattices of dihedral and dicyclic groups. We disprove a conjecture on non-abelian p-groups by producing an infinite family of non-abelian p-groups with Hamiltonian subgroup lattices. Finally, we provide a list of the Hamiltonicity of the subgroup lattices of every finite group up to order 35 barring two groups.

KEYWORDS: group theory, graph theory, Hamiltonian, subgroup lattice, group, graph, lattice

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Approved:

Les Reid, Ph.D., Thesis Committee Chair

Kishor Shah, Ph.D., Committee Member

fCameron Wickham, Ph.D., Committee Member

Julie Masterson, Ph.D., Dean of the Graduate College

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1 INTRODUCTION

Determining whether or not a given abstract graph has a Hamiltonian cycle, often know as the "Hamiltonian cycle problem", is suspected to be computationally difficult (since it is NP-complete [4]). It is a natural question to ask which groups have subgroup lattices that are Hamiltonian. This question was investigated in McLaughlin's master's thesis [3], and we continue that investigation in this thesis.

In McLaughlin's thesis he made a few conjectures. One was that all abelian p-groups of odd exponent and rank greater than two are Hamiltonian [3, Conjecture 2]. While a proof has not been forthcoming, we did find more evidence for this conjecture. Another conjecture McLaughlin made was that all non-abelian p-groups are non-Hamiltonian [3, Conjecture 3]. In Section 6 we disprove this conjecture by exhibiting an infinite family of Hamiltonian non-abelian p-groups.

In Section 3 we provide sufficient conditions for cube-free abelian groups to be Hamiltonian. In Sections 4 and 5 we show that all dihedral and dicyclic groups are non-Hamiltonian, while some direct products of these with cyclic groups are shown to be Hamiltonian. In Section 7 we display a number of Hamiltonian non-abelian groups. Finally in the Appendix we list the Hamiltonicity of (almost) all groups up to order 35.

2 PRELIMINARIES AND BASIC RESULTS

Throughout this paper, by "graph" we are referring to a simple, connected, undirected graph. We will refer to the vertex set and edge set of a graph Γ as V_Γ and E_Γ respectively.

Definition 2.1. Given a group G, the *subgroup lattice* of G is defined to be the graph $\Gamma(G)$ where the vertices are the subgroups of G and for $H_1, H_2 \leq G$, H_1H_2 is an edge in $\Gamma(G)$ if and only if $H_1 \leq H_2$ or $H_2 \leq H_1$ and there does not exist a $K \leq G$ such that $H_1 < K < H_2$ or $H_2 < K < H_1$.

In Figure 1 we see an example of the subgroup lattice of C_{60}, where we have a subgroup for each divisor of its order. Note that, for instance, the subgroup isomorphic to C_2 is not adjacent to the subgroup isomorphic to C_{12} (despite the former being a subgroup of the latter) since the subgroup isomorphic to C_6 is a supergroup of the former and a subgroup of the latter.

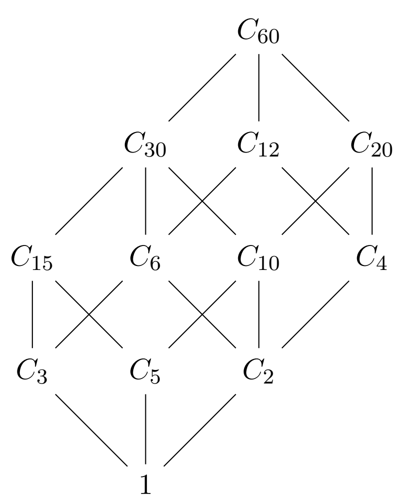

Figure 1: The subgroup lattice of C_{60}

Definition 2.2. The *order* of a graph Γ, denoted $|\Gamma|$, is the cardinality of its vertex set.

Definition 2.3. A *walk* in a graph is a sequence of (more than one) vertices $\{v_1, v_2, v_3, \ldots\}$ such that for all i, $v_i v_{i+1}$ is in the edge set of the graph.

Definition 2.4. A walk is said to be a *path* if all of its vertices are distinct.

Definition 2.5. A *cycle* is a finite walk where all of its vertices are distinct except that $v_i = v_n$ (for a walk of cardinality n).

Definition 2.6. A *Hamiltonian cycle* is a cycle containing every vertex of the graph. A graph containing a Hamiltonian cycle is said to be *Hamiltonian*. We say a group is *Hamiltonian* if its subgroup lattice is Hamiltonian.

In Figure 2, we see an example of a Hamiltonian cycle through the graph $\Gamma(C_{60})$. We then say that $\Gamma(C_{60})$ is a Hamiltonian graph and C_{60} is a Hamiltonian group.

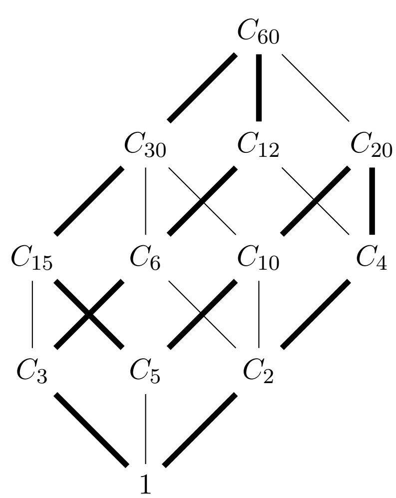

Figure 2: C_{60} is Hamiltonian

Definition 2.7. A set S of vertices is said to be *independent* if for all distinct $v_1, v_2 \in S$, we have that v_1v_2 is not in the edge set of the graph.

For an example of an independent set of vertices in a graph, consider the set $\{C_{30}, C_{20}, 1\}$ in Figure 1.

Definition 2.8. A graph is said to be *bipartite* if its vertex set can be partitioned into two independent sets. Furthermore, a group is said to be *bipartite* if its subgroup lattice is bipartite.

Figure 1 also provides an example of a bipartite graph. We can see this by considering the two independent sets

$$A := \{1, C_6, C_4, C_{15}, C_{10}, C_{60}\}$$
$$B := \{C_3, C_2, C_{12}, C_5, C_{30}, C_{20}\}.$$

This forms a bipartition of $\Gamma_{C_{60}}$ since $A \cap B = \varnothing$ and $A \cup B = V_{\Gamma_{C_{60}}}$.

Definition 2.9. A bipartite graph is said to be *balanced* if the two independent sets are of the same cardinality, otherwise it is said to be *unbalanced*. We say a group is *balanced/unbalanced* if its subgroup lattice is balanced/unbalanced.

Theorem 2.10. *All p-groups are bipartite.*

Proof. Let G be group such that $|G| = p^n$. It is well known that p-groups satisfy the converse to Lagrange's Theorem, hence we know that every non-trivial subgroup of order p^k has a subgroup of order p^{k-1}. It is also well known that every proper subgroup H of a p-group is a proper subgroup of its normalizer $N_G(H)$. Now Cauchy's Theorem yields a subgroup of order p of $N_G(H)/H$ and hence by the Lattice Isomorphism Theorem we have a supergroup of H of order p^{k+1}. This shows that we can partition the subgroup lattice by alternating indices (that is, subgroups of order $1, p^2, p^4, \ldots$ in one set and subgroups of order $p, p^3, p^5, \ldots$ in the other set). □

Theorem 2.11. *All unbalanced graphs are non-Hamiltonian.*

Proof. Let Γ be a bipartite graph with the bipartition A, B. Assume that Γ is Hamiltonian. Let $\{v_1, v_2, v_3, v_4, \ldots, v_{n-1}, v_n, v_1\}$ be a Hamiltonian cycle of Γ. Since A, B is a partition, v_1 belongs to either A or B. Without loss of generality, say $v_1 \in A$. We have that $v_1v_2 \in E_\Gamma$, so since A is an independent set of vertices $v_2 \notin A$ and hence $v_2 \in B$. Next, we have that $v_2v_3 \in E_\Gamma$ so since B is an independent set of vertices, $v_3 \notin B$ and hence $v_3 \in A$. We can continue this process inductively to see that $v_i \in A$ if and only if i is odd and $v_i \in B$ if and only if i is even. Finally, we have that $v_nv_1 \in E_\Gamma$, so since $v_1 \in A$, it must be that $v_n \in B$. This then tells us that n is even. Hence $|A| = |B| = \frac{n}{2}$ so that Γ is balanced. □

Note that the converse of this Theorem does not hold. Indeed, P_4 (as defined later in Definition 2.19) is non-Hamiltonian and yet is balanced.

This theorem, in conjunction with Theorem 2.10, has been used to determine that a multitude of p-groups are non-Hamiltonian. In particular, we were able to use GAP to quickly test every group of order 32 and determine which were unbalanced. This led to many groups being determined to be non-Hamiltonian (as can be seen in the Appendix).

Definition 2.12. The *degree* of a vertex v in a graph, denoted $\deg(v)$, is the number of distinct vertices w such that vw is in the edge set of the graph.

The following definition will lead to another non-Hamiltonian condition.

Definition 2.13. A graph Γ is said to have a *proper diamond* if it contains a cycle denoted $\{v_1, v_2, v_3, v_4, v_1\}$ (with each v_i unique) as a subgraph such that $\deg(v_2) = \deg(v_4) = 2$, and at least one vertex v_5 distinct from those in the cycle.

Theorem 2.14. *Any graph with a proper diamond is non-Hamiltonian.*

Proof. Let Γ have a proper diamond and assume that Γ is Hamiltonian. As seen in Figure 3, this would then force v_2 adjacent to v_1 and v_3 in the Hamiltonian cycle and v_4 adjacent to v_1 and v_3 in the Hamiltonian cycle. This then creates a cycle before v_5 has been included. □

Example 2.15. Consider the subgroup lattice of A_4. As seen in Figure 4, choose $v_1 = 1$, $v_3 = C_2^2$ and v_2, v_4 to be any two of the three products of two disjoint transpositions (and $v_5 = A_4$, for instance). This shows that A_4 has a proper diamond and hence is non-Hamiltonian by Theorem 2.14 (this is

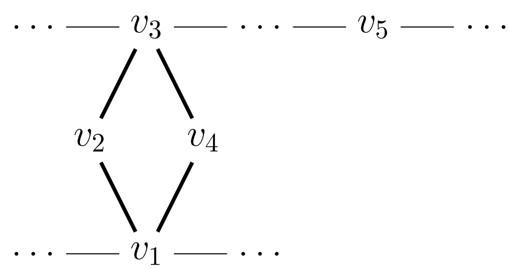

Figure 3: Proof that graphs with proper diamonds are non-Hamiltonian

certainly not our only choice of vertices to show this).

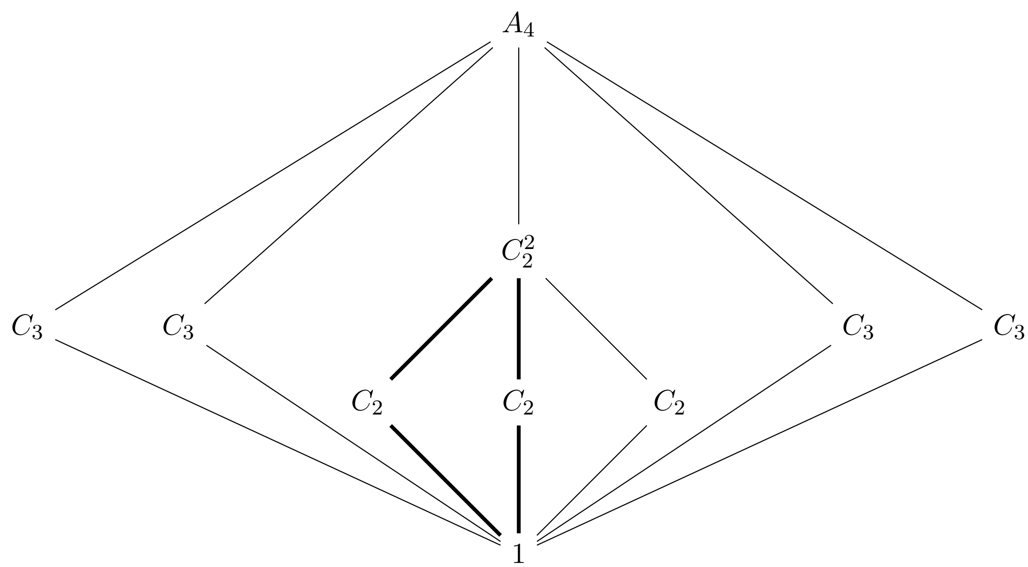

Figure 4: A diamond in the subgroup lattice of A_4

This is a useful method of eliminating candidates for Hamiltonian groups visually. There are many groups listed as being non-Hamiltonian in the Appendix with the justification of "Theorem 2.14" - these are groups in which we were able to visually find a diamond in the subgroup lattice produced by GAP.

We continue with our preliminary concepts.

Definition 2.16. A *tree* is a connected graph with no cycles.

Definition 2.17. A *spanning tree* of a graph Γ is a tree that is a subgraph of Γ containing all of its vertices.

Definition 2.18. Let Γ be a graph and let $\{T_i\}_{i=1}^n$ be the set of all spanning trees of Γ. Define

$$\mathscr{D}(\Gamma) := \min_{1\le i\le n}\left(\left\{\max_{v\in V_{T_i}}\left(\{\deg(v)\}\right)\right\}\right).$$

That is, $\mathscr{D}(\Gamma)$ is the degree of the vertex of maximum degree in the spanning tree of Γ that has the smallest degree of the vertex of maximum degree.

Definition 2.19. The *path graph* of order n, denoted P_n, is the graph with vertex set $\{v_i\}_{i=1}^n$ and edge set $\{v_iv_{i+1}\}_{i=1}^{n-1}$. See Figure 5 for an example.

$$v_1 — v_2 — v_3 — v_4 — v_5$$

Figure 5: The path graph P_5

Example 2.20. We can show that P_n is bipartite through the bipartition $A := \{v_i \big| i = 2k-1 \le n, k \in \mathbb{N}\}$ and $B := \{v_i \big| i = 2k \le n, k \in \mathbb{N}\}$. (If $n = 1$, then $B = \varnothing$). We can also see that $\mathscr{D}(P_n) = 1$ for $n = 1, 2$ and $\mathscr{D}(P_n) = 2$ for $n > 2$ since P_n itself is the only spanning tree of P_n and the maximum degree of its vertices is 1 when $n = 1, 2$ and 2 otherwise.

Definition 2.21. Given two graphs Γ_1, Γ_2, their *Cartesian product*, denoted $\Gamma_1 \square \Gamma_2$, is defined to be the graph with vertices $\{(v, w) \big| v \in \Gamma_1, w \in \Gamma_2\}$ and edge set $\{(v_1, w_1)(v_2, w_2) \Big| [v_1 v_2 \in E_{\Gamma_1} \text{ and } w_1 = w_2] \text{ or } [w_1 w_2 \in E_{\Gamma_2} \text{ and } v_1 = v_2]\}$.

It is easy to see that the Cartesian product is associative and commutative. It is also clear that $\Gamma \square P_1 \cong \Gamma$ for any graph Γ.

Let us investigate what the product of an arbitrary graph with a path graph looks like. Let Γ be a graph of order m such that

$$V_\Gamma = \{w_1, w_2, \ldots, w_m\}.$$

We then have $V_{\Gamma \square P_n} =$

$$\{(w_1, v_1), \ldots (w_m, v_1), (w_1, v_2), \ldots (w_m, v_2), \ldots, (w_1, v_n), \ldots (w_m, v_n)\}.$$

For brevity denote (w_i, v_j) as v_i^j so that

$$V_{\Gamma \square P_n} = \{v_1^1, v_2^1, \ldots, v_m^1, v_1^2, v_2^2, \ldots, v_m^2, \ldots, v_1^n, v_2^n, \ldots, v_m^n\}.$$

We can then see that the Cartesian product consists of n copies of Γ. From the definition we see that $v_i^k v_i^{k+1}$ is in the edge set of $\Gamma \square P_n$ for all $1 \leq i \leq m$ and $1 \leq k \leq n - 1$ (See Figure 6).

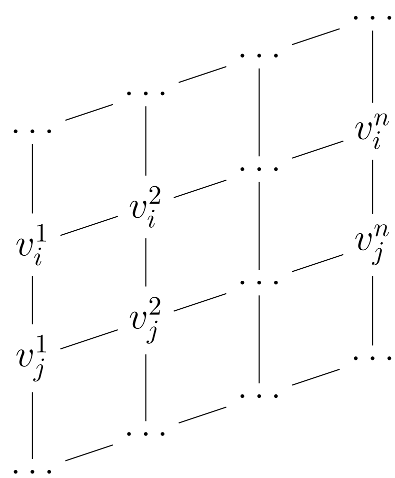

Figure 6: $\Gamma \,\Box\, P_n$ for arbitrary Γ

What then can we determine about the Hamiltonicity of $\Gamma \,\Box\, P_n$?

Proposition 2.22. *If $\Gamma \,\Box\, P_n$ is Hamiltonian for some fixed $n \in \mathbb{N}$, then $\Gamma \,\Box\, P_{kn}$ is also Hamiltonian for all $k \in \mathbb{N}$. In particular, if Γ is Hamiltonian, then $\Gamma \,\Box\, P_n$ is also Hamiltonian for all $n \in \mathbb{N}$.*

Proof. The case where $k = 1$ clearly holds by assumption. For $k = 2$, we have that $\Gamma \Box P_{2n}$ consists of two copies of $\Gamma \Box P_n$. Choose some v_i^n, v_j^n that are adjacent in the Hamiltonian cycle of $\Gamma \,\Box\, P_n$. To see that such an adjacency exists, choose an arbitrary vertex $v_{i_0}^n$ and note that it must be adjacent to exactly two other vertices. Since this is the n-th copy of Γ, there is only one vertex it could possibly be adjacent to that is not in the n-th copy of Γ, and so it must be adjacent to at least one other vertex in the n-th copy of Γ. Next, reflect the Hamiltonian cycle of $\Gamma \,\Box\, P_n$ across the plane passing through each $v_{i_0}^n v_{i_0}^{n+1}$ to make a second cycle through $\Gamma \,\Box\, P_{2n}$. This then makes v_i^{n+1}, v_j^{n+1}

adjacent in the cycle. If we instead make v_i^n adjacent to v_i^{n+1} and v_j^n adjacent to v_j^{n+1}, we connect the two cycles and create a Hamiltonian cycle through $\Gamma \,\Box\, P_{2n}$ (See Figure 7).

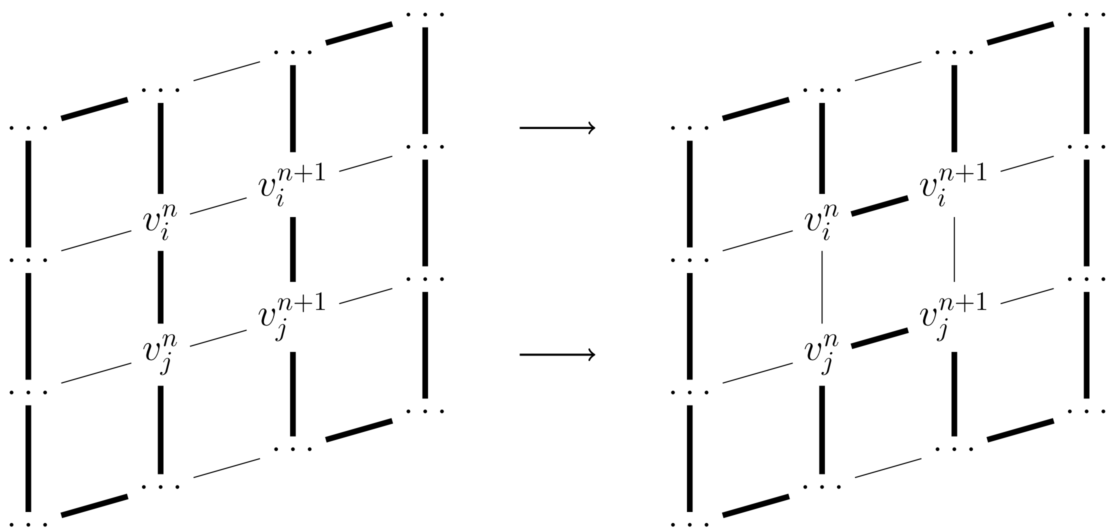

Figure 7: Constructing a Hamiltonian cycle through $\Gamma \,\Box\, P_{2n}$

For $k = 3$ we then repeat this process without reflecting the Hamiltonian cycle and make v_i^{2n} adjacent to v_i^{2n+1} and v_j^{2n} adjacent to v_j^{2n+1}. For $k = 4$ we follow the same process of $k = 2$, making v_i^{3n} adjacent to v_i^{3n+1} and v_j^{3n} adjacent to v_i^{3n+1}. Hence we can inductively repeat this process for arbitrary k by making $v_i^{(k-1)n}$ adjacent to $v_i^{(k-1)n+1}$ and $v_j^{(k-1)n}$ adjacent to $v_j^{(k-1)n+1}$ (reflecting the Hamiltonian cycle only when k is even).
If Γ itself is Hamiltonian, then $\Gamma \,\Box\, P_1 \cong \Gamma$ is also Hamiltonian and hence by this result $\Gamma \,\Box\, P_n$ is also Hamiltonian for all $n \in \mathbb{N}$. $\square$

The Cartesian products of path graphs also leads to another useful idea:

Definition 2.23. The *grid graph* of type $n_1, n_2, \ldots, n_k$, where $n_i > 1$ for

$i = 1, 2, \ldots, k$ or ($k = 1$ and $n_1 = 1$), denoted $P_{n_1,n_2,\ldots,n_k}$ is defined to be $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}$.

The following is then an immediate result of Definitions 2.21 and 2.4:

Proposition 2.24. $|P_{n_1,n_2,\ldots,n_k}| = \prod_{i=1}^{k} n_i$

To make use of grid graphs, we will first need a result from [1, p. 7] that connects the Cartesian product to the $\mathscr{D}$ from Definition 2.18.

Theorem 2.25. *If Γ_2 is Hamiltonian and $\mathscr{D}(\Gamma_1) \leq |V_{\Gamma_2}|$, then $\Gamma_1 \square \Gamma_2$ is also Hamiltonian.*

This can then be used to expand Proposition 2.22:

Theorem 2.26. *If $\Gamma \square P_n$ is Hamiltonian for some fixed $n \in \mathbb{N}$, then $\Gamma \square P_{kn,m_1,m_2,\ldots,m_l}$ is also Hamiltonian for any $k, m_1, m_2, \ldots, m_l \in \mathbb{N}$. In particular, if Γ is Hamiltonian, then $\Gamma \square P_{m_1,m_2,\ldots,m_l}$ is Hamiltonian for any $m_1, m_2, \ldots, m_l \in \mathbb{N}$.*

Proof. We know from Proposition 2.22 that $\Gamma \square P_{kn}$ is Hamiltonian. Note then that the smallest possible Hamiltonian graph is K_3, the complete graph on 3 vertices. So it must be that $|\Gamma \square P_{kn}| \geq 3$. Then since $\mathscr{D}(P_{m_1}) \leq 2$, by Theorem 2.25 we have that $\Gamma \square P_{kn,m_1} \cong (\Gamma \square P_{kn}) \square P_{m_1}$ is Hamiltonian. We can continue this process inductively showing that $|\Gamma \square P_{kn,m_1,m_2,\ldots,m_{i-1}}| \geq \mathscr{D}(P_{m_i})$ for each $i = 2, 3, \ldots, l$. Hence $\Gamma \square P_{kn,m_1,m_2,\ldots,m_l}$ is Hamiltonian.

If $\Gamma \cong \Gamma \,\Box\, P_1$ is Hamiltonian, then by this result $\Gamma \,\Box\, P_{k,m_1,m_2,\ldots,m_l}$ is Hamiltonian for any $k, m_1, m_2, \ldots, m_l \in \mathbb{N}$. Relabeling m_i to m_{i+1} for $i = 1, 2, \ldots, l$, l to $l+1$, and k to m_1 gives the desired result. $\Box$

We can extend this even further:

Corollary 2.27. *If $\Gamma \,\Box\, P_{n_1,n_2,\ldots,n_{l_1}}$ is Hamiltonian for some fixed $n_1, n_2, \ldots, n_{l_1} \in \mathbb{N}$, then*

$$\Gamma \,\Box\, P_{k_1 n_1, k_2 n_2, \ldots, k_{l_1} n_{l_1}, m_1, m_2, \ldots, m_{l_2}}$$

is also Hamiltonian for any $k_1, k_2, \ldots, k_{l_1}, m_1, m_2, \ldots, m_{l_2} \in \mathbb{N}$.

Proof. First note that $\Gamma \,\Box\, P_{n_1,n_2,\ldots,n_{l_1}} \cong (\Gamma \,\Box\, P_{n_2,\ldots,n_{l_1}}) \,\Box\, P_{n_1}$ so that by Proposition 2.22 $\cong (\Gamma \,\Box\, P_{n_2,\ldots,n_{l_1}}) \,\Box\, P_{k_1 n_1} \cong \Gamma \,\Box\, P_{k_1 n_1, n_2, \ldots, n_{l_1}}$ is Hamiltonian. Next since $\Gamma \,\Box\, P_{k_1 n_1, n_2, \ldots, n_{l_1}} \cong \Gamma \,\Box\, P_{k_1 n_1, n_3, \ldots, n_{l_1}} \,\Box\, P_{n_3}$ Proposition 2.22 tells us that $\Gamma \,\Box\, P_{k_1 n_1, k_2 n_2, n_3, \ldots, n_{l_1}}$ is Hamiltonian. We can continue this process to show that $\Gamma \Box P_{k_1 n_1, k_2 n_2, k_3 n_3, \ldots, k_{l_1 - 1} n_{l_1 - 1}, n_{l_1}}$ is Hamiltonian. Finally Theorem 2.26 tells us that $\Gamma \,\Box\, P_{k_1 n_1, k_2 n_2, \ldots, k_{l_1} n_{l_1}, m_1, m_2, \ldots, m_{l_2}}$ is also Hamiltonian. $\Box$

Proposition 2.28. *If Γ_1 and Γ_2 are bipartite, then $\Gamma_1 \Box \Gamma_2$ is also bipartite.*

Proof. Let A_i, B_i be a bipartition for Γ_i, $i = 1, 2$. Then

$$A := \{(u, v) | (u \in A_1 \text{ and } v \in A_2) \text{ or } (u \in B_1 \text{ and } v \in B_2)\}$$

and $B := V_{\Gamma_1 \square \Gamma_2} \setminus A$ is a bipartition of $\Gamma_1 \square \Gamma_2$. $\square$

We can directly apply this to grid graphs:

Example 2.29. By Example 2.20 and Proposition 2.28, every grid graph is bipartite.

The following example of this will lead to another useful Theorem:

Example 2.30. Let Γ be a bipartite graph and consider $\Gamma \square P_n$. Label each of the n copies of Γ as $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. Let A_i, B_i be a bipartition for Γ_i, $i = 1, 2, \ldots, n$. Then for n even,

$$A := A_1 \cup B_2 \cup A_3 \cup B_4 \cup \ldots \cup A_{n-1} \cup B_n$$
$$B := B_1 \cup A_2 \cup B_3 \cup A_4 \cup \ldots \cup B_{n-1} \cup A_n$$

defines a bipartition of $\Gamma \square P_n$ and for n odd,

$$A := A_1 \cup B_2 \cup A_3 \cup B_4 \cup \ldots \cup B_{n-1} \cup A_n$$
$$B := B_1 \cup A_2 \cup B_3 \cup A_4 \cup \ldots \cup A_{n-1} \cup B_n$$

defines a bipartition of $\Gamma \square P_n$.

Using this example, we can prove the following theorem that characterizes the balance of an arbitrary bipartite graph with a path graph:

Proposition 2.31. *If Γ is balanced, then $\Gamma \square P_n$ is balanced for all n. If Γ*

is unbalanced, then $\Gamma \,\square\, P_n$ *is balanced for* n *even and unbalanced for* n *odd.*

Proof. Assume that n is even. Using the bipartition of $\Gamma \,\square\, P_n$ defined in Example 2.30, we have that:

$$\begin{aligned}
|A| &= |A_1 \cup B_2 \cup A_3 \cup B_4 \cup \ldots \cup A_{n-1} \cup B_n| \\
&= |A_1| + |B_2| + |A_3| + |B_4| + \ldots + |A_{n-1}| + |B_n| \\
&= |A_1| + |B_1| + |A_1| + |B_1| + \ldots + |A_1| + |B_1| \\
&= \tfrac{n}{2}\left(|A_1| + |B_1|\right)
\end{aligned}$$

and similarly $|B| = \frac{n}{2}\left(|A_1| + |B_1|\right) = |A|$ so that $\Gamma \,\square\, P_n$ is balanced. For n odd we have:

$$\begin{aligned}
|A| &= |A_1 \cup B_2 \cup A_3 \cup B_4 \cup \ldots \cup B_{n-1} \cup A_n| \\
&= |A_1| + |B_2| + |A_3| + |B_4| + \ldots + |B_{n-1}| + |A_n| \\
&= |A_1| + |B_1| + |A_1| + |B_1| + \ldots + |B_1| + |A_1| \\
&= \tfrac{n+1}{2}|A_1| + \tfrac{n-1}{2}|B_1|
\end{aligned}$$

and

$$\begin{aligned}
|B| &= |B_1 \cup A_2 \cup B_3 \cup A_4 \cup \ldots \cup A_{n-1} \cup B_n| \\
&= |B_1| + |A_2| + |B_3| + |A_4| + \ldots + |A_{n-1}| + |B_n| \\
&= |B_1| + |A_1| + |B_1| + |A_1| + \ldots + |A_1| + |B_1|
\end{aligned}$$

$$= \tfrac{n+1}{2}|B_1| + \tfrac{n-1}{2}|A_1|.$$

If Γ is balanced, then

$$\begin{aligned} |A| &= \tfrac{n+1}{2}|A_1| + \tfrac{n-1}{2}|B_1| \\ &= \tfrac{n+1}{2}|B_1| + \tfrac{n-1}{2}|A_1| \\ &= |B| \end{aligned}$$

and hence $\Gamma \,\square\, P_n$ is balanced.

On the other hand, if Γ is unbalanced, by way of contradiction assume that $|A| = |B|$. Then

$$|A| = |B|$$

$$\tfrac{n+1}{2}|A_1| + \tfrac{n-1}{2}|B_1| = \tfrac{n+1}{2}|B_1| + \tfrac{n-1}{2}|A_1|$$

$$|A_1| = |B_1|,$$

but Γ is unbalanced by assumption. Hence $\Gamma \,\square\, P_n$ is unbalanced. □

We can then expand this to characterize when the Cartesian product of an arbitrary bipartite graph with a grid graph is balanced.

Theorem 2.32. *If Γ is balanced, then $\Gamma \,\square\, P_{n_1,n_2,\ldots,n_k}$ is balanced for all $n_1, n_2, \ldots, n_k \in \mathbb{N}$. If Γ is unbalanced, then $\Gamma \,\square\, P_{n_1,n_2,\ldots,n_k}$ is unbalanced if and only if each $n_1, n_2, \ldots, n_k$ is odd.*

Proof. If Γ is balanced, then $\Gamma \,\square\, P_{n_1}$ is balanced by Proposition 2.31. This then implies that $\Gamma \,\square\, P_{n_1,n_2} \cong (\Gamma \,\square\, P_{n_1}) \,\square\, P_{n_2}$ is also balanced. We can repeat this inductively to show that $\Gamma \,\square\, P_{n_1,n_2,\ldots,n_k}$ is balanced.

Next, assume that Γ is unbalanced. Suppose that each $n_1, n_2, \ldots, n_k$ is odd. Proposition 2.31 then tells us that $\Gamma \,\square\, P_{n_1}$ is unbalanced. This then implies that $\Gamma \,\square\, P_{n_1,n_2} \cong (\Gamma \,\square\, P_{n_1}) \,\square\, P_{n_2}$ is also unbalanced. Again, we can repeat this inductively to show that $\Gamma \,\square\, P_{n_1,n_2,\ldots,n_k}$ is unbalanced.

Finally suppose that at least one of $n_1, n_2, \ldots, n_k$ is even. Without loss of generality (since the Cartesian product is commutative) assume that n_1 is even. Proposition 2.31 then tells us that $\Gamma \,\square\, P_{n_1}$ is balanced. This then implies that $\Gamma \,\square\, P_{n_1,n_2} \cong (\Gamma \,\square\, P_{n_1}) \,\square\, P_{n_2}$ is also balanced. Once again, we can repeat this inductively to show that $\Gamma \,\square\, P_{n_1,n_2,\ldots,n_k}$ is balanced. □

Corollary 2.33. *$P_{n_1,n_2,\ldots,n_k}$ is unbalanced if and only if each $n_1, n_2, \ldots, n_k$ is odd.*

Proof. This follows immediately from Theorem 2.32 since P_1 is unbalanced and $P_{n_1,n_2,\ldots,n_k} \cong P_{n_1,n_2,\ldots,n_k} \,\square\, P_1$. □

This idea can then be used to characterize the Hamiltonicity of grid graphs:

Proposition 2.34. *The grid graph $\Gamma := P_{n_1,n_2,\ldots,n_k}$ is Hamiltonian if and only if $k > 1$ and at least one of $n_1, n_2, \ldots, n_k$ is even.*

Proof. If $k = 1$, then Γ is a path graph which is clearly non-Hamiltonian. If each $n_1, n_2, \ldots, n_k$ is odd, then our graph is unbalanced by Theorem 2.33 and hence non-Hamiltonian by Theorem 2.11.

Suppose then that $k > 1$ and at least one of $n_1, n_2, \ldots, n_k$ is even. Without loss of generality (since the Cartesian product is commutative) assume that n_1 is even. Consider $P_{n_1,n_2} \cong P_{n_1} \square P_{n_2}$. Let $u_1, u_2, \ldots, u_{n_1}$ be the vertices of P_{n_1} and $v_1, v_2, \ldots, v_{n_1}$ be the vertices of P_{n_2} as in Definition 2.19. Figure 8 shows how to define a Hamiltonian cycle through P_{n_1,n_2}. Then by Proposition 2.22, $P_{n_1,n_2,n_3} \cong P_{n_1,n_2} \square P_{n_3}$ is also Hamiltonian. This can be repeated inductively to show that Γ is Hamiltonian. $\square$

We have the following result from [5, p. 36] that will help us determine what the subgroup lattices of many groups are:

Theorem 2.35. *If the orders of a sequence of groups $G_1, G_2, \ldots G_k$ are pairwise relatively prime then*

$$\Gamma(G_1 \times G_2 \times \ldots \times G_k) \cong \Gamma(G_1) \square \Gamma(G_2) \square \ldots \square \Gamma(G_k)$$

holds.

We can use this to show that $\Gamma(C_n)$ is a grid graph for all n:

Example 2.36. First note that C_{p^α} for p prime is isomorphic to $P_{\alpha+1}$ since the subgroups of C_{p^α} are exactly $C_{p^\alpha}, C_{p^{\alpha-1}}, C_{p^{\alpha-2}}, \ldots, C_{p^2}, C_p, 1$. Write $n =$

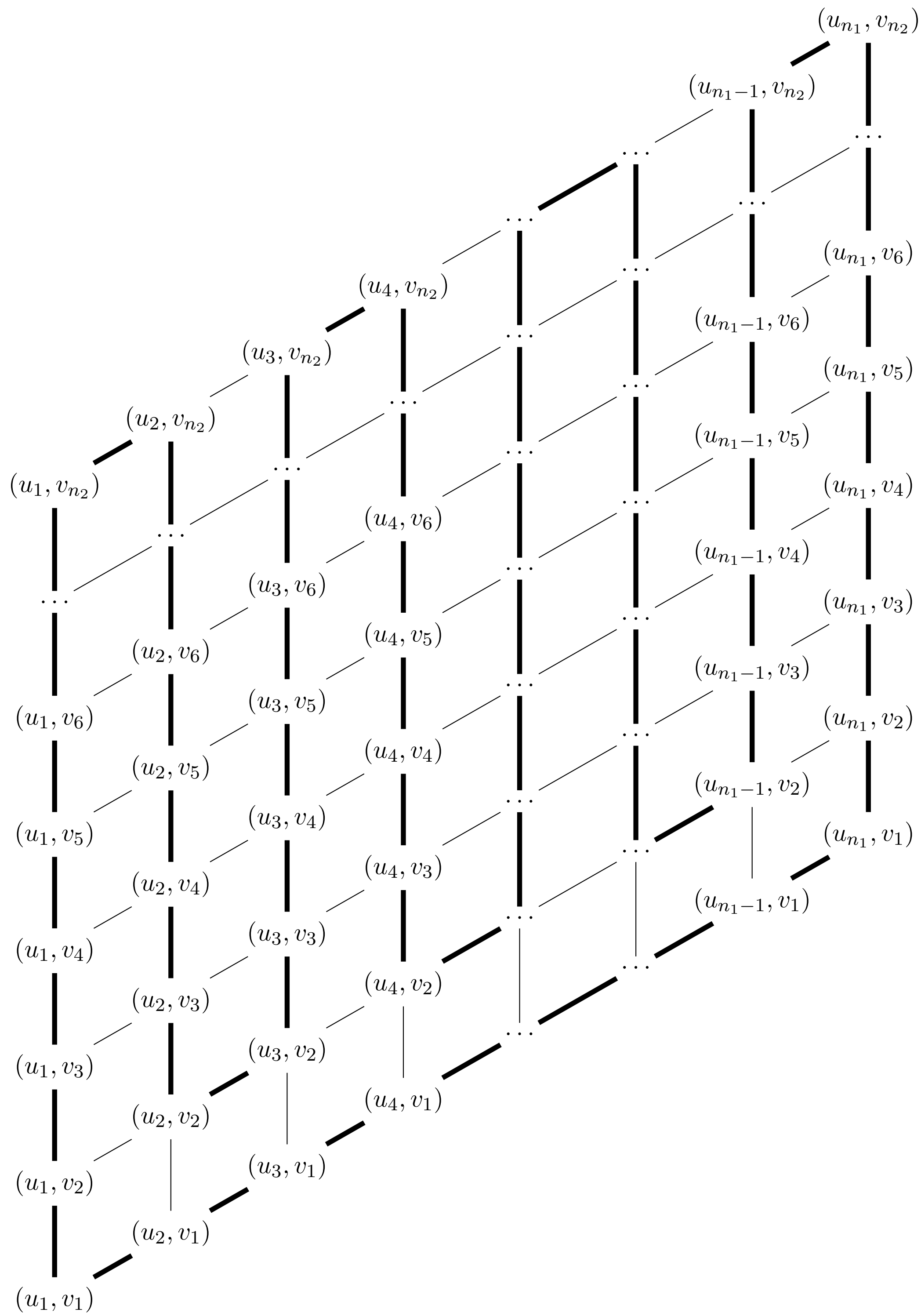

Figure 8: A Hamiltonian cycle through P_{n_1,n_2} with n_1 even

$p_{1_1^\alpha}p_{2_2^\alpha}\dots p_{k_k^\alpha}$ in its prime factorization. Then by the Chinese Remainder Theorem and Theorem 2.35 we have:

$$\begin{aligned}
\Gamma(C_n) &\cong \Gamma(C_{p_{1_1^\alpha}p_{2_2^\alpha}\dots p_{k_k^\alpha}}) \\
&\cong \Gamma(C_{p_{1_1^\alpha}} \times C_{p_{2_2^\alpha}} \times \dots \times C_{p_{k_k^\alpha}}) \\
&\cong \Gamma(C_{p_{1_1^\alpha}}) \,\square\, \Gamma(C_{p_{2_2^\alpha}}) \,\square\, \dots \,\square\, \Gamma(C_{p_{k_k^\alpha}}) \\
&\cong P_{\alpha_1+1} \,\square\, P_{\alpha_2+1} \,\square\, \dots \,\square\, P_{\alpha_k+1} \\
&\cong P_{\alpha_1+1,\alpha_2+1,\dots,\alpha_k+1}
\end{aligned}$$

as required.

Using this example, Proposition 2.34 then leads to a characterization of the Hamiltonicity of cyclic groups:

Theorem 2.37. *If $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$ in its prime factorization, then C_n is Hamiltonian if and only if $k > 1$ and at least one of $\alpha_1, \alpha_2, \dots, \alpha_k$ is odd.*

3 ABELIAN GROUPS

A proof of McLaughlin's conjecture that all non-abelian p-groups of rank larger than two are Hamiltonian has not been forthcoming. We did test multiple abelian p-groups to attempt to find a counterexample, but this yielded no results. We found that $C_2 \times C_4^2, C_2^5$ and $C_3 \times C_9^2$ are Hamiltonian which provides further evidence for this conjecture. However, Mathematica was unable to answer the Hamiltonian cycle problem for $C_2^3 \times C_4, C_3^3 \times C_9$ and C_3^5.

As for other abelian groups, we now provide sufficient conditions for cube-free abelian groups to be Hamiltonian.

Lemma 3.1. *For p prime, $\mathscr{D}\left(\Gamma\left(C_p^2\right)\right) \leq \frac{p+3}{2}$.*

Proof. Referring to Figure 9, $\Gamma\left(C_p^2\right)$ consists of $p+1$ independent copies of C_p, each connected to 1 and C_p^2. Label the C_p's as $C_{p,1}, C_{p,2}, \ldots, C_{p,p+1}$. We will now construct a spanning tree T_p for $\Gamma\left(C_p^2\right)$. For $p = 2$, adjoin $C_{2,1}$ and $C_{2,2}$ to C_2^2 and adjoin $C_{2,2}$ and $C_{2,3}$ to 1. In this case $\max_{v \in V_{T_2}}(\deg(v)) = 2$ so that $\mathscr{D}\left(\Gamma\left(C_2^2\right)\right) \leq \max_{v \in V_{T_2}}(\deg(v)) = 2 \leq \frac{5}{2}$. For $p > 2$, adjoin $C_{p,1}, C_{p,2}, \ldots, C_{p,\frac{p+3}{2}}$ to C_p^2 and adjoin $C_{p,\frac{p+3}{2}}, C_{p,\frac{p+3}{2}+1}, \ldots, C_{p,p+1}$ to 1. Then $\max_{v \in V_{T_p}}(\deg(v)) = \frac{p+3}{2}$ so that $\mathscr{D}\left(\Gamma\left(C_p^2\right)\right) \leq \max_{v \in V_{T_p}}(\deg(v)) = \frac{p+3}{2}$. □

Proposition 3.2. *If $p \neq 2$ is prime, then $C_2^2 \times C_p$ is Hamiltonian. Similarly if $p \neq 3$ is prime, then $C_3^2 \times C_p$ is Hamiltonian. If p and q are distinct primes*

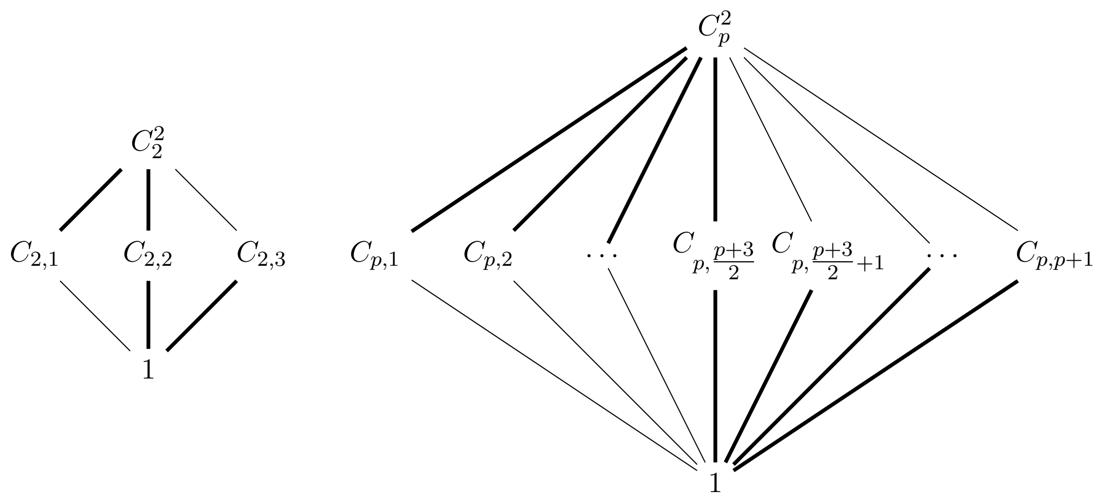

Figure 9: Spanning tree for C_p^2

and $q > 3$, then $C_q^2 \times C_{p^\alpha}$ is non-Hamiltonian for all $\alpha \in \mathbb{N}$.

Proof. Since $\gcd(2, p) = 1$, by Theorem 2.35 $\Gamma(C_2^2 \times C_p) \cong \Gamma(C_2^2) \,\Box\, \Gamma(C_p)$. Figure 10 is then a Hamiltonian cycle through $\Gamma(C_2^2 \times C_p)$.

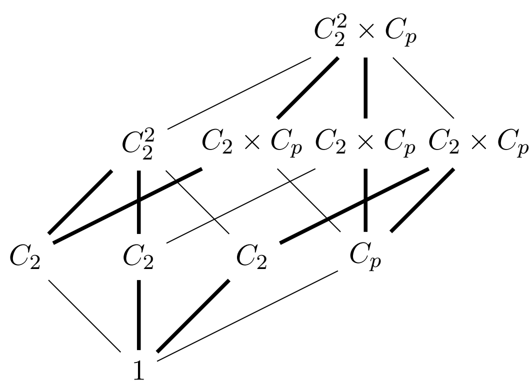

Figure 10: Hamiltonian cycle through $\Gamma(C_2^2 \times C_p)$

Similarly, $\Gamma(C_3^2 \times C_p) \cong \Gamma(C_3^2) \Box \Gamma(C_p)$. Figure 11 is then a Hamiltonian cycle through $\Gamma(C_3^2 \times C_p)$.

Now consider $\Gamma(C_q^2 \times C_{p^\alpha})$. Again, $\Gamma(C_q^2 \times C_{p^\alpha}) \cong \Gamma(C_q^2) \,\Box\, \Gamma(C_{p^\alpha})$. Furthermore by Example 2.36, $\Gamma(C_q^2) \,\Box\, \Gamma(C_{p^\alpha}) \cong \Gamma(C_q^2) \,\Box\, P_{\alpha+1}$. Use the

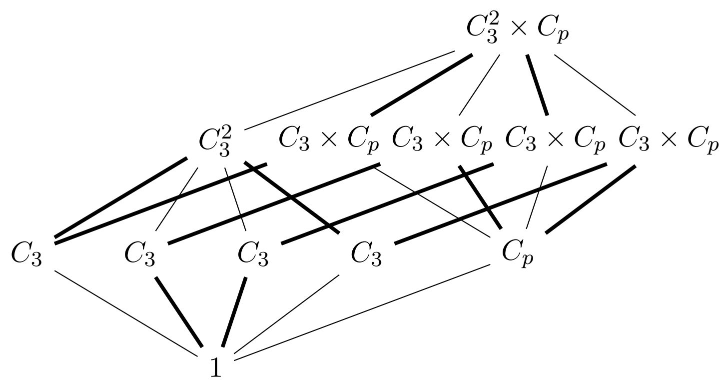

Figure 11: Hamiltonian cycle through $\Gamma(C_3^2 \times C_p)$

labeling in Figure 12 for the vertices of $\Gamma(C_q^2 \times C_{p^\alpha})$.

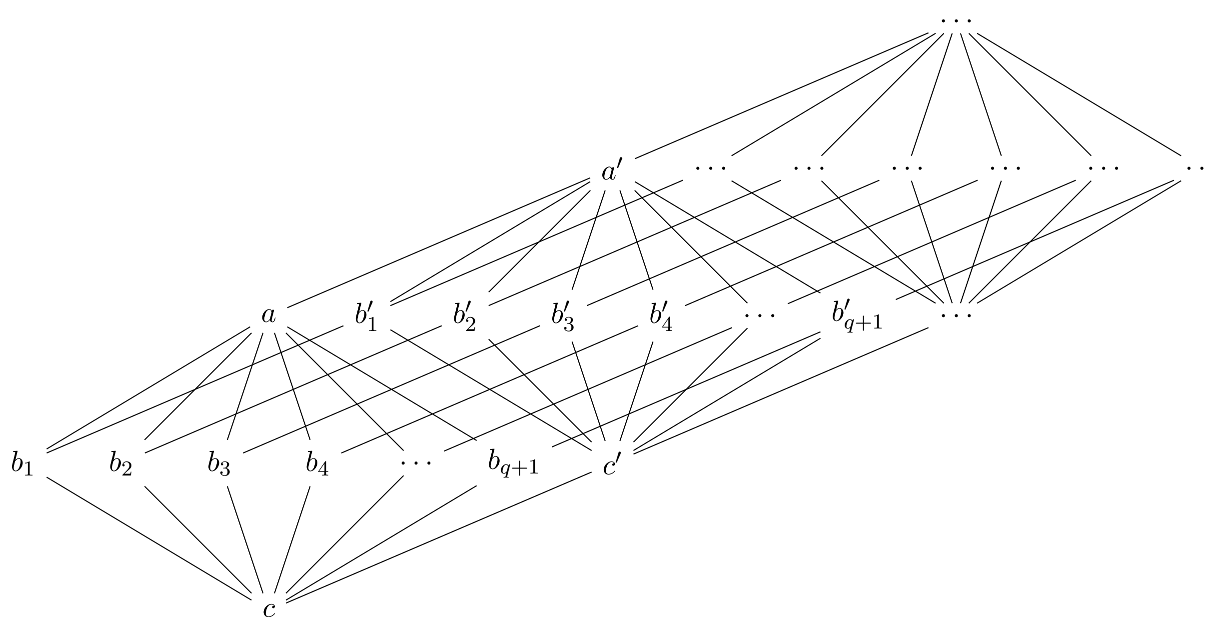

Figure 12: A labeling for the vertices of $\Gamma(C_q^2 \times C_{p^\alpha})$

By way of contradiction, assume that this graph is Hamiltonian. Consider the vertices b_i for $i = 1, 2, 3, 4$. Each b_i must either be adjacent to a and c in the Hamiltonian cycle, or adjacent to b_i and one of a or c.

Case 1: One of b_i for $i = 1, 2, 3, 4$ is adjacent to both a and c, without loss of generality say b_1 is adjacent to a and c (see Figure 13).

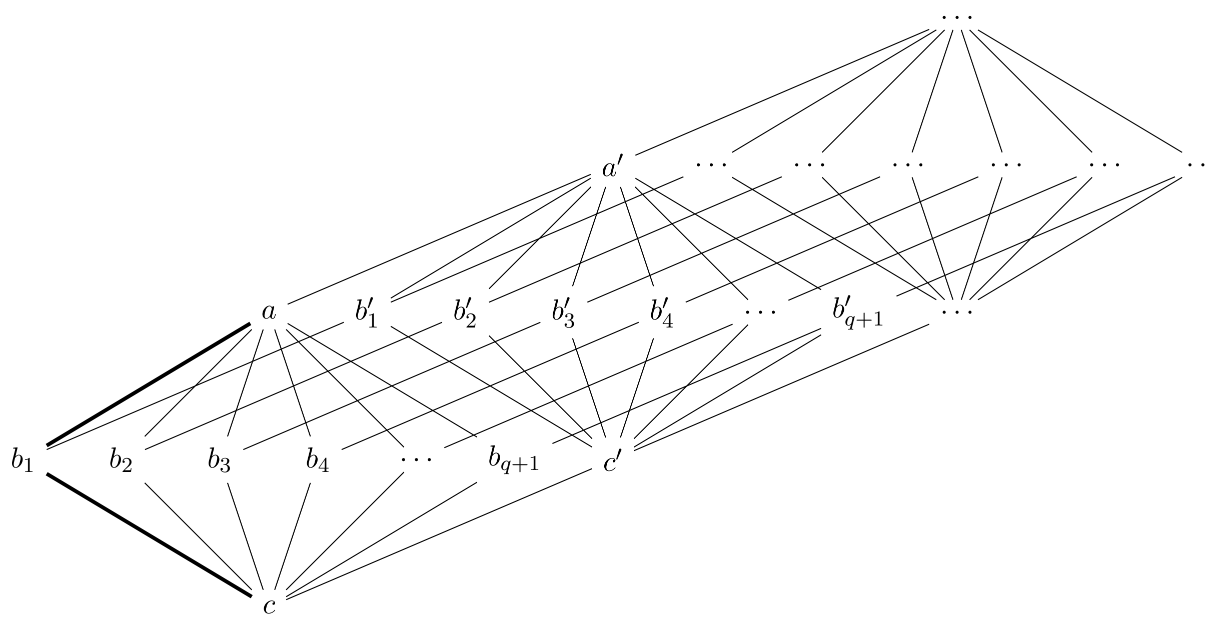

Figure 13: Case 1 in the proof of Proposition 3.2

We consider the same options for b_2. If b_2 were made adjacent to a and c, then this would close the cycle before many vertices have been reached. Thus it must be that b_2 is adjacent to b'_2 and, without loss of generality, a. We consider the same options for b_3, however, we are also forced to make b_3 adjacent to c and b'_3. Then, as in Figure 14, b_4 has only one vertex (b'_4) it can be made adjacent to, hence Case 1 cannot hold.

Case 2: Each of b_i for $i = 1, 2, 3, 4$ is adjacent to b'_i (See Figure 15).

Each has to be adjacent to a or c. At most two of these are adjacent to a and at most two are adjacent to c, hence exactly two must be adjacent to a and exactly two must be adjacent to c. Without loss of generality say b_1, b_2 are adjacent to a and b_3, b_4 are adjacent to c. Then, as in Figure 16, b_{q+1} has only one vertex (b'_{q+1}) it can be made adjacent to, hence Case 2 cannot hold.

This exhausts the cases and thus $C_q^2 \times C_{p^\alpha}$ is non-Hamiltonian for all

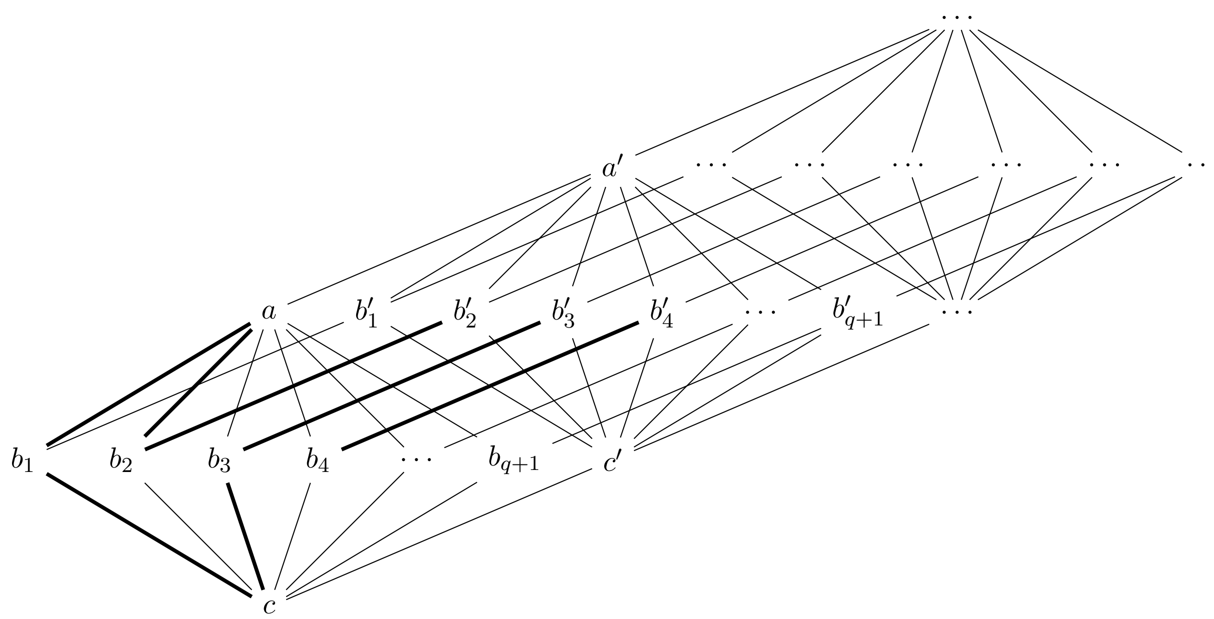

Figure 14: Case 1 in the proof of Proposition 3.2 (cont.)

$\alpha \in \mathbb{N}$. □

Proposition 3.3. *Let n be cube-free (i.e. $p^3 \nmid n$ for any prime p) such that when written in the form $n = p_1 p_2 \ldots p_m r_1^2 r_2^2 \ldots r_l^2$ where each p_i and r_j are distinct primes and $m > 0$. Let q be a prime such that $q \nmid n$. Then if $\frac{q+3}{2} \leq 2^m 3^l$ then $C_q^2 \times C_n$ is Hamiltonian.*

Proof. The minimum value for q is 2. For $\frac{2+3}{2} \leq 2^m 3^l$ to hold, $l > 0$ or $m > 1$. If $l = 0$, then it must be that $m > 1$ so that the rank of n is at least 2. Similarly if $m = 1$, then it must be that $l > 0$ so that again the rank of n is greater than 2. Thus in both cases C_n is Hamiltonian by Theorem 2.37. Then by Lemma 3.1, Example 2.36, and Proposition 2.24:

$$\mathscr{D}\left(\Gamma\left(C_q^2\right)\right) \leq \frac{q+3}{2} \leq 2^m 3^l = |\Gamma(C_n)|$$

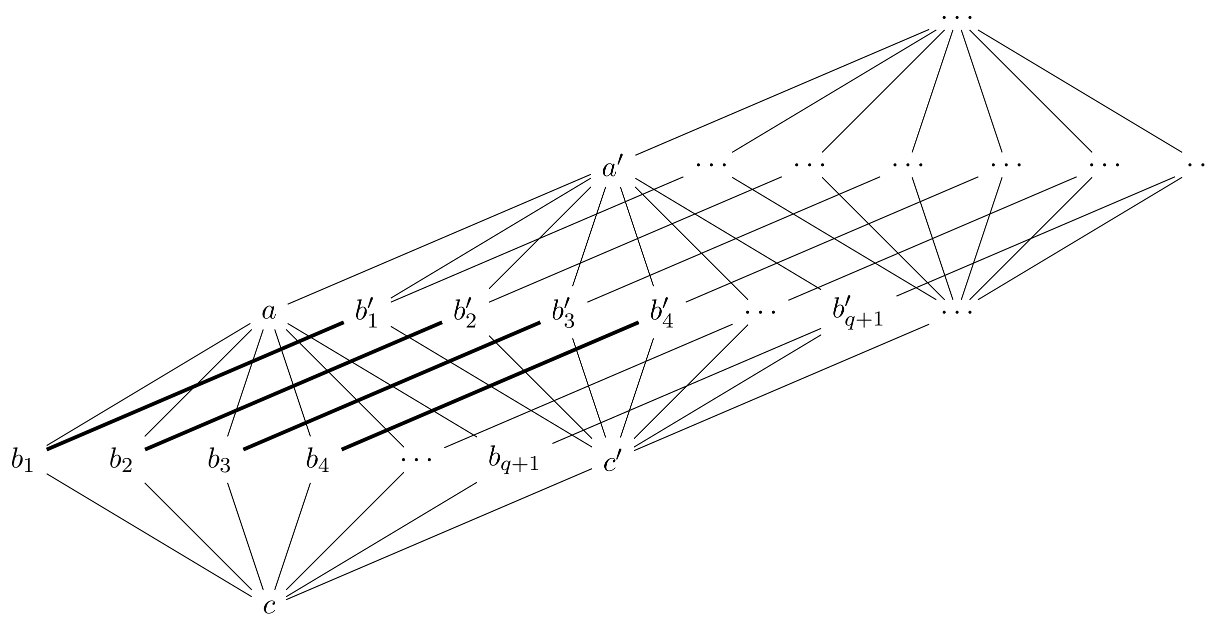

Figure 15: Case 2 in the proof of Proposition 3.2

so that $C_q^2 \times C_n$ is Hamiltonian by Theorem 2.25. $\square$

This can be generalized even further using the following theorem.

Theorem 3.4. *Let n be cube-free where $m > 0$ when written in the previous form and let $q_1 < q_2 < \ldots < q_k$ be primes such that $q_1, q_2, \ldots, q_k \nmid n$. If $\frac{q_k+3}{2} \leq 2^m 3^l$ then $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_k}^2 \times C_n$ is Hamiltonian.*

Proof. Now, by Proposition 3.3, $C_{q_k}^2 \times C_n$ is Hamiltonian. We have that

$$|\Gamma(C_{q_k}^2 \times C_n)| = |\Gamma(C_{q_k}^2) \,\square\, \Gamma(C_n)| = |\Gamma(C_{q_k}^2)||\Gamma(C_n)| = (q_k + 3)2^m 3^l$$

Since $q_1 < q_2 < \ldots < q_k$ we have that

$$\tfrac{q_1+3}{2} < \tfrac{q_2+3}{2} < \ldots < \tfrac{q_k+3}{2} \leq 2^m 3^l$$

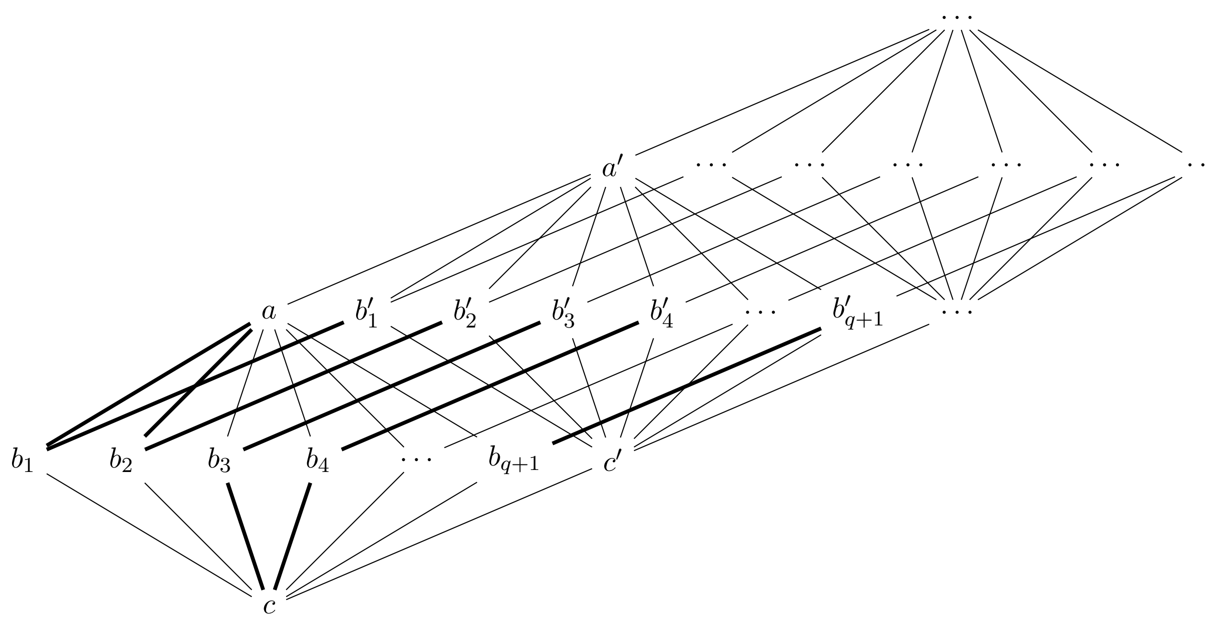

Figure 16: Case 2 in the proof of Proposition 3.2 (cont.)

so in particular

$$\mathscr{D}\left(\Gamma\left(C^2_{q_{k-1}}\right)\right) \leq \tfrac{q_{k-1}+3}{2} < 2^m 3^l < (q_k+3)2^m 3^l = |\Gamma(C^2_{q_k} \times C_n)|$$

and hence $C^2_{q_{k-1}} \times C^2_{q_k} \times C_n$ is Hamiltonian. We can continue by induction to show that for each $i = 2, \ldots, k-1$ (done in this order) that

$$\mathscr{D}\left(\Gamma\left(C^2_{q_{k-i}}\right)\right) \leq \tfrac{q_{k-i}+3}{2} < 2^m 3^l < \prod_{j=0}^{i}(q_{k-j}+3)2^m 3^l = \left|\Gamma\left(\prod_{j=0}^{i}(C^2_{q_{k-j}}) \times C_n\right)\right|$$

and hence that $\prod_{j=0}^{k-1}(C^2_{q_{k-j}}) \times C_n = C^2_{q_1} \times C^2_{q_2} \times \ldots \times C^2_{q_k} \times C_n$ is Hamiltonian.

□

This leads to an even more effective sufficient condition:

Corollary 3.5. *Let n be cube-free where $m > 0$ when written in the previous* *form and let* $q_1 < q_2 < \ldots < q_k$ *be primes such that* $q_1, q_2, \ldots, q_k \nmid n$. *Define* $q_0 := 1$. *If* $\frac{q_1+3}{2} \le 2^m 3^l$, $k > 1$ *and*

$$k = \max\left\{ i \middle| \tfrac{q_i+3}{2} \le 2^m 3^l \prod_{j=1}^{i-1}(q_j+3) \text{ and } \forall h < i, \tfrac{q_h+3}{2} \le 2^m 3^l \prod_{j=1}^{h-1}(q_j+3) \right\}$$

then $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_k}^2 \times C_n$ *is Hamiltonian.*

Proof. Define $k_0 := \max\{i | \frac{q_i+3}{2} \le 2^m 3^l\}$ and

$$k_1 := \max\left\{ i \middle| \tfrac{q_i+3}{2} \le 2^m 3^l \prod_{j=1}^{i-1}(q_j+3) \text{ and } \forall h < i, \tfrac{q_h+3}{2} \le 2^m 3^l \prod_{j=1}^{h-1}(q_j+3) \right\}$$

(the domains of these maximum functions are non-empty since $\frac{q_1+3}{2} \le 2^m 3^l = 2^m 3^l(1)$). Now, $k_0 \le k_1$ since $\frac{q_{k_0}+3}{2} \le 2^m 3^l \le 2^m 3^l \prod_{j=1}^{k_0-1}(q_j+3)$. By Theorem 3.4, $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_{k_0}}^2 \times C_n$ is Hamiltonian and of order $2^m 3^l \prod_{j=1}^{k_0-1}(q_j+3)$. So if $\frac{q_{k_0+1}+3}{2} \le 2^m 3^l \prod_{j=1}^{k_0}(q_j+3)$ then $k_0+1 \le k_1$ and by Theorem 2.25, $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_{k_0+1}}^2 \times C_n$ is Hamiltonian. We can continue this until we reach k_1 to conclude $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_{k_1}}^2 \times C_n$ is Hamiltonian. Hence if $k_1 = k$, then $C_{q_1}^2 \times C_{q_2}^2 \times \ldots \times C_{q_k}^2 \times C_n$ is Hamiltonian. $\square$

Example 3.6. We can use this to show that $C_{12} \times C_5^2 \times C_7^2 \times C_{1117}^2 \times C_{2000003}^2$ is Hamiltonian. In this instance $n = 12 = 2^2 3$ so that $|\Gamma(C_1 2)| = 2 \cdot 3^2 = 18$. We can clearly see that $q_4 = 2000003 > 18$, so we cannot directly apply Theorem 3.4. Instead we search for k_0 so that we may begin applying Corollary 3.5. We need $\frac{q_{k_0}+3}{2} \le 18$ so that $q_{k_0} \le 33$, so in this instance $k_0 = 2$

(where $q_{k_0} = 7$). Next we check each $k > 2$. We can see that

$$\mathscr{D}(\Gamma(C_{1117}^2)) = \tfrac{1117+3}{2} = 600 < 1920 = 18(5+3)(7+3)$$

and furthermore

$$\mathscr{D}(\Gamma(C_{2000003}^2)) = \tfrac{2000003+3}{2} = 1000003 < 1612800 = 1920(1117+3)$$

so that $k_1 = k$ and hence $C_{12} \times C_5^2 \times C_7^2 \times C_{1117}^2 \times C_{2000003}^2$ is Hamiltonian.

4 DIHEDRAL GROUPS

In this section we show that the dihedral groups are non-Hamiltonian. We will do this by showing that the dihedral groups are unbalanced and hence non-Hamiltonian by Theorem 2.11. We have

$$D_{2n} = \left\langle r, s | r^n = s^2 = 1, srs = r^{-1} \right\rangle.$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ be its prime factorization, $m > 0$ and each $\alpha_i > 0$. Define $\alpha(n) := \sum_{i=1}^{m} \alpha_i$.

Proposition 4.1. *D_{2n} is bipartite.*

Proof. Now, $\langle r \rangle$ is a cyclic group of order n, so for each $d|n$ it has a unique subgroup of order d of the form $\left\langle r^{\frac{n}{d}} \right\rangle$. Note that each $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$ where $0 \leq \beta_i \leq \alpha_i$ for $i = 1, 2, \dots, m$. Define $\beta(d) := \sum_{i=1}^{m} \beta_i$. Since each p_i is distinct, if $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$ and $d' = p_1^{\beta'_1} p_2^{\beta'_2} \dots p_m^{\beta'_m}$ where $\beta(d) = \beta(d_i)$, then either $d = d'$ or neither $\left\langle r^{\frac{n}{d}} \right\rangle$ nor $\left\langle r^{\frac{n}{d'}} \right\rangle$ is a subgroup of the other. Thus

$$\left\{ \left\langle r^{\frac{n}{d}} \right\rangle \middle| \beta(d) = k \right\}$$

forms an independent set of vertices in $\Gamma(D_{2n})$. Furthermore, if $\left\langle r^{\frac{n}{d}} \right\rangle < \left\langle r^{\frac{n}{d'}} \right\rangle$ such that $\beta(d) + l = \beta(d')$ where $l \geq 2$, then there must be some d'' where $\beta(d) < \beta(d'') < \beta(d')$ and $\left\langle r^{\frac{n}{d}} \right\rangle < \left\langle r^{\frac{n}{d''}} \right\rangle < \left\langle r^{\frac{n}{d'}} \right\rangle$. In particular, if $l = 2$, then there must be some d'' where $\beta(d) + 2 = \beta(d'') + 1 = \beta(d')$ and

$\left\langle r^{\frac{n}{d}}\right\rangle < \left\langle r^{\frac{n}{d''}}\right\rangle < \left\langle r^{\frac{n}{d'}}\right\rangle$. Hence $\left\langle r^{\frac{n}{d}}\right\rangle$ and $\left\langle r^{\frac{n}{d''}}\right\rangle$ are adjacent in $\Gamma(D_{2n})$ and $\left\langle r^{\frac{n}{d''}}\right\rangle$ and $\left\langle r^{\frac{n}{d'}}\right\rangle$ are adjacent in $\Gamma(D_{2n})$ but $\left\langle r^{\frac{n}{d}}\right\rangle$ and $\left\langle r^{\frac{n}{d'}}\right\rangle$ are not adjacent in $\Gamma(D_{2n})$. Thus

$$\left\{\left\langle r^{\frac{n}{d}}\right\rangle \middle| \beta(d) = 2k\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor} \text{ and } \left\{\left\langle r^{\frac{n}{d}}\right\rangle \middle| \beta(d) = 2k+1\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$

form two independent sets of vertices in $\Gamma(D_{2n})$. For each d, if $i \equiv j(\mod \frac{n}{d})$ then $\left\langle r^{\frac{n}{d}}, r^i s\right\rangle = \left\langle r^{\frac{n}{d}}, r^j s\right\rangle$. Then, in a similar manner as before,

$$\left\{\left\langle r^{\frac{n}{d}}, r^i s\right\rangle \middle| 0 \leq i < \tfrac{n}{d},\ \beta(d) = 2k\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$
$$\text{and } \left\{\left\langle r^{\frac{n}{d}}, r^i s\right\rangle \middle| 0 \leq i < \tfrac{n}{d},\ \beta(d) = 2k+1\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$

form independent sets of vertices in $\Gamma(D_{2n})$.

Now, if $\left\langle r^{\frac{n}{d}}\right\rangle < \left\langle r^{\frac{n}{d'}}, r^i s\right\rangle$ then $\left\langle r^{\frac{n}{d}}\right\rangle \leq \left\langle r^{\frac{n}{d'}}\right\rangle < \left\langle r^{\frac{n}{d'}}, r^i s\right\rangle$. So, $\left\langle r^{\frac{n}{d}}\right\rangle$ is only adjacent to $\left\langle r^{\frac{n}{d'}}, r^i s\right\rangle$ in $\Gamma(D_{2n})$ whenever $d = d'$. Finally, we know that it could never be that $\left\langle r^{\frac{n}{d}}, r^i s\right\rangle < \left\langle r^{\frac{n}{d'}}\right\rangle$. Thus we can partition the vertices of $\Gamma(D_{2n})$ in the following manner. Define $A_0 := \{\langle r, s\rangle\}$. If $\alpha(n)$ is odd then for $k = 1, 2, \ldots, \lceil\frac{\alpha(n)}{2}\rceil - 1$ define

$$A_k := \left\{\left\langle r^{\frac{n}{d}}\right\rangle, \left\langle r^{\frac{n}{d'}}, r^i s\right\rangle \middle| 0 \leq i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 1 = \alpha(n) - (2k-1)\right\}$$

and $A_{\lceil \frac{\alpha(n)}{2} \rceil} := \{1\}$. Furthermore, for $k = 0, 1, \ldots \lfloor \frac{\alpha(n)}{2} \rfloor$ set

$$B_k := \left\{ \left\langle r^{\frac{n}{d}} \right\rangle, \left\langle r^{\frac{n}{d'}}, r^i s \right\rangle \middle| 0 \le i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 1 = \alpha - 2k \right\}.$$

If $\alpha(n)$ is even then for $k = 1, 2, \ldots, \lceil \frac{\alpha}{2} \rceil$ define

$$A_k := \left\{ \left\langle r^{\frac{n}{d}} \right\rangle, \left\langle r^{\frac{n}{d'}}, r^i s \right\rangle \middle| 0 \le i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 1 = \alpha(n) - (2k-1) \right\}.$$

Furthermore for $k = 0, 1, \ldots \lfloor \frac{\alpha(n)}{2} \rfloor - 1$ set

$$B_k := \left\{ \left\langle r^{\frac{n}{d}} \right\rangle, \left\langle r^{\frac{n}{d'}}, r^i s \right\rangle \middle| 0 \le i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 1 = \alpha(n) - 2k \right\}$$

and $B_{\lfloor \frac{\alpha(n)}{2} \rfloor} := \{1\}$. Then in either case let

$$A := \bigcup_{k=0}^{\lceil \frac{\alpha(n)}{2} \rceil} A_k \quad \text{and} \quad B := \bigcup_{k=0}^{\lfloor \frac{\alpha(n)}{2} \rfloor} B_k.$$

Thus A and B form two independent sets of vertices and hence D_{2n} is bipartite. □

Now, we wish to prove that D_{2n} is unbalanced. First, we will need some lemmas:

Lemma 4.2. *For any $n_0 > 1$ and any prime p,*

$$\sum_{k=0}^{n_0} (-p)^k \neq 0$$

Proof. Suppose that this were the case. We would then have;

$$\begin{aligned}\sum_{k=0}^{n_0}(-p)^k &= 0\\ 1-p+p^2-\ldots\pm p^{n_0} &= 0\\ p-p^2+\ldots\pm p^{n_0} &= 1\\ p(1-p+p^2-\ldots\pm p^{n_0-1}) &= 1\end{aligned}$$

Since $p, 1-p+p^2-\ldots\pm p^{n_0-1} \in \mathbb{Z}$, it must be that

$$p = 1-p+p^2-\ldots\pm p^{n_0} = \pm 1,$$

but p was chosen to be prime, a contradiction. □

Lemma 4.3. *For any $n_0 > 1$ and any prime p,*

$$\sum_{k=0}^{n_0}(-p)^k \neq \pm 1.$$

Proof. Suppose that this sum is 1. We would then have;

$$\begin{aligned}\sum_{k=0}^{n_0}(-p)^k &= 1\\ 1-p+p^2-\ldots\pm p^{n_0} &= 1\end{aligned}$$

$$-p + p^2 - \ldots \pm p^{n_0} = 0$$

$$-p(1 - p + p^2 - \ldots \pm p^{n_0 - 1}) = 0$$

Since $p, 1-p+p^2-\ldots \pm p^{n_0-1} \in \mathbb{Z}$, it must be that either $p = 0$, contradicting the assumption that p is prime, or $1 - p + p^2 - \ldots \pm p^{n_0-1} = 0$, contradicting Lemma 4.2. Hence $\sum_{k=0}^{n_0}(-p)^k \neq 1$.

Now suppose that this sum is -1. We would then have;

$$\sum_{k=0}^{n_0}(-p)^k = -1$$

$$1 - p + p^2 - \ldots \pm p^{n_0} = -1$$

$$-p + p^2 - \ldots \pm p^{n_0} = -2$$

$$-p(1 - p + p^2 - \ldots \pm p^{n_0-1}) = -2$$

$$p(1 - p + p^2 - \ldots \pm p^{n_0-1}) = 2$$

$$p\sum_{k=0}^{n_0-1}(-p)^k = 2$$

Since $p, 1 - p + p^2 - \ldots \pm p^{n_0-1} \in \mathbb{Z}$, it must be that $p = \pm 1, \pm 2$. Since p was chosen to be prime the only valid option is that $p = 2$. This then implies that $\sum_{k=0}^{n_0-1}(-2)^k = 1$. If $n_0 - 1 = 1$, then this amounts to $1 - 2 = 1$, which is an obvious contradiction. If $n_0 - 1 > 1$ then this contradicts our findings in the first half of this proof. Thus $\sum_{k=0}^{n_0}(-p)^k \neq -1$. □

From the definition of A_k and B_k, for each d such that $\beta(d) = k$, we have that, for $\alpha(n)$ odd $\left\langle r^{\frac{n}{d}}\right\rangle \in A$ for k even and $\left\langle r^{\frac{n}{d}}\right\rangle \in B$ for k odd and for $\alpha(n)$ even $\left\langle r^{\frac{n}{d}}\right\rangle \in B$ for k even and $\left\langle r^{\frac{n}{d}}\right\rangle \in A$ for k odd. Define $A'(n) := \left\{\left\langle r^{\frac{n}{d}}\right\rangle \middle| d|n\right\} \setminus B$ and $B'(n) := \left\{\left\langle r^{\frac{n}{d}}\right\rangle \middle| d|n\right\} \setminus A$. Define $A'_k := A'(n) \cap A_k$ and $B'_k := B'(n) \cap B_k$ for the proper values of k. Define $s'(n) := |A'(n)| - |B'(n)|$.

Lemma 4.4. *If α_i is even for all $i = 1, 2, \ldots, m$ then $s'(n) = -1$, else $s'(n) = 0$.*

Proof. If $d, d'|n$ and $\beta(d) \equiv \beta(d') \mod 2$, then $\left\langle r^{\frac{n}{d}}\right\rangle, \left\langle r^{\frac{n}{d'}}\right\rangle \in A'$ or $\left\langle r^{\frac{n}{d}}\right\rangle, \left\langle r^{\frac{n}{d'}}\right\rangle \in B'$. Whereas if $\beta(d) \not\equiv \beta(d') \mod 2$ then $\left\langle r^{\frac{n}{d}}\right\rangle \in A'$ and $\left\langle r^{\frac{n}{d'}}\right\rangle \in B'$ or $\left\langle r^{\frac{n}{d}}\right\rangle \in B'$ and $\left\langle r^{\frac{n}{d'}}\right\rangle \in A'$. Note that for each $d|n$, we have that $\frac{n}{d}|n$ where $\beta(\frac{n}{d}) = \alpha - \beta(d)$.

Now $\beta(d)$ ranges from 0 to $\alpha(n)$. If $\alpha(n)$ is odd, (so that at least one α_i is odd), $\beta(\frac{n}{d}) = \alpha - \beta(d) \not\equiv \beta(d) \mod 2$ and hence $\left\langle r^{\frac{n}{d}}\right\rangle \in A'$ and $\langle r^d\rangle \in B'$ or $\left\langle r^{\frac{n}{d}}\right\rangle \in B'$ and $\langle r^d\rangle \in A'$. Note that it cannot be that $\beta(d) = \alpha(n) - \beta(d)$ since this would imply $2\beta(d) = \alpha(n)$, but $\alpha(n)$ is odd. So in either case, each element of A' corresponds to an element in B' and thus $s'(n) = 0$.

To cover the case where $\alpha(n)$ is even, we will proceed by induction on $\alpha(n)$. First, some base cases with even $\alpha(n)$'s:

Case 1: $\alpha(n) = 2$ and $m = 1$.

We then have $n = p_1^2$ so that all α_is are even. In this case $B'_0 = \{\langle r\rangle\}$, $A'_1 = \{\langle r^{p_1}\rangle\}$ and $B'_1 = \{1\}$. Hence $s'(n) = -1$.

Case 2: $\alpha(n) = 2$ and $m = 2$.

We then have $n = p_1 p_2$ so that at least one α_i is odd. In this case $B'_0 = \{\langle r \rangle\}$, $A'_1 = \{\langle r^{p_1} \rangle, \langle r^{p_2} \rangle\}$ and $B'_1 = \{1\}$. Hence $s'(n) = 0$.

Case 3: $\alpha(n) = 4$ and $m = 1$.

We then have $n = p_1^4$ so that all α_is are even. In this case $B'_0 = \{\langle r \rangle\}$, $A'_1 = \{\langle r^{p_1} \rangle\}$, $B'_1 = \left\{\left\langle r^{p_1^2} \right\rangle\right\}$, $A'_2 = \left\{\left\langle r^{p_1^3} \right\rangle\right\}$ and $B'_2 = \{1\}$. Hence $s'(n) = -1$.

Case 4: $\alpha(n) = 4$ and $m = 2$.

We then have $n = p_1^2 p_2^2$ so that all α_is are even. In this case $B'_0 = \{\langle r \rangle\}$, $A'_1 = \{\langle r^{p_1} \rangle, \langle r^{p_2} \rangle\}$, $B'_1 = \left\{\left\langle r^{p_1^2} \right\rangle, \langle r^{p_1 p_2} \rangle, \left\langle r^{p_2^2} \right\rangle\right\}$, $A'_2 = \left\{\left\langle r^{p_1^2 p_2} \right\rangle, \left\langle r^{p_1 p_2^2} \right\rangle\right\}$ and $B'_2 = \{1\}$. Hence $s'(n) = -1$.

Case 5: $\alpha(n) = 4$ and $m = 3$.

We then have $n = p_1^2 p_2 p_3$ so that at least one α_is is odd. In this case $B'_0 = \{\langle r \rangle\}$, $A'_1 = \{\langle r^{p_1} \rangle, \langle r^{p_2} \rangle, \langle r^{p_3} \rangle\}$, $B'_1 = \left\{\left\langle r^{p_1^2} \right\rangle, \langle r^{p_1 p_2} \rangle, \langle r^{p_1 p_3} \rangle, \langle r^{p_2 p_3} \rangle\right\}$, $A'_2 = \left\{\left\langle r^{p_1^2 p_2} \right\rangle, \left\langle r^{p_1^2 p_3} \right\rangle, \langle r^{p_1 p_2 p_3} \rangle\right\}$ and $B'_2 = \{1\}$. Hence $s'(n) = 0$.

Case 6: $\alpha(n) = 4$ and $m = 4$.

We then have $n = p_1 p_2 p_3 p_4$ so that at least one α_i is odd. In this case we have that

$$B'_0 = \{\langle r \rangle\},$$

$$A'_1 = \{\langle r^{p_1} \rangle, \langle r^{p_2} \rangle, \langle r^{p_3} \rangle, \langle r^{p_4} \rangle\},$$

$$B'_1 = \{\langle r^{p_1 p_2} \rangle, \langle r^{p_1 p_3} \rangle, \langle r^{p_1 p_4} \rangle, \langle r^{p_2 p_3} \rangle, \langle r^{p_2 p_4} \rangle, \langle r^{p_3 p_4} \rangle\},$$

$$A_2' = \{\langle r^{p_1p_2p_3}\rangle, \langle r^{p_1p_2p_4}\rangle, \langle r^{p_1p_3p_3}\rangle, \langle r^{p_2p_3p_4}\rangle\} \text{ and}$$

$$B_2' = \{1\}.$$

Hence we can easily compute that $s'(n) = 0$.

By use of strong induction, assume that the conclusion holds for all $0 < \alpha(n) \leq \alpha_0$ for some $\alpha_0 > 4$. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$ with $\alpha(n) = \alpha_0$, choose some prime p_{m+1} (so that $\alpha(np_{m+1}) = \alpha_0 + 1$, and hence they have different parities). Given how A' and B' are defined, all elements of A' for $\Gamma(D_{2n})$ become elements of B' for $\Gamma(D_{2p_{m+1}})$ and all elements of B' for $\Gamma(D_{2n})$ become elements of A' for $\Gamma(D_{2np_{m+1}})$. So changing the sign of $s'(n)$ for $\Gamma(D_{2n})$ will account for the sublattice C_n in $C_{np_{m+1}}$. The new elements of A' and B' are then of the form $\left\langle r^{\frac{np_{m+1}}{d}}\right\rangle$ where $p_{m+1} \nmid d$.

If $p_{m+1} \nmid n$ (and hence $\alpha_{m+1} = 1$, which is odd) then this forms a second copy of C_n wherein if $\beta(d) = k$ in $\Gamma(D_{2n})$ then $\beta(d) = k+1$ in $\Gamma(D_{2np_{m+1}})$. So we have two copies of C_n with different signs for the alternating sum. By our induction hypothesis, $s'(np_{m+1}) = s'(n) + (-1)s'(n) = 0$ as required.

Suppose then that $p_{m+1}|n$, that is $p_{m+1} = p_i$ for some $0 < i \leq m$, say p_{i_0}. Note then that $\alpha_{i_0}(np_{m+1}) = \alpha_{i_0}(n) + 1$. Write $np_{m+1} = p_{i_0}^{\alpha_{i_0}}l$. We then have a copy of C_n with the opposite sign on its alternating sum, and a copy of C_l. We then have the following cases:

Case 1: α_i is even for all $i \neq i_0$, and $\alpha_{i_0}(n)$ is even. In this case $\alpha(n)$ is even so that $\alpha(np_{m+1})$ is odd and hence $s'(np_{m+1}) = 0$ as shown previously.

Case 2: α_i is even for all $i \neq i_0$, and $\alpha_{i_0}(n)$ is odd. In this case $\alpha(n)$ is odd so as previously shown $s'(n) = 0$. We then have, by our induction hypothesis, $s'(np_{m+1}) = -s'(n) + s'(l) = -1$.

Case 3: α_{j_0} is odd for some $p_{j_0}|l$. Then by our induction hypothesis

$$s'(np_{m+1}) = -s'(n) + s'(l) = 0$$

as required. $\square$

Lemma 4.5. *If at least one of the α_i's is odd, then*

$$|A| - |B| = \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right)$$

or else

$$|A| - |B| = \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right) - 1$$

if all α_i's are even.

Proof. Again from the definition of A_k and B_k, for each d such that $\beta(d) = \alpha - k$ we have $\left\langle r^{\frac{n}{d}}, r^i s \right\rangle \in B$, $0 \leq i \leq \frac{n}{d}$ for k even and $\left\langle r^{\frac{n}{d}}, r^i s \right\rangle \in A$, $0 \leq i \leq \frac{n}{d}$ for k odd. Recalling then that each $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m}$, this accounts for $\frac{n}{d} = p_1^{\alpha_1 - \beta_1} p_2^{\alpha_2 - \beta_2} \dots p_m^{\alpha_m - \beta_m}$ elements in B for k even and $\frac{n}{d}$ elements in A for k odd. Define $A'' := A \setminus A'$ and $B'' := B \setminus B'$. We then have that

$|A'' \cup B''| = \sigma(n)$, the sum of the divisors of n. Now

$$\begin{aligned}\sigma(n) &= \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} p_i^j \right) \\ &= \prod_{i=1}^{m} (1 + p_i + p_i^2 + \ldots + p_i^{\alpha_i})\end{aligned}$$

Since we are looking to determine $|A''| - |B''|$, we need to assign negative values to those divisors d such that $\beta(d)$ is odd. So, if instead we consider

$$\prod_{i=1}^{m} (1 - p_i + p_i^2 - p_i^3 + \ldots + (-p_i)^{\alpha_i})$$

only those divisors with an odd number of prime factors with odd exponents will retain the negative sign, which is exactly what we are looking for. Hence

$$\begin{aligned}|A''| - |B''| &= \prod_{i=1}^{m} (1 - p_i + p_i^2 - p_i^3 + \ldots + (-p_i)^{\alpha_i}) \\ &= \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right).\end{aligned}$$

Now, A', A'', B' and B'' are all disjoint by definition. In addition, $A = A' \cup A''$ and $B = B' \cup B''$. So

$$\begin{aligned}|A| - |B| &= |A'| + |A''| - |B'| - |B''| \\ &= |A''| - |B''| + |A'| - |B'| \\ &= \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right) + s'(n)\end{aligned}$$

Lemma 4.4 completes the proof. □

Theorem 4.6. *D_{2n} is unbalanced and hence non-Hamiltonian.*

Proof. If at least one of the α_i's is odd, then by lemma 4

$$|A| - |B| = \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right).$$

By Lemma 4.2, none of these factors are 0, hence their product cannot be 0 so that $|A| \neq |B|$.

If all α_i's are even, then by Lemma 4.4

$$|A| - |B| = \prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right) - 1.$$

By Lemma 4.3, none of the factors of $\prod_{i=1}^{m} \left(\sum_{j=0}^{\alpha_i} (-p_i)^j \right)$ can be ± 1. Hence, their product cannot be 1 and thus $|A| \neq |B|$. Hence D_{2n} is non-Hamiltonian by Theorem 2.11. □

While a dihedral group on its own is non-Hamiltonian, that is not always the case for direct products of dihedral groups with other groups:

Proposition 4.7. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ in its prime factorization and $2 \nmid n$, $D_8 \times C_n$ and $D_{16} \times C_n$ are Hamiltonian if and only if at least one of $\alpha_1, \alpha_2, \dots, \alpha_k$ is odd.*

Proof. Figure 17 below shows a Hamiltonian cycle through $D_8 \times C_p$ for $2 \nmid p$

prime. Theorem 2.26, Example 2.36 and the Chinese Remainder Theorem then imply that if one of $\alpha_1, \alpha_2, \ldots, \alpha_k$ is odd, then $D_8 \times C_n$ is Hamiltonian. If instead each $\alpha_1, \alpha_2, \ldots, \alpha_k$ is even, then $D_8 \times C_{p^\alpha}$ is unbalanced by Theorem 2.32 and Theorem 4.6 and hence non-Hamiltonian by Theorem 2.11. Mathematica verifies that $D_{16} \times C_p$ for $2 \nmid p$ prime is also Hamiltonian, so this argument can be repeated to show the same result for $D_{16} \times C_n$. $\square$

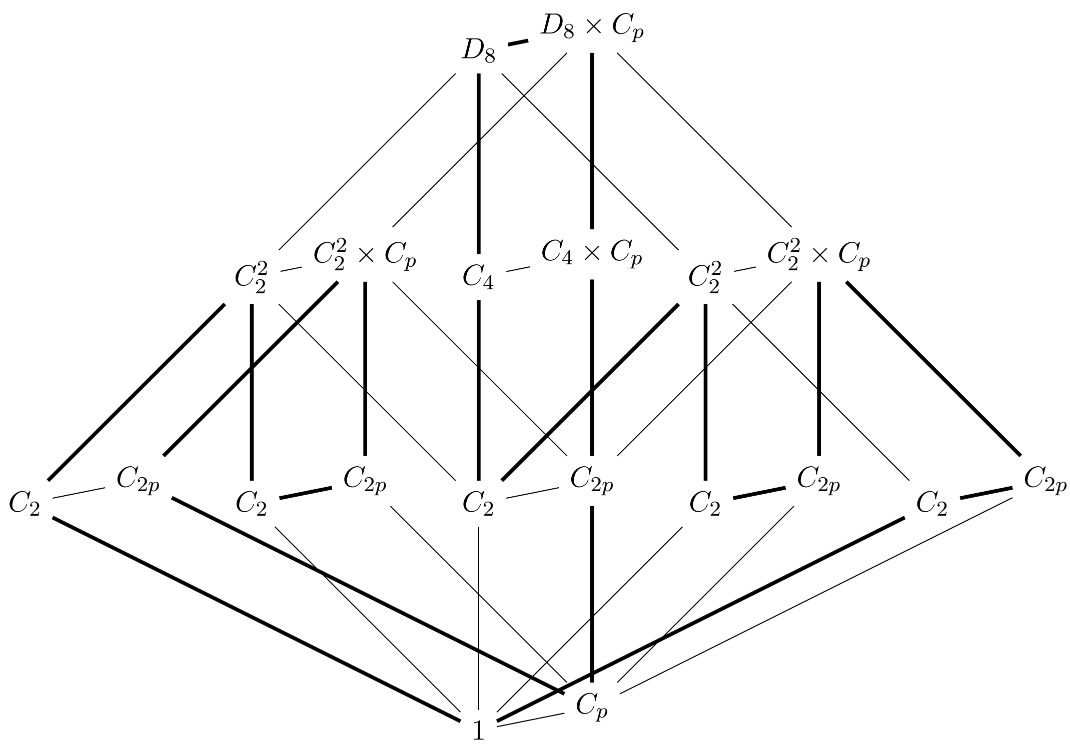

Figure 17: A Hamiltonian cycle through $\Gamma(D_8 \times C_p)$ for $p \neq 2$

Next, we look at D_{2p} where p is prime. As seen in Figure 18 , $\Gamma(D_{2p}) \cong \Gamma(C_p^2)$. Then by Proposition 3.2 and Theorem 2.35 we have:

Proposition 4.8. *If $p \neq q$ are prime, then $D_{2p} \times C_q$ is non-Hamiltonian for all $p > 3$, $q \neq 2$.*

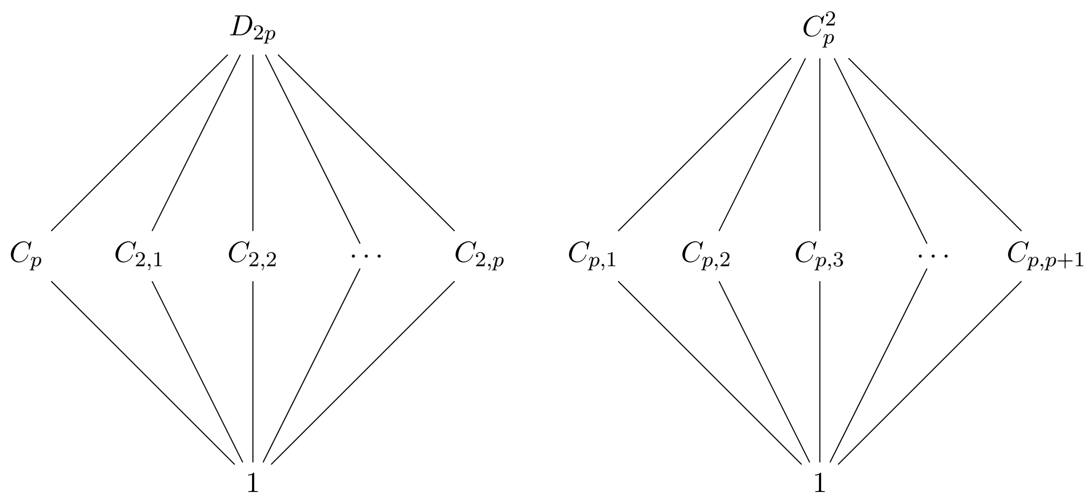

Figure 18: $\Gamma(D_{2p}) \cong \Gamma(C_p^2)$

5 DICYCLIC GROUPS

In this section we show that the dihedral groups are non-Hamiltonian. We will do this by showing that the dihedral groups are unbalanced and hence non-Hamiltonian by Theorem 2.11. We have

$$\mathrm{Dic}_n = \left\langle x, y | x^{2n} = 1, y^2 = x^n, yx = x^{-1}y \right\rangle .$$

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ be its prime factorization, $m > 0$ and each $\alpha_i > 0$. Define $\alpha(n) := \sum_{i=1}^{m} \alpha_i$ and $\eta(n) := \max\{\eta | 2^\eta \text{ divides } n\}$.

We will first need some lemmas:

Lemma 5.1. *For all $i, k \in \mathbb{Z}$, $\left\langle x^k, x^i y \right\rangle = \left\langle x^{\gcd(k,n)}, x^i y \right\rangle$.*

Proof. It is clear that $x^i y$ is an element of both of these subgroups. Thus $(x^i y)^2 = x^i y x^i y = x^i x^{-i} y^2 = x^n$ is also an element of both of these subgroups. It is also immediate that $x^k \in \left\langle x^{\gcd(k,n)}, x^i y \right\rangle$ so that $\left\langle x^k, x^i y \right\rangle \subseteq \left\langle x^{\gcd(k,n)}, x^i y \right\rangle$. To see that $x^{\gcd(k,n)} \in \left\langle x^k, x^i y \right\rangle$, note that by the Greatest Common Divisor Theorem we can write $\gcd(k, n) = m_1 a + m_2 n$ for some $m_1, m_2 \in \mathbb{Z}$. Then

$$x^{\gcd(k,n)} = x^{m_1 k + m_2 n} = (x^k)^{m_1} (x^n)^{m_2} \in \left\langle x^k, x^i y \right\rangle$$

so that $\left\langle x^k, x^i y \right\rangle = \left\langle x^{\gcd(k,n)}, x^i y \right\rangle$ as required. ☐

This can be directly used for the following lemma, the proof of which then follows immediately.

Lemma 5.2. *For all* $i \in \mathbb{Z}$*, if* $\gcd(2k,n) = \gcd(k,n)$*, then* $\langle x^k, x^i y\rangle = \langle x^{2k}, x^i y\rangle$ We first need to show that:

Proposition 5.3. Dic_n *is bipartite for all* $n \in \mathbb{N}$*.*

Proof. Now, $\langle x\rangle$ is a cyclic group of order $2n$. We first focus on the sublattice isomorphic to C_n. So similarly to the proof of Proposition 4.1 (and using the same notation),

$$\left\{\left\langle x^{\frac{2n}{d}}\right\rangle \middle| \beta(d) = 2k\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor} \text{ and } \left\{\left\langle x^{\frac{2n}{d}}\right\rangle \middle| \beta(d) = 2k+1\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$

form two independent sets of vertices in $\Gamma(\mathrm{Dic}_n)$. We also need to account for the subgroups of $\langle x\rangle$ not contained in the sublattice isomorphic to C_n, namely those subgroups of the form $\left\langle x^{\frac{2n}{2d}}\right\rangle$ such that $2^{\eta(n)} \mid d$. For these,

$$\left\{\left\langle x^{\frac{2n}{2d}}\right\rangle \middle| 2^{\eta(n)}|d, \beta(2d) = 2k\right\}_{k=\eta(n)}^{\lfloor\frac{\alpha(n)+1}{2}\rfloor}$$
$$\text{and } \left\{\left\langle x^{\frac{2n}{2d}}\right\rangle \middle| 2^{\eta(n)}|d, \beta(2d) = 2k+1\right\}_{k=\eta(n)}^{\lfloor\frac{\alpha(n)+1}{2}\rfloor}$$

form two independent sets of vertices in $\Gamma(\mathrm{Dic}_n)$.

Next, for each d, if $i \equiv j \pmod{\frac{2n}{d}}$ then $\left\langle x^{\frac{2n}{d}}, x^i y\right\rangle = \left\langle x^{\frac{2n}{d}}, x^j y\right\rangle$. In addition, if $2 \nmid d$ then $\gcd(\frac{2n}{d}, n) = \gcd(\frac{2n}{2d}, n)$ so that by Lemma 5.2 we have

$\left\langle x^{\frac{2n}{d}}, x^i y \right\rangle = \left\langle x^{\frac{n}{d}} \right\rangle = \left\langle x^{\frac{2n}{2d}}, x^i y \right\rangle$. Then, in a similar manner as before

$$\left\{\left\langle x^{\frac{n}{d}}, x^i s\right\rangle \Big| 0 \le i < \tfrac{n}{d},\ \beta(d) = 2k\right\}_{k=1}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$
$$\text{and } \left\{\left\langle x^{\frac{n}{d}}, x^i y\right\rangle \Big| 0 \le i < \tfrac{n}{d},\ \beta(d) = 2k+1\right\}_{k=0}^{\lfloor\frac{\alpha(n)}{2}\rfloor}$$

form independent sets of vertices in $\Gamma(\mathrm{Dic}_n)$. This amounts to all of the subgroups of Dic_n.

Now, if $\left\langle x^{\frac{2n}{d}} \right\rangle < \left\langle x^{\frac{2n}{d'}}, x^i y \right\rangle$ then $\left\langle x^{\frac{2n}{d}} \right\rangle \le \left\langle x^{\frac{2n}{d'}} \right\rangle < \left\langle x^{\frac{2n}{d'}}, x^i s \right\rangle$ so $\left\langle x^{\frac{2n}{d}} \right\rangle$ is only adjacent to $\left\langle x^{\frac{2n}{d'}}, x^i s \right\rangle$ in $\Gamma(\mathrm{Dic}_n)$ whenever $d = d'$. Finally, we know that it could never be that $\left\langle x^{\frac{2n}{d}}, x^i y \right\rangle < \left\langle x^{\frac{2n}{d'}} \right\rangle$. Hence we can partition the the vertices of $\Gamma(\mathrm{Dic}_n)$ in the following manner. From here on out, let d describe an arbitrary divisor of $2n$, rather than of n. Define $A_0 := \{\langle x, y\rangle\}$. If $\alpha(n)$ is odd then for $k = 1, 2, \ldots, \lfloor\frac{\alpha(n)}{2}\rfloor$ define

$$A_k := \left\{\left\langle x^{\frac{2n}{d}} \right\rangle, \left\langle x^{\frac{n}{d'}}, x^i s\right\rangle \Big| 0 \le i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 2 = \alpha(n) - (2k-1)\right\}$$

and define

$$A_{\lceil\frac{\alpha(n)}{2}\rceil} := \left\{\left\langle x^{\frac{2n}{d}} \right\rangle \Big| \beta(d) = 1\right\}.$$

Furthermore, for $k = 0, 1, \ldots \lfloor\frac{\alpha(n)}{2}\rfloor$ define $B_k :=$

$$\left\{\left\langle x^{\frac{2n}{d}} \right\rangle, \left\langle x^{\frac{n}{d'}}, x^i s\right\rangle \Big| 0 \le i < \tfrac{n}{d'},\ 2^{\eta} | d',\ \beta(d) = \beta(d') + 2 = \alpha(n) - 2k\right\}$$

and define $B_{\lceil \frac{\alpha(n)}{2} \rceil} := \{1\}$.

If $\alpha(n)$ is even then for $k = 1, 2, \ldots, \frac{\alpha(n)}{2}$ define

$$A_k := \left\{ \left\langle x^{\frac{2n}{d}} \right\rangle, \left\langle x^{\frac{n}{d'}}, x^i s \right\rangle \middle| 0 \leq i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 2 = \alpha(n) - (2k-1) \right\}$$

and define $A_{\frac{\alpha(n)}{2}+1} := \{1\}$.

Furthermore, for $k = 0, 1, \ldots \frac{\alpha(n)}{2} - 1$ define

$$B_k := \left\{ \left\langle x^{\frac{2n}{d}} \right\rangle, \left\langle x^{\frac{n}{d'}}, x^i y \right\rangle \middle| 0 \leq i < \tfrac{n}{d'},\ \beta(d) = \beta(d') + 2 = \alpha(n) - 2k \right\}$$

and define

$$B_{\frac{\alpha(n)}{2}} := \left\{ \left\langle r^{\frac{n}{d}} \right\rangle \middle| \beta(d) = 1 \right\}.$$

Then, in either case, define

$$A := \bigcup_{k=0}^{\lfloor \frac{\alpha(n)}{2} \rfloor + 1} A_k \quad \text{and} \quad B := \bigcup_{k=0}^{\lceil \frac{\alpha(n)}{2} \rceil} B_k$$

Thus A and B form two independent sets of vertices and hence Dic_n is bipartite. $\square$

If we define $A'(2n), B'(2n)$ and $s'(2n)$ in the same manner as was done in Lemma 4.4, we achieve the same result with the roles of "odd" and "even" reversed since $\alpha(2n) = \alpha(n) + 1$. Then if we define A'' and B'' in the same manner as in Lemma 4.5 and let $2n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_l^{\alpha_l}$ in its prime factorization we can follow a similar technique in the proof of Lemma 4.5 and the proof of Theorem 4.6 to show:

Theorem 5.4. *Dic_n is unbalanced and hence non-Hamiltonian.*

It is well know that $\text{Dic}_{2^k} \cong Q_{2^{k+2}}$, the generalized quaternion group. While the quaternion groups are non-Hamiltonian, much like for D_{2n} this may not be the case for direct products of cyclic groups and quaternion groups.

Proposition 5.5. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ in its prime factorization such that $2 \nmid n$. Then*

(i) $Q_8 \times C_n$ and $Q_{16} \times C_n$ are Hamiltonian if and only if there exists an $i = 1, 2, \dots, k$ such that α_i is odd.

(ii) $Q_{32} \times C_n$ and $Q_{64} \times C_n$ are Hamiltonian if there exists two distinct $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, k$ such that both α_i and α_j are odd.

both hold.

Proof. It has been shown [3, p. 22] that $Q_8 \times C_3$ is Hamiltonian, and Mathematica verifies that $Q_{16} \times C_3$ is Hamiltonian. This extends to telling us that $Q_8 \times C_p$ and $Q_{16} \times C_p$ are Hamiltonian for $p > 2$ prime. Then since Q_8 and Q_{16} are unbalanced by Theorem 5.4, Theorem 2.26 and 2.32 verify (i).

Mathematica verifies that $Q_{32} \times C_{15}$ and $Q_{64} \times C_{15}$ are Hamiltonian. This extends to telling us that $Q_{32} \times C_{pq}$ and $Q_{64} \times C_{pq}$ are Hamiltonian for distinct primes $p, q > 2$. Then by Corollary 2.27, if there exists two distinct

$i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, k$ such that both α_i and α_j are odd, (ii) is verified. $\square$

Note that $Q_{32} \times C_3$ and $Q_{64} \times C_3$ were found by Mathematica to be non-Hamiltonian and that $Q_{32} \times C_{3^3}$ was also found to be non-Hamiltonian (whereas $Q_{64} \times C_{p^3}$'s Hamiltonicity could not be determined by Mathematica). While it is possible a priori that there may be some cyclic p-group with exponent larger than 3 that can be crossed with Q_{32} or Q_{64} to result in a Hamiltonian graph, we do know that this order cannot be even by Theorem 2.32 since Q_{32} and Q_{64} are unbalanced. We also determined that $Q_{128} \times C_3$ is non-Hamiltonian but $Q_{128} \times C_{15}$'s Hamiltonicity could not be determined by Mathematica.

6 NON-ABELIAN p-GROUPS

McLaughlin conjectured in [3, Conjecture 3] that all non-abelian p-groups are non-Hamiltonian. To disprove this, we show that $C_p \times (C_{p^3} \rtimes C_p)$ is Hamiltonian for all prime p. First note that in the case where $p = 2$, the four semi-direct products give us: $C_2^2 \times C_8,\ C_2 \times QD_{16},\ C_2 \times M_4(2)$ and $C_2 \times D_{16}$. Gap and Mathematica verify that these four groups are Hamiltonian. When $p > 2$, we have two semidirect products:

$$G_1 := \left\langle x, y, z \,\middle|\, x^p = y^{p^3} = z^p, xy = yx, xz = zx, yz = zy \right\rangle \cong C_p^2 \times C_{p^3}$$
$$G_2 := \left\langle a, b, c \,\middle|\, a^p = b^{p^3} = b^p, ab = ba, ac = ca, cb = b^{p^2+1}c \right\rangle$$

By [3, Theorem 8], G_1 is Hamiltonian. We claim that $\Gamma(G_2) \cong \Gamma(G_1)$ and hence G_2 is Hamiltonian. We will show this through the following theorem from [5, p. 17]:

Theorem 6.1. *Let $\phi : G_1 \to G_2$ be bijective and $\phi(H) := \{\phi(h)|h \in H\}$ for $H \subseteq G_1$. Then $H \leq G_1 \iff \phi(H) \leq G_2$ implies that $\Gamma(G_1) \cong \Gamma(G_2)$.*

Hence we can show that $\Gamma(G_1) \cong \Gamma(G_2)$ through the construction of a projectivity between G_1 and G_2. Before we create such a function, we will need a few lemmas.

Lemma 6.2. *For* $k \in \mathbb{N} \cup \{0\}$

$$(p^2+1)^k \equiv kp^2+1 \mod p^3.$$

In particular, $(p^2+1)^p \equiv 1 \mod p^3$*.*

Proof. First note that $(p^2+1)^0 = 1 \equiv (0)p^2+1 \mod p^3$. Inductively assume that the statement holds for $k \in \mathbb{N} \cup \{0\}$. Then:

$$\begin{aligned}
(p^2+1)^k &\equiv kp^2+1 \mod p^3 \\
(p^2+1)(p^2+1)^k &\equiv (p^2+1)(kp^2+1) \mod p^3 \\
(p^2+1)^{k+1} &\equiv kp^4+p^2+kp^2+1 \mod p^3 \\
(p^2+1)^{k+1} &\equiv (k+1)p^2+1 \mod p^3.
\end{aligned}$$

We can then see that $(p^2+1)^p \equiv p^3+1 \equiv 1 \mod p^3$. □

Lemma 6.3. *For* $n, m \in \mathbb{N} \cup \{0\}$*, we have (in* G_2*),* $c^n b^m = b^{m(p^2+1)^n} c^n$*.*

Proof. This holds trivially for $n = m = 0$. Let $n = 1$. For $m = 1$ this is as appears in the definition of G_2. Suppose this holds for some $m \in \mathbb{N}$. Then:

$$\begin{aligned}
cb^{m+1} &= cb^m b \\
&= b^{m(p^2+1)} cb
\end{aligned}$$

$$= b^{m(p^2+1)}b^{p^2+1}c$$

$$= b^{(m+1)(p^2+1)}c$$

Then, let the statement hold for some $n \in \mathbb{N}$ and fix $m \in \mathbb{N}$. Then:

$$\begin{aligned} c^{n+1}b^m &= cc^nb^m \\ &= cb^{m(p^2+1)^n}c^n \\ &= b^{m(p^2+1)^n(p^2+1)}cc^n \\ &= b^{m(p^2+1)^{n+1}}c^{n+1} \end{aligned}$$

as required. □

Lemma 6.4. *For all $k \in \mathbb{N}$ and $l, m, n \in \mathbb{Z}$,*

$$(a^lb^mc^n)^k = a^{kl}b^{m(k+\frac{np^2k(k-1)}{2})}c^{kn}$$

holds.

Proof. We first claim that

$$(a^lb^mc^n)^k = a^{kl}b^{m\sum_{i=1}^{k}(p^2+1)^{n(i-1)}}c^{kn}$$

This holds trivially for $k = 1$. Suppose this holds for some $k \in \mathbb{N}$. Then by

Lemma 6.3:

$$\begin{aligned}(a^l b^m c^n)^{k+1} &= (a^l b^m c^n)^k (a^l b^m c^n)\\ &= a^{kl} b^{m\sum_{i=1}^{k}(p^2+1)^{n(i-1)}} c^{kn} (a^l b^m c^n)\\ &= a^{(k+1)l} b^{m\sum_{i=1}^{k}(p^2+1)^{n(i-1)}} c^{kn} b^m c^n\\ &= a^{(k+1)l} b^{m\sum_{i=1}^{k}(p^2+1)^{n(i-1)}} b^{m(p^2+1)^{kn}} c^{(k+1)n}\\ &= a^{(k+1)l} b^{m\left(\sum_{i=1}^{k}(p^2+1)^{n(i-1)}+(p^2+1)^{kn}\right)} c^{(k+1)n}\\ &= a^{(k+1)l} b^{m\sum_{i=1}^{k+1}(p^2+1)^{n(i-1)}} c^{(k+1)n}\end{aligned}$$

Then by Lemma 6.2:

$$\begin{aligned}m\sum_{i=1}^{k+1}(p^2+1)^{n(i-1)} &\equiv m\sum_{i=1}^{k}(n(i-1)p^2+1) \mod p^3\\ &\equiv m(k+np^2\sum_{i=1}^{k}(i-1)) \mod p^3\\ &\equiv m(k+np^2\sum_{i=1}^{k-1} i \mod p^3\\ &\equiv m(k+np^2\tfrac{(k-1)(k-1+1)}{2}) \mod p^3\\ &\equiv m(k+\tfrac{np^2k(k-1)}{2}) \mod p^3\end{aligned}$$

Hence

$$(a^l b^m c^n)^k = a^{kl} b^{m\sum_{i=1}^{k}(p^2+1)^{n(i-1)}} c^{kn} = a^{kl} b^{m\left(k+\frac{np^2k(k-1)}{2}\right)} c^{kn}$$

□

Theorem 6.5. *For p prime,*

$$G_2 := C_p \times (C_{p^3} \rtimes C_p) = \left\langle a, b, c | a^p = b^{p^3} = b^p, ab = ba, ac = ca, cb = b^{p^2+1}c \right\rangle$$

is Hamiltonian. This serves as a counterexample to McLaughlin's Conjecture 3.

Proof. Given the relations prescribed in their definitions, for $g_1 \in G_1$ and $g_2 \in G_2$ we may write $g_1 = x^{l_1}y^{m_1}z^{n_1}$ and $g_2 = a^{l_2}b^{m_2}c^{n_2}$ for some $l_1, l_2, m_1, m_2, n_1, n_2 \in \mathbb{Z}$ such that $0 \le l_1, l_2, n_1, n_2 < p$ and $0 \le m_1, m_2 < p^3$. Define $\phi : G_1 \to G_2$ by $\phi(x^l y^m z^n) := a^l b^m c^n$. Furthermore define $\phi(H) := \{\phi(h) | h \in H\}$ for $H \subseteq G_1$. If $x^{l_1}y^{m_1}z^{n_1} = x^{l_2}y^{m_2}z^{n_2}$ then $l_1 \equiv l_2 \mod p$, $m_1 \equiv m_2 \mod p^3$ and $n_1 \equiv n_2 \mod p$ so that necessarily

$$\phi(x^{l_1}y^{m_1}z^{n_1}) = a^{l_1}b^{m_1}c^{n_1} = a^{l_2}b^{m_2}c^{n_2} = \phi(x^{l_2}y^{m_2}z^{n_2}).$$

Hence ϕ is well-defined. If $a^l b^m c^n \in G_2$, then $x^l y^m z^n \in G_1$ and $\phi(x^l y^m z^n) = a^l b^m c^n$ so that ϕ is surjective. Furthermore, if $\phi(x^{l_1}y^{m_1}z^{n_1}) = \phi(x^{l_2}y^{m_2}z^{n_2})$, then $a^{l_1}b^{m_1}c^{n_1} = a^{l_2}b^{m_2}c^{n_2}$. Hence $l_1 \equiv l_2 \mod p$, $m_1 \equiv m_2 \mod p^3$ and $n_1 \equiv n_2 \mod p$ so that $x^{l_1}y^{m_1}z^{n_1} = x^{l_2}y^{m_2}z^{n_2}$. Therefore ϕ is injective.

It remains to be shown that $H \le G_1 \iff \phi(H) \le G_2$. First, assume that $H \le G_1$. Let $a^{l_1}b^{m_1}c^{n_1}, a^{l_2}b^{m_2}c^{n_2} \in \phi(H)$. Then $x^{l_1}y^{m_1}z^{n_1}, x^{l_2}y^{m_2}z^{n_2} \in$

H. Since $H \leq G_1$, $\left(x^{l_2}y^{m_2}z^{n_2}\right)^{-1} = x^{-l_2}y^{-m_2}z^{-n_2} \in H$. And furthermore $(x^{-l_2}y^{-m_2}z^{-n_2})^{(p^2+1)^{n_1}} \in H$. Now

$$\begin{aligned}(x^{-l_2}y^{-m_2}z^{-n_2})^{(p^2+1)^{n_1}} &= x^{-l_2(p^2+1)^{n_1}}y^{-m_2(p^2+1)^{n_1}}z^{-n_2(p^2+1)^{n_1}} \\ &= x^{-l_2(1)^{n_1}}y^{-m_2(p^2+1)^{n_1}}z^{-n_2(1)^{n_1}} \\ &= x^{-l_2}y^{-m_2(p^2+1)^{n_1}}z^{-n_2}\end{aligned}$$

So that finally

$$\begin{aligned}&\left(x^{l_1}y^{m_1}z^{n_1}\right)\left(x^{-l_2}y^{-m_2(p^2+1)^{n_1}}z^{-n_2}\right) \\ &= x^{l_1-l_2}y^{m_1-m_2(p^2+1)^{n_1}}z^{n_1-n_2} \in H.\end{aligned}$$

Through Lemma 6.3 we see:

$$\begin{aligned}\phi\left(x^{l_1-l_2}y^{m_1-m_2(p^2+1)^{n_1}}z^{n_1-n_2}\right) &= a^{l_1-l_2}b^{m_1-m_2(p^2+1)^{n_1}}c^{n_1-n_2} \\ &= a^{l_1}a^{-l_2}b^{m_1}b^{-m_2(p^2+1)^{n_1}}c^{n_1}c^{-n_2} \\ &= a^{l_1}a^{-l_2}b^{m_1}c^{n_1}b^{-m_2}c^{-n_2} \\ &= \left(a^{l_1}b^{m_1}c^{n_1}\right)\left(a^{-l_2}b^{-m_2}c^{-n_2}\right) \in \phi(H)\end{aligned}$$

Hence $\phi(H) \leq G_2$.

Next assume that $\phi(H) \leq G_2$. Let $x^{l_1}y^{m_1}z^{n_1}, x^{l_2}y^{m_2}z^{n_2} \in H$. Then

$$a^{l_1}b^{m_1}c^{n_1}, a^{l_2}b^{m_2}c^{n_2} \in \phi(H).$$

Since $\phi(H) \leq G_2$, by Lemmas 6.2 and 6.3:

$$\begin{aligned}
\left(a^{l_2}b^{m_2}c^{n_2}\right)^{-1} &= c^{-n_2}b^{-m_2}a^{-l_2} \\
&= a^{-l_2}c^{-n_2}b^{-m_2} \\
&= a^{-l_2}b^{-m_2(p^2+1)^{-n_2}}c^{-n_2} \in \phi(H).
\end{aligned}$$

Furthermore, $\left(a^{-l_2}b^{-m_2(p^2+1)^{-n_2}}c^{-n_2}\right)^{(p^2+1)^{n_2-n_1}} \in \phi(H)$. Finally by Lemmas 6.2, 6.3, and 6.4 we have that:

$$\begin{aligned}
&\left(a^{l_1}b^{m_1}c^{n_1}\right)\left(a^{-l_2}b^{-m_2(p^2+1)^{-n_2}}c^{-n_2}\right)^{(p^2+1)^{n_2-n_1}} \\
&= a^{l_1}b^{m_1}c^{n_1}a^{-l_2(p^2+1)^{n_2-n_1}} \\
&\quad b^{-m_2(p^2+1)^{-n_2}\left((p^2+1)^{n_2-n_1}+\frac{n_2p^2(p^2+1)^{n_2-n_1}((p^2+1)^{n_2-n_1}-1)}{2}\right)}c^{-n_2(p^2+1)^{n_2-n_1}} \\
&= a^{l_1}b^{m_1}c^{n_1}a^{-l_2(1)^{n_2-n_1}} \\
&\quad b^{-m_2(p^2+1)^{-n_2}\left((p^2+1)^{n_2-n_1}+\frac{n_2p^2(p^2+1)^{n_2-n_1}((p^2+1)^{n_2-n_1}-1)}{2}\right)}c^{-n_2(1)^{n_2-n_1}} \\
&= a^{l_1-l_2}b^{m_1}c^{n_1}b^{-m_2(p^2+1)^{-n_2}\left((p^2+1)^{n_2-n_1}+\frac{n_2p^2(p^2+1)^{n_2-n_1}((p^2+1)^{n_2-n_1}-1)}{2}\right)}c^{-n_2} \\
&= a^{l_1-l_2}b^{m_1}b^{-m_2(p^2+1)^{n_1}(p^2+1)^{-n_2}\left((p^2+1)^{n_2-n_1}+\frac{n_2p^2(p^2+1)^{n_2-n_1}((p^2+1)^{n_2-n_1}-1)}{2}\right)}c^{n_1}c^{-n_2} \\
&= a^{l_1-l_2}b^{m_1-m_2(p^2+1)^{n_1-n_2}\left((p^2+1)^{n_2-n_1}+\frac{n_2p^2(p^2+1)^{n_2-n_1}((p^2+1)^{n_2-n_1}-1)}{2}\right)}c^{n_1-n_2} \\
&= a^{l_1-l_2}b^{m_1-m_2((n_1-n_2)p^2+1)\left(((n_2-n_1)p^2+1)+\frac{n_2p^2((n_2-n_1)p^2+1)((n_2-n_1)p^2+1-1)}{2}\right)}c^{n_1-n_2} \\
&= a^{l_1-l_2}b^{m_1-m_2((n_1-n_2)p^2+1)\left((n_2p^2-n_1p^2+1)+\frac{n_2p^2(n_2p^2-n_1p^2+1)(n_2p^2-n_1p^2)}{2}\right)}c^{n_1-n_2} \\
&= a^{l_1-l_2}b^{m_1-m_2(n_1p^2-n_2p^2+1)\left((n_2p^2-n_1p^2+1)+\frac{(n_2^2p^4-n_1n_2p^4+n_2p^2)(n_2p^2-n_1p^2)}{2}\right)}c^{n_1-n_2}
\end{aligned}$$

$$= a^{l_1-l_2} b^{m_1-m_2(n_1p^2-n_2p^2+1)\left((n_2p^2-n_1p^2+1)+\frac{n_2p^2(n_2p^2-n_1p^2)}{2}\right)} c^{n_1-n_2}$$

$$= a^{l_1-l_2} b^{m_1-m_2(n_1p^2-n_2p^2+1)\left((n_2p^2-n_1p^2+1)+\frac{n_2^2p^4-n_1n_2p^4}{2}\right)} c^{n_1-n_2}$$

$$= a^{l_1-l_2} b^{m_1-m_2(n_1p^2-n_2p^2+1)(n_2p^2-n_1p^2+1)} c^{n_1-n_2}$$

$$= a^{l_1-l_2} b^{m_1-m_2(n_1n_2p^4-n_2^2p^4+n_2p^2-n_1^2p^4+n_1n_2p^4-n_1p^2+n_1p^2-n_2p^2+1)} c^{n_1-n_2}$$

$$= a^{l_1-l_2} b^{m_1-m_2(n_2p^2-n_1p^2+n_1p^2-n_2p^2+1)} c^{n_1-n_2}$$

$$= a^{l_1-l_2} b^{m_1-m_2} c^{n_1-n_2} \in \phi(H).$$

This implies that $x^{l_1-l_2}y^{m_1-m_2}z^{n_1-n_2} = x^{l_1}y^{m_1}z^{n_1}x^{-l_2}y^{-m_2}z^{-n_2} \in H$. Therefore $H \leq G_1$. Thus $H \leq G_1 \iff \phi(H) \leq G_2$ so that $\Gamma(G_1) \cong \Gamma(G_2)$. Then since G_1 is Hamiltonian, G_2 is also Hamiltonian. $\square$

7 OTHER NON-ABELIAN GROUPS

There are many ways we can go about determining the Hamiltonicity of some non-abelian groups that are not p-groups. First, we will consider a class of non-abelian groups whose subgroup lattices are isomorphic to another class of groups whose Hamiltonicity has already been determined.

Consider the group

$$C_p^2 \rtimes C_2 = \left\langle r, s, t | r^p = s^p = t^2 = 1, sr = rs, tr = r^{-1}t, ts = s^{-1}t \right\rangle$$

for $p > 2$ prime. The non-trivial proper subgroups of $C_p^2 \rtimes C_2$ consist of the following classes:

(a) The $p+1$ subgroups isomorphic to C_p of the form $\left\langle r^i s \right\rangle$ for $0 \leq i < p$ and $\langle r \rangle$.

(b) The p^2 subgroups isomorphic to C_2 of the form $\langle r^y s^x t \rangle$ for $0 \leq x, y < p$.

(c) The unique subgroup isomorphic to C_p^2, namely $\langle r, s \rangle$.

(d) The $p^2 + p$ subgroups isomorphic to D_{2p} of the form $\left\langle r^m s, r^b t \right\rangle$ for $0 \leq m, b < p$ and $\langle r, s^c t \rangle$ for $0 \leq c < p$.

Lemma 7.1. *The geometry Δ where subgroups of the class (d) are lines, subgroups of the class (b) are points, and incidence is defined by a point being a subset of a line is isomorphic to the affine plane defined over $\mathbb{F}_p$.*

Proof. We can coordinatize this in the following manner. Since points are of the form $\langle r^y s^x t\rangle$, let $\langle r^y s^x t\rangle \leftrightarrow (x, y)$. Lines that are of the form $\left\langle r^m s, r^b t\right\rangle$ contain the points $\left\langle r^{nm+b} s^n t\right\rangle$ for each $0 \leq n < p$. Given our coordinatization, $\left\langle r^{nm+b} s^n t\right\rangle \leftrightarrow (n, nm + b)$ that is, if $n = x$, $y = mx + b$. In this way $\left\langle r^m s, r^b t\right\rangle = \{(x, y)|y = mx + b\} \leftrightarrow [m, b]$. Finally, we have our vertical lines that are of the form $\langle r, s^c t\rangle$ which contain the points $\langle r^n s^c t\rangle$ for each $0 \leq n < p$. That is, $\langle r, s^c t\rangle = \{(x, y)|x = c\} \leftrightarrow [c]$. □

Lemma 7.2. *The geometry Δ' containing Δ as well as the subgroups of class (a) as points at infinity and $\langle r, s\rangle$ as the line at infinity forms the projective plane defined over $\mathbb{F}_p$.*

Proof. Our lines of the form $[m, b]$, with m fixed, are parallel in Δ. Indeed, each of these now also now also contain the point $\langle r^m s\rangle$ in Δ'. Furthermore each vertical line is parallel in Δ and indeed they now also contain the point $\langle r\rangle$ in Δ'. Finally $\langle r, s\rangle$, the line at infinity, contains each of the newly added points at infinity. □

This leads us to our first class of Hamiltonian non-abelian non-p-groups:

Theorem 7.3. *$C_p^2 \rtimes C_2$ is Hamiltonian for $p > 2$ prime.*

Proof. From Lemma 7.2 we have that $\Gamma(GC_p^2 \rtimes C_2)\backslash\{C_p^2 \rtimes C_2, 1\}$ is isomorphic to the projective plane defined over $\mathbb{F}_p$. Then, $\Gamma(C_p^2 \rtimes C_2)$ consists of the graph of the projective plane defined over $\mathbb{F}_p$ with a vertex added that connects to every line and a vertex added that connects to every point. It is well know that $\Gamma(C_p^3)$ consists of the graph of the projective plane defined over $\mathbb{F}_p$ with a vertex added that connects to every line and a vertex added that connects to every point. Hence $\Gamma(C_p^2 \rtimes C_2) \cong \Gamma(C_p^3)$. By [3, Theorem 7], C_p^3 is Hamiltonian. Hence $C_p^2 \rtimes C_2$ is Hamiltonian for $p > 2$ prime. $\square$

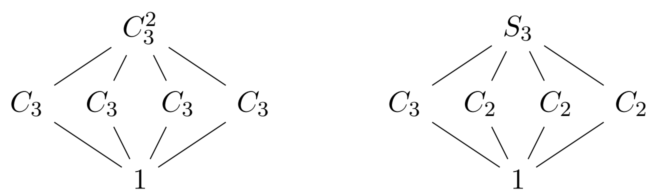

Figure 19: Subgroup lattices of C_3^2 and S_3

In order to obtain other classes of non-abelian groups that are Hamiltonian, we look to direct products of cyclic groups with non-abelian groups. Our first observation comes from taking the direct product cyclic groups and symmetric groups. Consider $S_3 \times C_q$ for $q \neq 2, 3$ prime. By Theorem 2.35 we have that $\Gamma(S_3 \times C_q) \cong \Gamma(S_3) \,\square\, P_2$. In Figure 19 it is easy to see that $\Gamma(S_3) \cong \Gamma(C_3^2)$ and hence that $\Gamma(S_3 \times C_q) \cong \Gamma(C_3^2 \times C_q)$. It is also clear that S_3 is unbalanced. By Proposition 3.2, $C_3^2 \times C_q$ is Hamiltonian so that $S_3 \times C_q$ is also Hamiltonian. Then by Theorem 2.26, 2.32, and 2.11 we have that if $2, 3 \nmid n$, then $S_3 \times C_n$ is Hamiltonian if and only if one exponent in the prime factorization of n is odd. Furthermore, Mathematica verifies that $S_4 \times C_5$ and $S_5 \times C_7$ are Hamiltonian. Using Theorems 2.35 and 2.26, we can come to similar conclusions (though they are not bi-conditionals as neither S_4 nor

S_5 is bipartite). To summarize:

Proposition 7.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ in its prime factorization. Then*

i) If $2, 3 \nmid n$, then $S_3 \times C_n$ is Hamiltonian if and only if there exists an $i = 1, 2, \dots, k$ such that α_i is odd.

ii) If $2, 3 \nmid n$, then $S_4 \times C_n$ is Hamiltonian if there exists an $i = 1, 2, \dots, k$ such that α_i is odd.

iii) If $2, 3, 5 \nmid n$, then $S_5 \times C_n$ is Hamiltonian if there exists an $i = 1, 2, \dots, k$ such that α_i is odd.

all hold.

Our attempts to find similar results for S_6 were unfruitful as the complexity of the Hamiltonian question for these graphs was too great to be processed by Mathematica.

As for the cases where $2 \mid n$ or $3 \mid n$, the Hamiltonicity of many groups of the form $C_n \times S_3$ have been determined by Mathematica and are visible in Table 1 (where $p > 3$ is prime). These where determined by testing the Hamiltonicity of $C_n \times S_3$ for n up to 60 (which proved to be the first n in which Mathematica was unable to determine the Hamiltonicity).

This leads to the following conjectures:

Conjecture 7.5. *$C_{2^\alpha 3^\beta} \times S_3$ is Hamiltonian if and only if both α and β are*

Table 1: Hamiltonicity of $C_n \times S_3$ where $2 \mid n$ or $3 \mid n$

Group	Hamiltonian?	Group	Hamiltonian?
$C_2 \times S_3$	No	$C_{2^2p} \times S_3$	Yes
$C_3 \times S_3$	No	$C_{2^33} \times S_3$	Yes
$C_{2^2} \times S_3$	No	$C_{3^3} \times S_3$	No
$C_{2\cdot3} \times S_3$	Yes	$C_{2\cdot3p} \times S_3$	Yes
$C_{2^3} \times S_3$	No	$C_{2^5} \times S_3$	No
$C_{3^2} \times S_3$	No	$C_{2^23^2} \times S_3$	No
$C_{2p} \times S_3$	Yes	$C_{2^3p} \times S_3$	Yes
$C_{2^23} \times S_3$	No	$C_{2^43} \times S_3$	No
$C_{3p} \times S_3$	Yes	$C_{2p^2} \times S_3$	No
$C_{2^4} \times S_3$	No	$C_{2*3^3} \times S_3$	Yes
$C_{2\cdot3^2} \times S_3$	No		

odd.

Conjecture 7.6. *$C_{2^\alpha 3^\beta p^\eta} \times S_3$ is Hamiltonian if and only if η is odd.*

We can similarly investigate the direct products of cyclic groups and alternating groups. For instance, while Mathematica verifies that A_5 is non-Hamiltonian, it also verifies that $A_5 \times C_7$ is Hamiltonian. While we have already seen in Example 2.15 that A_4 is non-Hamiltonian, Mathematica verifies that $A_4 \times C_{5^\alpha}$ is non-Hamiltonian for $\alpha = 1, 2, 3, 4$. However, $A_4 \times C_{35}$

is Hamiltonian. This leads us to our next result:

Proposition 7.7. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ in its prime factorization. Then*

(i) If $2, 3 \nmid n$, then $A_4 \times C_n$ is Hamiltonian if there exists two distinct $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, k$ such that both α_i and α_j is odd.

(ii) If $2, 3, 5 \nmid n$, then $A_5 \times C_n$ is Hamiltonian if there exists an $i = 1, 2, \dots, k$ such that α_i is odd.

both hold.

Proof. Since $A_4 \times C_{35}$ is Hamiltonian, so too is $A_4 \times C_{pq}$ for distinct primes $p, q > 3$. Then by Corollary 2.27, if there exists two distinct $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, k$ such that both α_i and α_j are odd, $A_4 \times C_n$ is Hamiltonian.

Since $A_5 \times C_7$ is Hamiltonian, so too is $A_5 \times C_p$ for $p > 5$ prime. Then by Theorem 2.26 $A_5 \times C_n$ is Hamiltonian if there exists an $i = 1, 2, \dots, k$ such that α_i is odd. □

Again, Mathematica was unable to determine the Hamiltonicity of the graphs necessary to find similar results for A_6.

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APPENDIX

There are some groups whose Hamiltonicity has already been determined by the following findings not included in the body of this thesis.

Theorem A. *If p is prime, then $C_{p^\alpha} \times C_{p^\beta}$ is non-Hamiltonian for all $\alpha, \beta \in \mathbb{N}$. [3, Theorem 3]*

Theorem B. *All finite p-groups with even exponent are non-Hamiltonian. [3, Theorem 4]*

Theorem C. *All non-abelian p-groups of order p^3 are non-Hamiltonian. [3, Theorem 5]*

Theorem D. *If p is prime, then $C_p^{2k+1} \times C_p \times C_p$ is Hamiltonian for all $k \geq 0$. [3, Theorem 7]*

The following table lists the Hamiltonicity of every finite group up to order 35, barring two groups whose Hamiltonicity could not be determined by Mathematica (a '?' appears on these). Any finding whose justification is listed as "Mathematica" was determined by using GAP to convert the subgroup lattice to a graph and then using Mathematica to determine whether or not the graph was Hamiltonian. The structure descriptions (if useful enough to be provided) and GAP IDs were referenced from [2].

Gap ID	**Structure**	**Hamiltonian?**	**Justification**
(1,1)	1	No	Definition 2.3
(2,1)	C_2	No	Theorem 2.37
(3,1)	C_3	No	Theorem 2.37
(4,1)	C_4	No	Theorem B
(4,2)	C_2^2	No	Theorem B
(5,1)	C_5	No	Theorem 2.37
(6,1)	S_3	No	Theorem 4.6
(6,2)	C_6	Yes	Theorem 2.37
(7,1)	C_7	No	Theorem 2.37
(8,1)	C_8	No	Theorem 2.37
(8,2)	$C_2 \times C_4$	No	Theorem A
(8,3)	D_8	No	Theorem 4.6
(8,4)	Q_8	No	Theorem 5.4
(8,5)	C_2^3	Yes	Theorem D
(9,1)	C_9	No	Theorem B
(9,2)	C_3^2	No	Theorem B
(10,1)	D_{10}	No	Theorem 4.6
(10,2)	C_{10}	Yes	Theorem 2.37
(11,1)	C_{11}	No	Theorem 2.37
(12,1)	Dic_3	No	Theorem 5.4
(12,2)	C_{12}	Yes	Theorem 2.37

Gap ID	Structure	Hamiltonian?	Justification
(12,3)	A_4	No	Example 2.15
(12,4)	D_{12}	No	Theorem 4.6
(12,5)	$C_2^2 \times C_3$	Yes	Proposition 3.2
(13,1)	C_{13}	No	Theorem 2.37
(14,1)	D_{14}	No	Theorem 4.6
(14,2)	C_{14}	Yes	Theorem 2.37
(15,1)	C_{15}	Yes	Theorem 2.37
(16,1-14)	-	No	Theorem B
(17,1)	C_{17}	No	Theorem 2.37
(18,1)	D_{18}	No	Theorem 4.6
(18,2)	C_{18}	Yes	Theorem 2.37
(18,3)	$C_3 \times S_3$	No	Mathematica
(18,4)	$C_3^2 \rtimes C_2$	Yes	Theorem 7.3
(18,5)	$C_2 \times C_3^2$	Yes	Proposition 3.2
(19,1)	C_{19}	No	Theorem 2.37
(20,1)	Dic_5	No	Theorem 5.4
(20,2)	C_{20}	Yes	Theorem 2.37
(20,3)	$C_5 \rtimes C_4$	No	Mathematica
(20,4)	D_{20}	No	Theorem 4.6
(20,5)	$C_2^2 \times C_5$	Yes	Proposition 3.2
(21,1)	$C_7 \rtimes C_3$	No	Theorem 2.14

Gap ID	**Structure**	**Hamiltonian?**	**Justification**
(21,2)	C_{21}	Yes	Theorem 2.37
(22,1)	D_{22}	No	Theorem 4.6
(22,2)	C_{22}	Yes	Theorem 2.37
(23,1)	C_{23}	No	Theorem 2.37
(24,1)	$C_3 \rtimes C_8$	No	Theorem 2.14
(24,2)	C_{24}	Yes	Theorem 2.37
(24,3)	$SL(2,3)$	No	Theorem 2.14
(24,4)	Dic_6	No	Theorem 5.4
(24,5)	$C_4 \times S_3$	No	Mathematica
(24,6)	D_{24}	No	Theorem 4.6
(24,7)	$C_2 \times \text{Dic}_3$	No	Mathematica
(24,8)	$(C_2^2 \times C_3) \rtimes C_2$	No	Mathematica
(24,9)	$C_2 \times C_3 \times C_4$	Yes	Mathematica
(24,10)	$C_3 \times D_8$	Yes	Proposition 4.7
(24,11)	$C_3 \times Q_8$	Yes	Proposition 5.5
(24,12)	S_4	No	Mathematica
(24,13)	$C_2 \times A_4$	No	Mathematica
(24,14)	$C_2^2 \times S_3$	No	Mathematica
(24,15)	$C_2^3 \times C_3$	Yes	Theorem D and 2.25
(25,1)	C_{25}	No	Theorem B
(25,2)	C_5^2	No	Theorem B

Gap ID	**Structure**	**Hamiltonian?**	**Justification**
(26,1)	D_{26}	No	Theorem 4.6
(26,2)	C_{26}	Yes	Theorem 2.37
(27,1)	C_{27}	No	Theorem 2.37
(27,2)	$C_3 \times C_9$	No	Theorem A
(27,3)	$C_3^2 \rtimes C_3$	No	Theorem C
(27,4)	$C_9 \rtimes C_3$	No	Theorem C
(27,5)	C_3^3	Yes	Theorem D
(28,1)	Dic_7	No	Theorem 5.4
(28,2)	C_{28}	Yes	Theorem 2.37
(28,3)	D_{28}	No	Theorem 4.6
(28,4)	$C_2^2 \times C_7$	Yes	Proposition 3.2
(29,1)	C_{29}	No	Theorem 2.37
(30,1)	$C_5 \times S_3$	Yes	Proposition 7.4
(30,2)	$C_3 \times D_{10}$	No	Proposition 4.8
(30,3)	D_{30}	No	Theorem 4.6
(30,4)	C_{30}	Yes	Theorem 2.37
(31,1)	C_{31}	No	Theorem 2.37
(32,1)	C_{32}	No	Theorem 2.37
(32,2)	$(C_2 \times C_4) \rtimes C_4$	No	Theorem 2.10 and 2.11
(32,3)	$C_4 \times C_8$	No	Theorem A
(32,4)	$C_8 \rtimes C_4$	No	Theorem 2.14

Gap ID	Structure	Hamiltonian?	Justification
(32,5)	$(C_2 \times C_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,6)	$C_2^3 \rtimes C_4$	No	Theorem 2.10 and 2.11
(32,7)	$(C_8 \rtimes C_2) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,8)	-	No	Theorem 2.10 and 2.11
(32,9)	$(C_2 \times C_8) \rtimes C_2$	No	Theorem 2.14
(32,10)	$Q_8 \rtimes C_4$	No	Theorem 2.14
(32,11)	$C_4^2 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,12)	$C_4 \rtimes C_8$	No	Theorem 2.14
(32,13)	$C_8 \rtimes C_4$	No	Theorem 2.10 and 2.11
(32,14)	$C_8 \rtimes C_4$	No	Theorem 2.10 and 2.11
(32,15)	-	No	Theorem 2.10 and 2.11
(32,16)	$C_2 \times C_{16}$	No	Theorem A
(32,17)	$M_5(2)$	No	Theorem 2.14
(32,18)	D_{32}	No	Theorem 4.6
(32,19)	QD_{32}	No	Mathematica
(32,20)	Q_{32}	No	Theorem 5.4
(32,21)	$C_2 \times C_4^2$	Yes	Mathematica
(32,22)	$C_2 \times (C_2^2 \rtimes C_4)$	No	Theorem 2.10 and 2.11
(32,23)	$C_2 \times (C_4 \rtimes C_4)$	Yes	Mathematica
(32,24)	$C_4^2 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,25)	$C_4 \times D_8$	No	Theorem 2.14

Gap ID	Structure	Hamiltonian?	Justification
(32,26)	$C_4 \times Q_8$	No	Theorem 2.10 and 2.11
(32,27)	$C_2^4 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,28)	$(C_2^2 \times C_4) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,29)	$(C_2 \times Q_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,30)	$(C_2^2 \times C_4) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,31)	$C_4^2 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,32)	-	No	Theorem 2.10 and 2.11
(32,33)	$C_4^2 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,34)	$C_4^2 \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,35)	$C_4 \rtimes Q_8$	No	Theorem 2.10 and 2.11
(32,36)	$C_2^2 \times C_8$	Yes	Theorem D
(32,37)	$C_2 \times M_4(2)$	Yes	Theorem 6.5
(32,38)	$(C_2 \times C_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,39)	$C_2 \times D_{16}$	Yes	Mathematica
(32,40)	$C_2 \times QD_{16}$	Yes	Mathematica
(32,41)	$C_2 \times Q_{16}$	No	Theorem 2.14
(32,42)	$(C_2 \times C_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,43)	$C_8 \rtimes C_2^2$	No	Theorem 2.10 and 2.11
(32,44)	$(C_2 \times Q_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,45)	$C_2^3 \times C_4$	?	-
(32,46)	$C_2^2 \times D_8$	No	Theorem 2.10 and 2.11

Gap ID	Structure	Hamiltonian?	Justification
(32,47)	$C_2^2 \times Q_8$	No	Theorem 2.10 and 2.11
(32,48)	$D_8 \rtimes C_2^2$	No	Theorem 2.10 and 2.11
(32,49)	$D_8 \rtimes C_2^2$	?	-
(32,50)	$(C_2 \times Q_8) \rtimes C_2$	No	Theorem 2.10 and 2.11
(32,51)	C_2^5	Yes	Mathematica
(33,1)	C_{33}	Yes	Theorem 2.37
(34,1)	D_{34}	No	Theorem 4.6
(34,2)	C_{34}	Yes	Theorem 2.37
(35,1)	C_{35}	Yes	Theorem 2.37