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Proper Sum Graphs

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PROPER SUM GRAPHS

A Master's Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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Mathematics

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ABSTRACT

The Proper Sum Graph of a commutative ring with identity has the prime ideals as vertices, with two ideals adjacent if their sum is a proper ideal. This thesis expands upon the research of Dhorajia. We will cover the groundwork to understanding the basics of these graphs, and gradually narrow our efforts into the minimal prime ideals of the ring.

KEYWORDS: proper, sum, graph, ideal, maximal, minimal, comaximal

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I dedicate this thesis to my parents.

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INTRODUCTION

Over the past 30 years, the study of graphs based on algebraic structures has been a growing area of research. It began with Beck creating the zero divisor graph of a ring R, denoted $\Gamma(R)$. This is the graph whose vertices are the nonzero zero divisors of R and edges exist between vertices if their product is 0 in R. Inspired by the work of Mulay [6], Wickham and Spiroff developed the graph of equivalence classes of zero divisors of a ring R, which is created from the classes of zero divisors based on the annihilator ideals, rather than the individual zero divisors [8]. D.F. Anderson and Badawi also introduced the total graph of R, denoted $T(\Gamma(R))$, as the graph with all elements of R as vertices, and for distinct $x, y \in R$, the vertices x and y are adjacent if and only if $x + y \in \mathbb{Z}(R)$ [1]. This can be combined with some work from Sharma and Bhatwadekar in 1995 [1]. This paper established another construction for a graph of a commutative ring R known as the a co-maximal graph. For this graph, vertices are still elements of the ring, but there is an edge between two vertices x and y in R if $xR + yR = R$. This idea would be followed up by Dhorajia in [3]. In this paper he defined the S-proper sum graph as a graph of commutative ring with identity that has the prime ideals as vertices, with two ideals adjacent if their sum is a proper ideal. We explore this graph further in this thesis. We begin with some background results. Then we improve upon some of the results in [3].

GRAPH THEORY BACKGROUND

The following are some graph theory basics and definitions, followed by a more focused discussion on diameter, star graphs, and subgraphs. For more information on any object discussed, please refer to Diestel.

A *graph* $G = (V, E)$ is a pair of sets such that $E \subseteq [V]^2$. That is to say, the elements of our edge set, E, are 2-element subsets of our vertex set, V. A vertex is considered to be *incident* to an edge if $v \in e$, then we have an edge e at the vertex v. Now consider an *isomorphism* between graphs. Let $G = (V, E)$ and $G' = (V', E')$. Then we consider these two graphs to be isomorphic, written $G \cong G'$ if there exists a bijection $\phi : V \to V'$ with $xy \in E \iff \phi(x)\phi(y) \in E'$ for all $x, y \in V$. Consider Figure 1 and Figure 2.

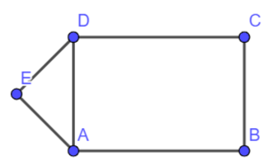

Figure 1: Graph J

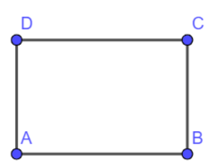

Figure 2: Graph J'

It is readily apparent that $J = (V = \{A, B, C, D, E\}, E = \{AB, BC, CD, DE, EA, DA\})$. Hence, we can say that A is incident to AB, EA, and DA. We can also tell that $J' = (V = \{A, B, C, D\}, E = \{AB, BC, CD, DA\})$. We can also tell that no isomorphism exists between J and J' as there are more vertices in J than in J'.

A *path* is the nonempty graph $P = (V, E)$ where $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where each x_i are distinct. The number of edges of a path is its *length*. The *distance* in G of two vertices x, y, notated as $d_G(x, y)$, is the length of a shortest path from x to y in G, if no such path exists, then $d(x, y) = \infty$. The *diameter* of G is the greatest distance between any two vertices in G, notated as diam$G$.

Example 1. Consider again the graphs J and J'. In Figure 1, a path exists from E to B of length 4, $EA * AD * DC * CB$. However, the distance between E and B is not 4, as a shorter path exists; namely $EA * AB$, which has a length of 2. After completing this process with every other pair of vertices, we can find that the diameter of J is 2. It can be shown that the distance between any two points in J' is at most 2, so its diameter will also be 2.

A *r-partite* (with $r \geq 2$) graph is a graph $G = (V, E)$ such that V admits a partition into r classes such that every edge has its ends in different classes. That is to say, vertices in the same class must not be adjacent. If $r = 2$ we call the graph *bipartite*. The complete r-partite graph is denoted by $K_{n_1,\ldots n_r}$. Graphs of the form $K_{1,n}$ are called *star graphs*. The vertex in the singleton partition class of $K_{1,n}$ is the center of the star, and all star graphs are bipartite. Consider again graph J'. This graph is bipartite, as it can be partitioned into 2 classes, with the first being A, C and the second being B, D. However, graph J is not bipartite. This can be seen in the triangle formed by vertices A, D, and E. As all three points share edges, they are all adjacent to each other. As such, the graph cannot be bipartite. Now consider Figure 3, which is a new graph that we will name $S = K_{1,6}$.

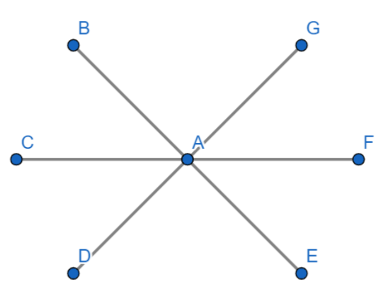

Figure 3: Graph of $S = K_{1,6}$

This is an example of a star graph. Hence A is the center of the star. It is easy to see this graph is bipartite, where A is in a class by itself, and all the other vertices are in a single class. It is important to note that while our example is finite, that is not always the case. It is possible to have the star graph $K_{1,\infty}$.

Consider $G = (V, E)$ and $G' = (V', E')$. If $V' \subseteq V$ and $E' \subseteq E$ then G' is a *subgraph* of G, this is written as $G' \subseteq G$. If $G' \subset G$, then we say we have a *proper subgraph.* If $G' \subseteq G$ and G' contains all the edges $xy \in V'$ then G' is an *induced subgraph* of G.

Example 2. Consider again our graphs J and J'. As each vertex of J' can be found in J (but not all vertices of J are in J'), then we know J' is a proper subgraph of J. We can also see that J' is an induced subgraph of J, as all vertices in J' come from J and J' has all the edges from J that are made up solely of vertices in J'.

RING THEORY BACKGROUND

Now let us discuss ring theory, and remember we only consider commutative rings. We need to specifically cover rings, ideals, prime ideal, krull dimension, artinian rings, and then an example: $\mathbb{Z}_{30}$.

A *ring* R is a set together with two binary operations $+$ and $\times$ (called addition and multiplication) satisfying the following axioms:

1. $(R, +)$ is an abelian group.
2. $\times$ is associative.
3. The distributive law holds in R.

Example 3. Let us consider the integers, $\mathbb{Z}$. This common set of numbers is actually a ring. We will go ahead and take for granted that $(\mathbb{Z}, +)$ is an abelian group. We also know that for any $a, b, c \in \mathbb{Z}, (a \times b) \times c = a \times (b \times c)$. Lastly we know that for any $a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$. Thus we have satisfied the requirements to be a ring.

In general, a ring does not need to be commutative. That is, for $a, b \in R, ab = ba$. For this paper, we will only consider rings that are commutative.

Example 4. We can use a similar argument to conclude that also $\mathbb{Q}$ and $\mathbb{R}$ are also rings.

Now that we some basic examples of a ring, let us jump into a slightly more complex example, polynomial rings. Let R be a commutative ring with identity. The sum,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $n \geq 0$ and each $a_i \in R$ is called a polynomial in x with coefficients a_i in R. The set of all such polynomials that can be written as above is called the ring of polynomials in the variable x with coefficients in R and is denoted $R[x]$. Addition is defined component wise. That is to say, for $(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) + (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_0 + b_0)$. Multiplication is defined by $(ax^i)(bx^j) = abx^{i+j}$ for polynomials with only one nonzero term, and then extended to all polynomials by distributive laws. It then becomes clear that this a ring when using the above definitions of addition and multiplication. It is also important that the ring R is contained in $R[x]$, as it is the constant polynomials (all $a_n = 0$ for $n > 0$).

A *subring* of the ring R is a subgroup of R that is closed under multiplication.Now let I be a subset of R and let $r \in R$.

1. $rI = \{ra|a \in I\}$ and $Ir = \{ar|a \in I\}$.
2. A subset I of R is an *ideal* of R if
   (a) I is a subring of R, and
   (b) I is closed under multiplication by elements from R, i.e. $rI \subseteq I$ for all $r \in R$.

Theorem 5. Let R be a ring, and let I be an ideal of R. The correspondence $A \leftrightarrow A/I$ is an inclusion preserving bijection between the set of subrings A of R that contain I and the set if subrings of R/I. Furthermore, A (a subring containing I) is an ideal of R if and only if A/I is an ideal of R/I. This is known as the Fourth or Lattice Isomorphism Theorem for Rings.

We also have other types of rings known as integral domains, *Principal Ideal Domains* (PID), and *fields*. An integral domain is a commutative ring with identity $1 \neq 0$ that has no zero divisors, an element such that there exists a nonzero x in R such that $ax = 0$. A principal ideal domain is an integral domain where every ideal is principal, which means every proper ideal can be generated by a single element. A field is a commutative ring R with identity not equal to 1 where every nonzero element has a multiplicative inverse. That is to say for every nonzero element $b \in R$, there exist $a \in R$ such that $ab = ba = 1$.

Lemma 6. A ring R is a field if and only if its only ideals are 0 and R.

Proof. If R is a field, then every nonzero ideal of R contains some unit, and so the only nonzero ideal of R must be R itself. Conversely, if R is a the only nonzero ideal of R, then for any nonzero $u \in R$, we have that $(u) = R$. Hence, $1 \in (u)$, and so we know there must be a $v \in R$ such that $uv = 1$. Thus every nonzero element of R is a unit, and hence R is a field. $\square$

Lemma 7. If M is an ideal of a ring R, then M is a maximal ideal if and only if the ring R/M is a field.

Proof. By definition, M is a maximal ideal if and only if there are no proper ideals, I, that contain M. Then by the Fourth Isomorphism Theorem, we have that the ideals of R that contain M correspond bijectively with the ideals of R/M. If R/M is a field, then the only ideals of R/M are (0) and R/M by the previous lemma. Hence M is maximal exactly when R/M is a field. $\square$

It is important to note that usually, an ring does not need to be commutative or even have an identity, but for our purposes we will assume the our ring has both qualities. Assume R is commutative. An ideal P is called a *prime ideal* if $P \neq R$ and whenever the product ab of two elements $a, b \in R$ is an element of P, then at least one of a and b is an element of P. A minimal prime ideal is a prime ideal that does not strictly contain another prime idea. The set of minimal prime ideals of a ring R will be denoted $Min(R)$. Using this definition, we get the following theorem.

Theorem 8. Every prime ideal of a commutative ring contains a minimal prime ideal.

Proof. Let P be a prime ideal of R. Let A be the set of all prime ideals contained in P. Since $P \subseteq P$, A is nonempty. Now let $p \supseteq p_1 \supseteq p_2 \supseteq p_3 \ldots$ be a chain in A. Let $Q = \cap p_i$. Suppose $ab \in Q$, but $a \notin Q$ and $b \notin Q$. Then $a \notin p_i$ for some i and $b \notin p_j$ for some j. Without loss of generality, assume $p_i \subseteq p_j$. Then $b \in p_i$. So we have $ab \in p_i$ with $a \in p_i$ and $b \notin p_i$, contradicting that p_i is prime. So we must have that either $a \in Q$ or $b \in Q$, hence Q is prime. By Zorn's Lemma, A has a minimal element, thus it contains a minimal prime ideal. $\square$

In contrast to the idea of a minimal prime, an ideal M in an arbitrary ring S is called a maximal ideal if $M \neq S$ and the only ideals containing M are M and S. Now that we know what a maximal ideal is, we can use *Krull's Theorem.*

Theorem 9. Every proper ideal of a ring is contained in some maximal ideal.

For any commutative ring R, the *Krull dimension* (also simply referred to as the *dimension*) of R is the maximum possible length of a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of distinct primes ideals in R. For example consider the ring of integers, $\mathbb{Z}$. This ring has dimension 1, as any prime ideal will only contain the zero ideal. We can also consider that all fields are of dimension zero. This is because a field only has one ideal, namely the (0) ideal. We can also use the fact that if the Krull dimension of a ring R is m, then the Krull dimension of $R[x] = m + 1$.

A commutative ring R is said to be *Artinian* or to satisfy the *descending chain condition on ideals* (or *D.C.C on ideals*) if there is no infinite decreasing chain of ideals in R, i.e., whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a decreasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$.

Theorem 10. The Chinese Remainder Theorem states that if we let $I_1, I_2, \dots, I_k$ are ideals in a ring R then the map $R \to R/I_1 \times R/I_2 \times \dots R/I_k$ defined by $r \to (r + I_1, I_2, \dots, r + I_k)$ is a ring homomorphism with the kernel being the intersection of all the ideals. If for each pair ideals, J and K, we have that $J + K = R$, then the map is surjective and $I_1 \cap I_2 \cap \cdots \cap I_k = I_1 I_2 I_3 \dots I_k$. Hence we have that $R/(I_1 I_2 \dots I_3) \cong R/I_1 \times R/I_2 \times \dots R/I_k$.This gives us an incredibly hand corollary; that is m, n are relatively prime, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$.

Now we can talk about an S-proper sum graph. To define this, let us consider a ring R, still with our parameters from earlier.

Definition 11. $Spec(R)$ is the set of all prime ideals of R. $Max(R)$ is the set of all maximal ideals of R.

Definition 12. Let S be a nonempty subset of $Max(R)$. Then we define the S-proper sum graph on the set of prime ideals of R, denoted as $\Gamma_S(Spec(R), S)$, as an undirected graph whose vertex set is the set of all prime ideals and two distinct vertices, P and Q are connected if $P + Q \subseteq M$, where M is some maximal ideal in S. We denote the connection by $P \sim S$. If $S = Max(R)$, we denote our S-proper sum graph $\Gamma_S(Spec(R))$ as $\Gamma(Spec(R))$ and we call it the proper sum graph.

Definition 13. A *reduced ring* is a ring R that has no non-zero nilpotent elements (an element x such that there exists $a \in R$ such that $x^a = 0$). .

Remark 14. For this paper, we will be assume the ring is reduced, as we will show later that every proper sum graph is isomorphic to a proper sum graph that is reduced.

Example 15. Let us consider the ring $\mathbb{Z}$. We know that $Spec(\mathbb{Z}) = \{0, (p) | p \text{ is a prime}\}$. We also can show that $(0) + (p) = (p)$. This tells us that $0 \sim (p)$. But what about the case of p and q being two distinct primes? Well if p and q are distinct, then $gcd(p, q) = 1$, as they are primes. Then, by the Euclidean Algorithm, we have $ap + bq = 1$. This implies implies $(p) + (q) = \mathbb{Z}$. Thus (p) and (q) are not connected. To graph this would give us Figure 4. Note, there will be infinite p's.

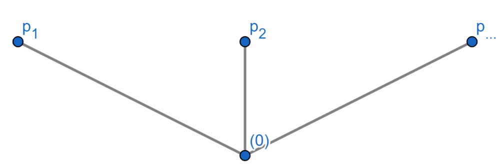

Figure 4: Proper Sum Graph of $\mathbb{Z}$

Now remember our definition of the dimension of a ring. Our dimension is the length of the longest chain $P_0 \subset P_1 \subset ... \subset P_n$, where P_i is a prime ideal. In our example above, it is apparent that $dim(\mathbb{Z}) = 1$ as we only have the chain $(0) \subset (p)$, where p is a prime.

Theorem 16. Let $R = \mathbb{Z}_{p^k}[x]$, where p is a prime integer and $k \in \mathbb{Z}^+$. Then (p) is a prime ideal of R and all other prime ideals of R are of the form $(p, \overline{f(x)})$, where the image $\overline{f(x)}$ of $f(x)$ in $\mathbb{Z}_p[x]$ is irreducible.

Proof. Since $R/(p) \cong \mathbb{Z}_{p^k}[x]$ is an integral domain, (p) is a prime ideal.Let $P \neq (p)$ be a prime ideal of R.

In $\mathbb{Z}_{p^k}$, elements are either multiples of p or units, and as P is a proper ideal, P must contain a polynomial of positive degree. Let $f(x) = a_n x^n + (a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$ be a polynomial of minimal positive degree in $P - (p)$. If a_n is a multiple of p, then $a_n x^n \in (p) \subset P$ and therefore $f_1(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in P$. By choice of $f(x)$, this would imply $n - 1 = 0$ and hence $f_1(x) = a_0$. But then $a_0 \in (p)$ implying $f(x) \in (p)$, a contradiction. Thus a_n is a unit and without loss of generality we may assume $f(x)$ is monic.

Choose any $g(x) \in P$. Since $f(x)$ is monic, we may use the division algorithm to write $g(x) = f(x)q(x) + r(x)$ with $r(x) = 0$ or deg $r(x) <$ deg $f(x)$. By choice of $f(x)$, this forces $r(x) \in (p)$. We thus have that $g(x) \in (f(x), p)$ and hence $P = (f(x), p)$.

Now $R/P = \mathbb{Z}_{p^k}[x]/(f(x), p) \cong \mathbb{Z}_p[x]/(\overline{f(x)})$, where $\overline{f(x)}$ is the image of $f(x)$ in $\mathbb{Z}_p[x]$. As this quotient ring is a domain, it follows that $(\overline{f(x)})$ is a prime ideal of $\mathbb{Z}_p[x]$. Since $\mathbb{Z}_p$ is a field, all prime ideals of $\mathbb{Z}_p[x]$ are maximal and thus $\overline{f(x)}$ is irreducible in $\mathbb{Z}_p[x]$.

Thus (p) is a prime ideal of R and all other primes ideals of R are of the form $(p, \overline{f(x)})$, where the image $\overline{f(x)}$ of $f(x)$ in $\mathbb{Z}_p[x]$ is irreducible. □

Example 17. Consider the ring $R = \mathbb{Z}_9[x]$. We can rewrite this as $R = \mathbb{Z}_{3^2}[x]$. Then (3) is a prime ideal of R. All other ideals of the ring are of the form $(3, \overline{f(x)})$, where the image $\overline{f(x)}$ of $f(x)$ in $\mathbb{Z}_3[x]$ is irreducible, for example $(3, x^2 + 2x + 2)$.

Now that we have a basic understanding of proper sum graphs, let us take a look at some definitions that will allow us to go further into our study.

Theorem 18. Let R be a PID. Then the prime ideals of $R[y]$ are precisely the zero ideal (0), the ideals $(f(y))$ where $f(y)$ is irreducible in $R[y]$, and the maximal ideals $(p, f(y))$ where $p \in R$ is prime and $f(y)$ is irreducible in $(R/p)[y]$.

Proof. Let P be a prime ideal of $R[y]$. Then $P \cap R$ is a prime ideal of R. Since R is a PID, we have that $P \cap R = (p)$ for some $p \in R$. Since $(p) \subseteq P$, we know that the image of P in $R[y]/pR[y] \cong (R/(p))[y]$ is a prime ideal of $(R/(p))[y]$. If $pR[y] = P$, then P must be a principal ideal of $R[y]$ generated by p and we can write $P = (p)$. Now assume that we have proper containment, that is that $pR[y] \subset P$ and so the image of P in $(R/(p))[y]$ is nonzero.

Suppose $p \neq 0$. Recall that in a principal ideal domain, every nonzero prime ideal is a maximal ideal. It follows then that $R/(p)$ is a field, and that $(R/(p))[y]$ is a polynomial ring over a field. Since nonzero prime ideals of $(R/(p))[y]$ are precisely the principal ideals generated by the irreducible polynomials, the image of P in $(R/(p))[y]$ is principal with an irreducible generator. If $f(y)$ is an irreducible pre-image of the generator, then $P = (p, f(y))$ and the quotient of $R[y]$ by P is a finite extension of $R/(p)$. So $P = (p, f(y))$ is a maximal ideal and this describes completely the set of prime ideals of $R[y]$ containing some nonzero prime ideal of R.

Now suppose that $p = 0$, so that P contains no nonzero element of R. Recall that if R is a UFD, then so is $R[y]$. It follows that if P is principal, then $P = (f(y))$ where $f(y)$ is irreducible. Now suppose that P is not principal and let $f(y)$ and $g(y)$ be two irreducible generators of P. By Gauss' Lemma, two polynomials of $R[y]$ with no common factor in $R[y]$ also have no common factor in $K[y]$, where K is the fraction field of R. But we also know that there exist $a(y), b(y) \in K[y]$ such that $a(y)f(y) + b(y)g(y) = 1$, thus clearing denominators we obtain a nonzero element of R in P, a contradiction. $\square$

Example 19. The non maximal prime ideals of $\mathbb{Z}[y]$ are (0) and $(f(y))$ where $f(y) \in \mathbb{Z}[y]$ is irreducible. It is important to note that any prime number $p \in \mathbb{Z}$ is an irreducible constant polynomial in $\mathbb{Z}[y]$ so (p) is a prime ideal of $\mathbb{Z}[y]$. The maximal ideals of $\mathbb{Z}[y]$ are of the form $(p, f(y))$ where p is a prime number and $f(y)$ is irreducible in $\mathbb{Z}_p[y]$.

Example 20. Let F be a field. We can view $F[x, y]$ as $R[y]$ where $R = F[x]$ and apply Theorem 23. Since $R = F[x]$ is a PID, the prime elements are precisely the irreducible polynomials. So the non maximal prime ideals of $F[x, y]$ are (0) and $(f(x, y))$ where $f(x, y) \in F[x, y]$ is irreducible. The maximal ideals of $F[x, y]$ are of the form $(g(x), f(x, y))$ where $g(x)$ is irreducible in $F[x]$ and $f(x, y)$ is irreducible in $(F[x]/g(x))[y]$.

Now let us discuss more on our Artinian Rings. Consider a ring that can be written as $\mathbb{Z}_p[x]$. The prime ideals are (p) and $p = (p, f(x))$ where $f(x)$ in $\mathbb{Z}_p[x]$ is irreducible. We can also see that as $p + (p, f(x)) = (p, f(x))$, where $f(x)$ still has the qualities mentioned above.

Example 21. Consider a ring $R[x]$ where R has only one prime ideal. For example, $\mathbb{C}[x, y]$. The prime ideals of this ring are $(0), (f(x, y))$ where $f(x, y)$ is a prime element (and thus irreducible), and $(x - a, y - b)$ where $a, b \in \mathbb{C}$, as these will be maximal ideals.

Example 22. Let $R = \frac{\mathbb{C}[x,y]}{xy}$. Then the prime ideals of R correspond to the prime ideals of $\mathbb{C}[x, y]$ that contain (xy). So if we know $(xy) \subset (f(x, y))$ then $xy = f(x, y)g(x, y) \in \mathbb{C}[x, y]$, since the ring is a unique factorization domain. Thus, we have that $f(x, y) = x$ or y. Then both (x) and (y) are prime ideals of R. Then we can also consider the case where $(xy) \subset (x - a, y - b)$, so that we have $a = 0$ or $b = 0$. In other words, we now have that $(x, y - b)$ and $(x - a, y)$ are maximal ideals. Hence, that will give us the following graph on Figure 5. Note that there are infinite vertices of the form $(x, y - b)$ and $(x - a, y)$, though they are only listed once each on the graph.

Definition 23. The *Nilradical* of a set, R is the set of all nilpotent elements. Denoted by N_R.

Theorem 24. $N_R = \cap p$, for all p where p is a prime ideal of of the ring R. This could

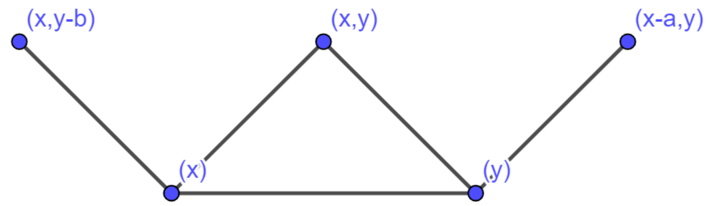

Figure 5: Graph of $R = \frac{\mathbb{C}[x,y]}{xy}$

also be written as $N_R = \cap m$ for all m where m is a minimal prime ideal.

Example 25. Consider the ring $\mathbb{Z}_6[y] \cong \mathbb{Z}[y]/(6)$.

The prime ideals of $\mathbb{Z}[y]$ are:

- (0)
- $(f(y))$ where f is irreducible
- $(p, f(y))$ where p is prime and image of f is irreducible over $\mathbb{Z}_p$

Thus for our ring, $(6) \subset (f(y))$. Then $6 = f(y)g(y)$ in $\mathbb{Z}[y]$. Hence $f(y) = 2$ or 3. Thus, we know that the minimal prime ideals are (2) and (3). Our maximal ideals would be of the form $(2, f(y))$ and $(3, f(y))$ where $f(y)$ is irreducible over $\mathbb{Z}_p$. As there are infinite choices for $f(y)$, there will be infinite vertices of these forms, though we only list one of each in our graph. Figure 6 is the proper sum graph of this example.

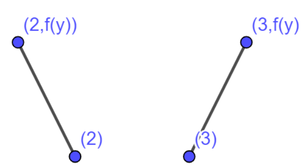

Figure 6: Graph of $\mathbb{Z}_6[y]$

Theorem 26. If a ring R has $\dim(R) = 1$ and a unique minimal prime ideal, then $\Gamma(R)$ is a star graph.

Proof. R has $\dim(R) = 1$ and a unique minimal prime ideal. Then the unique minimal prime is the Nilradical, and is contained in every other prime ideal. We also have that $\dim(R) = 1$, hence every non minimal prime ideal is a maximal prime ideal. As

$M_1 + M_2 = R$, for any two maximal prime ideals M_1, M_2, we have that $M_1 \not\sim M_2$. Hence we have a star graph. $\square$

Lemma 27. A Principle Ideal Domain that is not a field has dimension 1.

Definition 28. For any subset A of R define $V(A) = \{P \in X | A \subseteq P\} \subseteq Spec(R)$ the collection of prime ideals containing A. It is immediate that $V(A) = V(I)$, where $I = (A)$ is an ideal generated by A so there is no loss simply in considering $V(I)$ where I is an ideal if R.

Lemma 29. If R is a ring and I is a ideal of R, then $V(I) = \emptyset$ if and only if $I = R$.

Proof. As we know every prime ideal is properly contained, we know that no prime ideal can contain R. Thus $V(I) = \emptyset$. Now, by way of contradiction, consider if for some ideal I we had that $V(I) = \emptyset$. If I was proper, then it must be contained in some maximal ideal M of R, so $V(I)$ could not be empty. But since every maximal ideal is prime, then $M \in V(I)$. But we assumed that $V(I) = \emptyset$, so we have a contradiction as I cannot be a proper ideal of R and thus $I = R$. $\square$

Lemma 30. For any ideals I, J of R, $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

Proof. As $IJ \subset J$ and $IJ \subset I$, we have that $V(I) \cup V(J) \subset V(IJ)$.

Now let p be a prime ideal containing IJ. Then p contains either I or J, by definition of a prime ideal. Hence $V(I) \cup V(J) \supset V(IJ)$ $\square$

Lemma 31. If $\{I_j\}$ is an arbitrary collection of ideals of R, then $V(\cup I_j) = \cap V(I_j)$.

Proof. Let $K \in V(\cup I_j)$. Then $K \supseteq I_j$ for each j. Hence $K \in V(I_j)$ for each j. Therefore $K \in \cap V(I_j)$.

Now let $K \in \cap V(I_j)$. Then $K \in V(I_j)$ for each j. Hence $K \supseteq I_j$ for each j. Thus $K \in V(\cup I_j)$. $\square$

Lemma 32. For a reduced ring R and the ideal K of R, $Spec(R) = V(K)$ if and only if $K = 0$.

Proof. Let $K = 0$. Then $V(K) = V(0) = Spec(R)$.

Now let $Spec(R) = V(K)$. Then since we are in an integral domain, $0 \in Spec(R)$. Since $O \in V(K)$, $K \subseteq 0$. Hence $0 = K$. $\square$

Lemma 33. *$V(K) = \emptyset$ if and only if $K = R$ for an ideal K of a ring R.*

Proof. Let $V(K) = \emptyset$. Then there are no proper ideals that contain K. Hence K must be the entire ring. Let $K = R$. Then $V(K) = V(R)$. As no proper ideals of a ring contain the whole ring, $V(K) = \emptyset$. $\square$

Now we can see that, for a ring R, $V(R) = \emptyset$ and $V(0) = Spec(R)$.

Definition 34. A *topology (on a set)* is any set X together with a collection of subsets T of X, called the *closed sets* in X, satisfying the following axioms:

- An arbitrary intersection of closed sets is closed: if $S_i \in T$ for i in any index set, then $\cap S_i \in T$,
- A finite union of closed sets is closed: if $S_1, \dots S_q \in T$ then $S_1 \cup \dots \cup S_q \in T$, and
- The empty set and the whole space are closed: $\emptyset, X \in T$.

Definition 35. The topology on $Spec(R)$ defined by the closed sets $V(I)$ for the ideals I of R is called the *Zariski topology* in $Spec(R)$.

Definition 36. Two topological spaces X and Y are *homeomorphic* if there is a continuous bijection $f : X \to Y$ with a continuous inverse.

Lemma 37. *$Spec(R) = \bigcup_{p \in Min(R)} V(p)$.*

Theorem 38. *Let P and Q be prime ideals of a ring R. Then $V(P) \cap V(Q) \neq \emptyset$ if and only if $P \sim Q$.*

Proof. Let $V(P) \cap V(Q) \neq \emptyset$. Then we have that $V(P) \cap V(Q) \neq M$, for some maximal ideal, M in R. Then $V(P) \subset M$ and $V(Q) \subset M$. Then there is an ideal that contains both P and Q. Hence $P \sim Q$.

Now let $P \sim Q$. Then there is a prime ideal such that $P + Q = M$, for some maximal ideal M in R. Then there is an ideal that contains both P and Q. Hence $V(P) \cap V(Q) \neq \emptyset$. □

Theorem 39. For a ring R, where Γ (Min(R)) is connected and $|\text{Min}(R)| < \infty$, $\text{diam}(\Gamma(R)) \leq \text{diam}(\text{Min}(R)) + 2$.

Proof. Let $\text{diam}(\Gamma(R))$ $=\max\{d(P,Q)|P,Q \in Spec(R)\}$. For this to be a maximum distance, we need P and Q to be non-maximal prime ideals. Then each must contain a minimal prime ideal, P_1 and Q_1 (respectively). Then there exist a path from P_1 to Q_1 of length $\leq \text{diam}(\text{Min}(R))$. Hence $d(P,Q) \leq \text{diam}(\text{Min}(R)) + 2$. So $\text{diam}(\Gamma(R)) \leq \text{diam}(\text{Min}(R)) + 2$. □

Compare this to Dhorajia's proposition from earlier, where he states that given our initial conditions, $\text{diam}(\Gamma(Spec(R), S)) \leq 2|\text{Min}(R)|$[3].

Lemma 40. Let R be a ring, not necessarily reduced. Recall that N_R is the nilradical of a ring R. Then we have the following:

- $N_R = \cap p = \cap m$, for all p where p is a prime ideal of the R, for all m where m is a minimal prime ideal.
- $N_R \subseteq p$ for all prime ideals, then $N_R = \{x \in R | x^n = 0 \text{ for some } n \in \mathbb{Z}^+\}$.
- $Spec(R)$ is homeomorphic to $Spec(R/N)$.
- $\Gamma(R) \cong \Gamma(R/N)$.
- (R/N) is reduced, so every proper sum graph is isomorphic to a proper sum graph that is reduced.

Corollary 41. A reduced ring R with a unique minimal prime ideal is a domain.

Definition 42. $Spec(R)$ is *connected* if it cannot be written as $Spec(R) = V(I) \cup V(J)$ with $V(I) \cap V(J) = \emptyset$

Fact 43. Every prime ideal P contains a minimal prime p, that is, $P \in V(p)$, for some minimal prime p. Thus it follows that $Spec(R) = \bigcup_{p \in Min(R)} V(p)$.

Fact 44. If $P \in V(p)$, then $p \subseteq P$ so $p + P = P \neq R$, and thus in a proper sum graph, $p \sim P$ in $\Gamma(R)$.

Lemma 45. Let R and S be rings. Then $\Gamma(Spec(R \times S))$ and $\Gamma(\mathrm{Min}(R \times S)$ are both disconnected.

Proof. First, we need to show that the ideals of $R \times S$ are of the form $I \times J$ where I is an ideal of R and J is an ideal of S. Then we need to show that $\frac{R\times S}{I\times J} = R/I \times S/J$.

Any nontrivial direct product of rings contain zero divisors, so the product cannot be an integral domain. Thus $I \times J$ is a prime ideal ideal if and only if either I is prime and $J = S$ or $I = R$ and J is a prime ideal.

So let p be a minimal prime ideal of R and q a minimal prime ideal of S. Then $p \times S$ and $R \times q$ are minimal primes of $R \times S$, and all minimal primes of $R \times S$ are of this form. But, $(R \times q) + (p \times S) = R \times S$ for all minimal primes $R \times q$ and $p \times S$. So this implies that no vertex of the form $R \times q$ is connected to a vertex of the form $p \times S$ in $\Gamma(\mathrm{Min}(R \times S))$ and thus $\Gamma(\mathrm{Min}(R \times S))$ is disconnected. $\square$

For the following theorem and its corollaries, we will be assuming that the ring R has finitely many minimal prime ideals.

Theorem 46. *$Spec(R)$ is connected as a topological space if and only if $\Gamma(\mathrm{Min}(R))$ is a connected subgraph of $\Gamma(R)$.*

Proof. Suppose $Spec(R)$ is not connected in the Zariski topology, but $\Gamma(\mathrm{Min}(R))$ is. Then $Spec(R) = V(I) \cup V(J)$ for some ideals I, J where $V(I) \cap V(J) = \emptyset$.

We have that $V(I) \cap V(J) = V(I + J)$ and $V(I) \cup V(J) = V(IJ)$ for prime ideals I, J in a ring R from Lemma 29 and Lemma 30.

Thus we now know that $Spec(R) = V(IJ)$ and $V(I + J) = \emptyset$. Now consider Lemma 31 and Lemma 32, which gives us that $Spec(R) = V(K)$ if and only if $K = 0$ and that $V(K) = \emptyset$ if and only if $K = R$.

Thus we have that $IJ = 0$ and $I + J = R$. It is important to note that $I + J = R$ implies that $IJ = I \cap J$. Hence, we can now use the Chinese Remainder Theorem by creating the isomorphism $R \cong R/I \times R/J$. This implies that $\Gamma(\mathrm{Min}(R))$ is not connected, which contradicts our assumption.

Suppose $\Gamma(\mathrm{Min}(R))$ is not connected and let $G_1, G_2, \ldots, G_K$ be the connected components of $\Gamma(\mathrm{Min}(R))$. Each G_i is a set of minimal primes $G_i = \{p_{1i}, p_{2i}, \ldots, p_{m_i i}\}$. Note that if $i \neq j$ then $V(p_{si}) \cap V(p_{tj}) = \emptyset$ since p_{si} is not adjacent to any p_{tj}. But then $(\bigcup_{1 \leq s \leq m_i} V(p_{si})) \bigcap (\bigcup_{1 \leq t \leq m_j} V(p_{tj})) = \emptyset$ and hence $Spec(R) = \bigcup_{1 \leq j \leq k} (\bigcup_{1 \leq s \leq m_i} V(p_{si}))$ is a disjoint union. This contradicts that $Spec(R)$ is connected. $\square$

Compare this to Dhorajia's Proposition 2.14, where he states that $Spec(R)$ is connected if and only if $\Gamma(Spec(R), S)$ is connected[3].

CONCLUSION

In this thesis, we explored proper sum graphs. We investigated multiple examples of different rings and theorems helping to generalize results. We also took notice that the minimal prime ideals of a ring can be used to evaluate if the proper sum graph is connected. Finally, we were able to outline our results in comparison to Dhorajia. Specifically we were able to strengthen $Spec(R)$ is connected if and only if $\Gamma(Spec(R), S)$ is connected to $Spec(R)$ is connected as a topological space if and only if $\Gamma(\mathrm{Min}(R))$ is a connected subgraph of $\Gamma(R)$[3]. We also were able to strengthen $\mathrm{diam}(\Gamma(Spec(R), S)) \leq 2|\mathrm{Min}(R)|$ to $\mathrm{diam}(\Gamma(R)) \leq \mathrm{diam}(\mathrm{Min}(R)) + 2$. [3]

REFERENCES

[1] D. F. Anderson and A. Badawi, The total graph of a commutative ring, J. Algebra 320 (2008), 2706–2719.

[2] I. Beck, Coloring of commutative rings. J. Algebra, 116 (1988), 208–226.

[3] A. Dhorajia, Some graph on prime ideals of a commutative ring. International Electronic Journal of Algebra 23 (2018), 157-166. 10.24330/ieja.373659.

[4] R. Diestel, *Graph Theory* Springer-Verlag Heidelberg, New York, 2005.

[5] D. S. Dummit and R. M. Foote, *Abstract Algebra* John Wiley & Sons, Inc., Reading, USA, 2004.

[6] S. B. Mulay, Cycles and symmetries of zero-divisors. Comm. Algebra 30 (2002), 3533–3558.

[7] P. K. Sharma and S. M. Bhatwadekar, A note on graphical representation of rings, J. Algebra, 176(1) (1995), 124-127.

[8] S. Spiroff & C. Wickham, A Zero Divisor Graph Determined by Equivalence Classes of Zero Divisors, Communications in Algebra, 39:7 (2011), 2338-2348