Fall 2021

Stability Theory of Nonlinear Differential Equations

Jiaxiao Wei
Missouri State University, jw9938s@MissouriState.edu

STABILITY THEORY OF NONLINEAR DIFFERENTIAL EQUATIONS

A Master's Thesis

Presented to

The Graduate College of

Missouri State University

In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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December 2021

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ABSTRACT

Nonlinear differential equations are often effective tools in modeling some important phenomena in nature. However, most of the nonlinear ordinary differential equation cannot be solved by analytical methods. A more effective way is to explore the properties of critical point and the trajectory around it. In this study, I will focus on systems of autonomous differential equations, linear as well as nonlinear. I will not only focus on the stability of equilibria but also the orbital stability of nonlinear differential equations. I will introduce various approaches to the study of equilibrium points of the system in terms of their stability properties, illustrated by some applied models. The theory and applications show the great power of mathematical analysis.

KEYWORDS: stability, nonlinear differential equations, critical points, linearization, Liapunov function, limit circle.

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Approved:

Shouchuan Hu, Ph.D., Thesis Committee Chair

William Bray, Ph.D., Committee Member

Songfeng Zheng, Ph.D., Committee Member

Julie J. Masterson, Ph.D., Graduate College Dean

ACKNOWLEDGEMENTS

First and foremost, I would like to give my sincere gratitude to my distinguished and respectable supervisor, Dr.Hu, for his supports during not only the research process, but the whole academic year. He is the one who gives me inspiration of the thesis topic, and instructs me in analyzing the problems and taking the right approaches.

Secondly, I also express my sincere gratitude to other professor and teachers who guide my learning in the past two years. Without their precious encouragement, I could not have come this far in the writing of the thesis.

Last but not the least, I would like to give my deepest gratitude to my parents, my fiance and friends, who have always supported me spiritually with their generous encouragements and love.

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1. INTRODUCTION

Nonlinear differential equations are often more reflective of the nature of the process, however, most of nonlinear ordinary differential equations cannot be solved by analytical methods. Here comes the problem: how to analyse nonlinear differential equations and their solutions?

The qualitative theory of differential equations goes back to Henri Poincaré in 1880s. Instead of finding the solutions of differential equations, it seeks to study the qualitative properties of the solutions based on the characteristics of the differential equations. Accordingly, it has become an effective way to do research on differential equations. At the same time, Russian mathematician Liapunov researched into the stability of the solutions of differential equations, which is another landmark in the qualitative theory.

Nowadays in the study of the structure of solutions, people not only focus on the stability of solutions of differential equations under initial value conditions with a parameter disturbance, but also the periodic solutions and chaos when stability is destroyed.

In this study, we will focus on systems of autonomous differential equations, linear as well as nonlinear. We will introduce various approaches to the study of equilibrium points of the system in terms of their stability properties, illustrated by applications and interpretations of some applied models. The presentation of the theory and applications show the great power of mathematical analysis.

2. CONCEPT OF STABILITY

Before we get into the topics of stability, let us introduce some basic concepts, including the vector spaces, normed spaces and Banach spaces.

DEFINITION 2.1 (Vector Space) Given a nonempty set V (elements of V are called vectors) and a field $\mathbb{F}$ (elements of $\mathbb{F}$ are called scalars), V is a vector space on which two operations are defined, called vector addition and multiplication by scalars which satisfies the two closure axioms as well as eight vector space axioms :

For any u, v, w, in V and for any scalars α, β in $\mathbb{F}$,

- (Closure under vector addition) $u + v \in V$.
- (Closure under scalar multiplication) $\alpha v \in V$.
- (Commutativity) $u + v = v + u$.
- (Associativity of vector addition) $(u + v) + w = u + (v + w)$.
- (Additive identity) There exists $0 \in V$ such that for any u, $0 + u = u + 0 = u$.
- (Existence of additive inverses) For any u, there exists a $-u$ such that $u+(-u) = 0$.
- (Associativity of scalar multiplication) $\alpha(\beta v) = (\alpha\beta)v$.
- (Distributivity of scalar sums) $(\alpha + \beta)u = \alpha u + \beta u$.
- (Distributivity of vector sums) $\alpha(u + v) = \alpha u + \alpha v$.
- (Scalar multiplication identity) $1u = u$.

DEFINITION 2.2 [5]. A normed vector space consists of an underlying vector space V over a field of scalars (the real or complex numbers), together with a norm $\|\cdot\| : V \to R^+$ that satisfies:

1. $\|v\| \geq 0$ for any $v \in V$.

2. $\|v\| = 0$ if and only if $v = 0$.

3. $\|\alpha v\| = |a|\|v\|$, whenever α is a scalar and $v \in V$.

4. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, m \in V$.

The space V is said to be complete if whenever v_n is a Cauchy sequence in V , that is, $\|v_n - v_m\| \to 0$ as $n, m \to \infty$, then there exists a $v \in V$ such that $\|v_n - v\| \to 0$ as $n \to \infty$.

A complete normed vector space is called a Banach space. Here again, we stress the importance of the fact that Cauchy sequences converge to a limit in the space itself, hence the space is "closed" under limiting operations.

EXAMPLE 2.3 For $1 \leq p < \infty$, the sequence space $l^p(\mathbb{N})$ consists of all infinite sequences $x = (x_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty$$

with the p-norm,

$$\|x\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$$

For $p = \infty$, the sequence space $l^p(\mathbb{N})$ consists of all bounded sequences, with

$$\|x\|_{\infty} = \sup\{|x_n| : n = 1, 2, \cdots\}.$$

Then $l^p(\mathbb{N})$ is an infinite-dimensional Banach space for $1 \leq p \leq \infty$.

There are basically three categories of the stability concept: Laplace stability, Liapunov stability and Poincare stability. In the sense of Laplace, a system is stable if it exhibits only finite motions; i.e., all solutions of the differential equations are bounded as $t \to \infty$. Since Laplace is a boundedness concept of a very general nature and distinguishes variational effects only as to their being finite or infinite, it is rarely useful for qualititative matters in variational problems.

Liapunov stability requires that solutions which are once near together remain near together for future time as functions of the time.

A trajectory Γ is stable in the sense of Poincare, i.e., orbital stability, if neighboring half paths which are once near Γ remain near Γ.

DEFINITION 2.4 (Liapunov Stability) For the system

$$\frac{dx}{dt} = f(x,t), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $f = (f_1, f_2, \cdots, f_n)^T$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$,

(a) we say that an equilibrium solution x_0 of (2.1) is stable in the sense of Liapunov, if for each $\epsilon > 0$, there exists a $\delta > 0$ such that any solution $x = \phi(t)$ of the system (2.1) satisfying

$$\|\phi(0) - x_0\| < \delta$$

must also satisfy

$$\|\phi(t) - x_0\| < \epsilon$$

for all $t \geq 0$.

(b) The critical point x_0 of the above system is said to be asymptotically stable if it is Liapunov stable and there exists a $\delta > 0$ such that if

$$\|\phi(0) - x_0\| < \delta,$$

then

$$\lim_{t \to \infty} \phi(t) = x_0.$$

(c) The solution is unstable if it is not stable in the sense of Liapunov.

3. AUTONOMOUS SYSTEMS

3.1. Autonomous Differential Equations

An autonomous system is a collection of equations that do not explicitly contain the independent variable, which has the form

$$
\begin{aligned}
y_1' &= f_1(y_1, y_2, \cdots, y_n), \\
y_2' &= f_2(y_1, y_2, \cdots, y_n), \\
&\vdots \\
y_n' &= f_n(y_1, y_2, \cdots, y_n),
\end{aligned}
$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$. By choosing $y = (y_1, y_2, \cdots, y_n)^T$, $f = (f_1, f_2, \cdots, f_n)^T$, we have an equivalent vector equation

$$y' = f(y). \tag{3.2}$$

The solutions of $f(y) = 0$ are equilibrium points of equation (3.2). If c is such a point, we say that (3.2) is stable at c if $y = c$ is a stable solution of (3.2). We also refer to a singular point c as a stable (or unstable) equilibrium.

When the variable is time, they are also called time-invariant systems. For the time-invariant system, since differential equations can omit time, that is to say that submitting t to $t-t_0$ makes no difference to the system. Many physical laws can be described as a time-invariant system because most of the physical laws hold true no matter now, in the past or in the future. This explains why trajectories of the autonomous system do not intersect. However, this property cannot be applied to non-autonomous system, which is the key difference for these two systems.

REMARK 3.1 A solution curve of $y' = f(y)$ rigidly moved to the left or right will remain a solution, i.e., the horizontal translation of $y(t - t_0)$ of a solution to $y' = f(y)$ is also a solution.

Proof. Let $y_1(t)$ be a unique solution of initial value problem for an autonomous system

$$\frac{d}{dt}y(t) = f(y(t)), \quad y(0) = y_0. \tag{3.3}$$

Then $y_2(t) = y_1(t - t_0)$ solves

$$\frac{d}{dt}y(t) = f(y(t)), \quad y(t_0) = y_0.$$

Indeed, if we denote $s = t - t_0$, we will have

$$y_1(s) = y_2(t) \quad and \quad ds = dt,$$

thus

$$\frac{d}{dt}y_2(t) = \frac{d}{dt}y_1(s) = f(y_1(s)) = f(y_2(t)).$$

For the initial condition, the verification is trivial,

$$y_2(t_0) = y_1(t_0 - t_0) = y_1(0) = y_0$$

□

3.2. Equilibrium Analysis

A equilibrium solution, also called critical point, is any constant function $y(t) \equiv c$ that is a solution to the system differential equations, where $c = (c_1, c_2, \cdots, c_n)^T$ is a constant vector. So equilibrium points are the solutions which can be found by solving the equation

$$f(y) = 0$$

For autonomous equations, according to the Existence and Uniqueness Theorem, if $f(y)$ and $\frac{df}{dy}$ are continuous, then no other solution can intersect an equilibrium solution in finite time. We can classify the equilibrium solutions as follows:

- *Stable:* The equilibrium solution $y(t) = c$ is stable if all solutions with initial conditions y_0 'near' c stays near c as $t \to \infty$.

- *Asymptotically stable:* The equilibrium solution $y(t) = c$ is asymptotically stable if all the solutions with initial condition y_0 not only stays near c as $t \to \infty$ but also eventually converge to the equilibrium.

- *Unstable:* The equilibrium solution $y(t) = c$ is unstable if not all solutions with initial conditions y_0 'near' $y = c$ approach c as $t \to \infty$.

More precisely, we have $\epsilon - \delta$ definition of Liapunov stability:

DEFINITION 3.2 (a) A equilibrium point $y_0^* \in \mathbb{R}$ is Liapunov stable if for any $\epsilon > 0$, there exits a δ such that if $\|y_(t_0) - y_0^*\| < \delta$, then we have $\|y(t) - y_0^*\| < \epsilon$ for ant $t \geq t_0 > 0$.

(b) The equilibrium point y_0^* is asymptotically stable if it is Liapunov stable and for every initial state $\lim_{t\to\infty} y(t) = y_0^*$ as $t \to \infty$.

4. LINEARIZATION AND STABILITY OF NONLINEAR SYSTEMS

4.1. Linearization

Recall some facts of the stability of systems of linear differential equations:

$$x' = Ax, \text{where } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

When the eigenvalues are real and positive, then the critical point $\binom{0}{0}$ is an unstable node. When the eigenvalues are complex with positive real part, the critical point $\binom{0}{0}$ turns out to be an unstable spiral point. When the eigenvalues are real with opposite sign, the critical point is an unstable saddle point.

And for the asymptotically stable case, when the eigenvalues are negative real numbers or complex numbers with negative real parts, we have that the equilibrium point is asymptotically stable. The equilibrium point is stable only when the eigenvalues are pure imaginary.

When it comes to the applied problems, there are always some errors of the measurement or some uncertainties which result in small perturbations of the coefficients. So, whether the small perturbation to a linear system will significantly affect the stability of the critical point and trajectory is what we need to find out next.

Let's take a look at the stable case, that is, when the eigenvalues are pure imaginary. Any tiny perturbation could easily push the pure imaginary eigenvalues to the left or right of the imaginary axis, leading to eigenvalues will either negative real part or positive real part, respectively. Thus, it will break the stability of the system. The critical point $(0, 0)$ here is a center so all the trajectories are ellipses centered at origin. Suppose that there is a small perturbation of the coefficients. This slight change will alter eigenvalues $r_1 = i\mu, r_2 = -i\mu$ into new values $r_1' = \lambda' + i\mu', r_2' = \lambda' - i\mu'$, where λ' is in small magnitude and $\mu' \simeq \mu$. For the new perturbed system, the critical point is not a center any more. Instead the type of the critical point becomes spiral point. When $\lambda' > 0$, the trajectory is departing from the critical point, hence the critical point is unstable. When $\lambda' < 0$, the trajectory is approaching the critical point, hence the critical point is asymptotically stable. Therefore in this sensitive system case a little perturbation may result in the change of both the type and stability of the critical point.

There also exists a less sensitive system whose stability remains the same but the types of equilibrium point change when small perturbation occurs. For example, when the eigenvalues r_1, r_2 are equal, the origin is a node. Then slight change may set them apart so the trajectory will bifurcate and divide into two parts which go in different directions. If the new eigenvalues r_1', r_2' are still real numbers, the equilibrium of the perturbed system is a node too. However, when the eigenvalues become complex conjugates, that is $r_1' = \lambda + i\mu, r_2' = \lambda - i\mu$, then the critical point turns to be a spiral point.

For the other cases of linear differential equations, the stability and the type of the critical point are unaffected by small perturbations in the coefficients of the system. Now consider the differential equations in $\mathbb{R}^2$

$$\frac{dx}{dt} = f(x). \tag{4.4}$$

We notice that the trajectory near each critical point of nonlinear system resembles the trajectory of a corresponding linear system. Suppose x_0 is an equilibrium point of (4.4). Instead of looking at the equilibrium x_0, we can do a translation $u = x - x_0$ so the critical point is converted to the origin. We may express the right-hand part of (4.4) in the form

$$f(x) = A(x - x_0) + g(x) \tag{4.5}$$

where A is an 2×2 matrix with $\det A \neq 0$. Here $g(x)$ is merely the difference $f(x) - A(x - x_0)$. In order to have the nonlinear system (4.4) very close to linear system $x' = Ax$, we must let $g(x)$ to be small enough to satisfy

$$\lim_{x \to x_0} \frac{\|g(x)\|}{\|x - x_0\|} = 0, \tag{4.6}$$

that is to say, $\|g(x)\|$ is very small compared with $\|x - x_0\|$. Then the equation

$$\frac{dx}{dt} = A(x - x_0) \tag{4.7}$$

is called linear approximation of (4.5), where A is a Jacobian matrix of f at x_0 given by

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}.$$

$g(x)$ is a nonlinearity constraint for the system, however, condition (4.6) makes the linear term in (4.5) to be the dominant part of $f(x)$ when x is near x_0.

4.2. Stability of Nonlinear System

Given the linearization, instead of analysing the stability at the equilibrium of the nonlinear differential equations, we may focus on the linear system.

If all the eigenvalues of A have nonzero real parts, then the behavior of the orbits of nonlinear system (4.4) near x_0 is similar to the behavior of the orbits of the linearized system

$$x' = Ax$$

near the origin. In fact, we have

THEOREM 4.1 If the linear approximation (4.7) is asymptotically stable at x_0, then (4.4) is asymptotically stable at x_0.

To present a proof, we first introduce the variation of constant formula, which plays a very important role in the study of the stability, existence of the bounded solutions and the asymptotic behavior of nonlinear differential equations.

THEOREM 4.2 (Variation of Constant Formula) [2]. For the initial value problem

$$\begin{cases} x'(t) &= A(t) + f(t,x), x \in \mathbb{R}^n \\ x(0) &= x_0 \end{cases}$$

it follows that its solution is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)f(s,x(s))ds$$

where Φ is the fundamental matrix of the linear system

$$x'(t) = A(t)x,$$

namely, $\Phi' = A\Phi$ and $\Phi(0) = I$.

COROLLARY 4.3 For the above initial value problem, the solution can also be given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s,x(s))ds$$

LEMMA 4.4 (Gronwall inequality) Let $v : [t_0, t_1] \to \mathbb{R}^+$ be continuous and non-negative, suppose that v satisfies the integral inequality

$$v(t) \leq A + \int_{t_0}^t B(s)v(s)ds$$

for all $t \in [t_0, t_1]$, where $A \geq 0$ and $B : [t_0, t_1] \to \mathbb{R}^+$ is continuous, then

$$v(t) \leq A \exp\Big(\int_{t_0}^{t} B(s)ds\Big)$$

for all $t \in [t_0, t_1]$.

Proof. (of Theorem 4.1) Our proof will imply that solutions of (4.4) once near x_0 remain near x_0 and hence exist for all $t \geq 0$. If we introduce a variable $y = x - x_0$, then equation (4.4) may be expressed in the form

$$\frac{dy}{dt} = f(y + x_0) = Ay + g(y + x_0) \tag{4.8}$$

and the nonlinear condition (4.6) may be rewritten as

$$\lim_{y \to 0} \frac{\|g(y + x_0)\|}{\|y\|} = 0 \tag{4.9}$$

According to the Variation of Constants Formula, we can infer that the solution of equation (4.8) satisfies an integral equation of the form

$$y(t) = \Phi(t)x_0^* + \int_0^t \Phi(t - s)g(y(s) + x_0)ds \tag{4.10}$$

where $\Phi(t)$ is a fundamental matrix solution of the associated matrix equation $\frac{dX}{dt} = AX$ and x_0^* is an initial vector. Since the system is asymptotically stable and the eigenvalues have negative real part, then we can take α as the minimum of all the eigenvalues such that

$$\|\Phi(t)\| \leq Ce^{-\alpha t} \tag{4.11}$$

for suitable positive C. Hence

$$\|y(t)\| \le C\|x_0^*\|e^{-\alpha t} + C\int_0^t e^{-\alpha(t-s)}\|g(y(s)+x_0)\|ds, \tag{4.12}$$

$$\|y(t)\|e^{\alpha t} \le C\|x_0^*\| + C\int_0^t e^{\alpha s}\|g(y(s)+x_0)\|ds. \tag{4.13}$$

From condition (4.9), it follows that for any $\epsilon > 0$ there exists a positive number δ such that

$$\frac{\|g(y+x_0)\|}{\|y\|} \le \epsilon \quad \text{for } \|y\| \le \delta. \tag{4.14}$$

Here we take $\epsilon = \frac{\alpha}{2C}$. Then

$$\|g(y+x_0)\| \le \frac{\alpha}{2C}\|y\| \quad \text{for } \|y\| \le \delta. \tag{4.15}$$

From Eq.(4.13), we further have

$$\|y(t)\|e^{\alpha t} \le C\|x_0^*\| + \frac{\alpha}{2}\int_0^t e^{\alpha s}\|y(s)\|ds \tag{4.16}$$

provided $\|y(s)\| \le \delta$ for $0 \le s \le t$. For $t = 0$, we have $\|y(0)\| \le C\|x_0^*\|$ and so if x_0^* is required to satisfy

$$\|x_0^*\| < \frac{\delta}{C} \tag{4.17}$$

then the inequality $\|y(s)\| \le \delta$ will be satisfied on some interval $0 \le s \le t$ with $t > 0$. Applying Gronwall's lemma, Eq.(4.16) becomes

$$\|y(t)\|e^{\alpha t} \le C\|x_0^*\|e^{\frac{\alpha t}{2}}, \tag{4.18}$$

$$\|y(t)\| \le C\|x_0^*\|e^{-\frac{\alpha t}{2}} \tag{4.19}$$

However, this shows that the inequality $\|y(t)\| \le C\|x_0^*\| < \delta$ once initiated is maintained and thus applies throughout as t move forward to ∞. Thus (4.19) applies for all

$t \geq 0$ and so $\lim_{t\to\infty} \|y(t)\| = 0$. Note that the inequality (4.17) is merely a restriction on the initial perturbation of $x(t)$ relative to the equilibrium point x_0. Thus the theorem is proved. □

THEOREM 4.5 [4]. Suppose that f is defined by (4.5) and g satisfy the condition (4.6).

(a) If one of the eigenvalues of A has positive real part, then x_0 is unstable for system (4.4).

(b) If all of the eigenvalues of A have negative real parts, then x_0 is asymptotically stable for system (4.4).

Although the type of the critical point $(0, 0)$ may be the same for linear case and nonlinear case, the actual appearance of the paths may be somewhat different.

5. POPULATION DYNAMICS

In this section we will apply the theory and the strategy of system linearization to some mathematical models, in particular, the models of population dynamics.

5.1. Competing Species

When it comes to two related species, there exist two common relationships: coexistence or competition. Suppose that there are two kinds of species which do not prey on each other but compete for a limited food supply. Let x, y represent the populations of these two species with respect to time t. Assume that the equation of the population is governed by a logistic equation. Also, we need to consider the influence of coexistence of these two species because when they are both present, one can diminish the limited food supply for the other. That is to say, each of them will reduce the other's growth rate to some extent by competing for the food. Thus we have

$$\frac{dx}{dt} = x(\alpha_1 - \beta_1 x - \gamma_1 y), \tag{5.20}$$
$$\frac{dy}{dt} = y(\alpha_2 - \beta_2 y - \gamma_2 x). \tag{5.21}$$

Here the coefficient $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ are nonnegative. $\alpha_{1,2}$ represent the growth rate, $\beta_{1,2}$ are self-inhibition parameters, and $\gamma_{1,2}$ represent the interaction parameters. When we set $x', y' = 0$, we will have two lines

$$\alpha_1 - \beta_1 x - \gamma_1 y = 0, \tag{5.22}$$
$$\alpha_2 - \beta_2 y - \gamma_2 x = 0, \tag{5.23}$$

which are called the x- and y-nullclines.

So there are up to four steady states in the first quadrant: $S_1 = (0,0)$, $S_2 = (0, \frac{\alpha_2}{\beta_2})$, $S_3 = (\frac{\alpha_1}{\beta_2}, 0)$ and critical point $S_4 = (X, Y)$. We can conclude the nonzero values of critical point by solving equation (5.22),(5.23), so we have

$$X = \frac{\alpha_1\beta_2 - \alpha_2\gamma_1}{\beta_1\beta_2 - \gamma_1\gamma_2}, \quad Y = \frac{\alpha_2\beta_1 - \alpha_1\gamma_2}{\beta_1\beta_2 - \gamma_1\gamma_2} \tag{5.24}$$

In order to conclude the stability in the neighborhood of the critical point, we can take a look at the linearization of nonlinear system (5.20),(5.21):

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha_1 - 2\beta_1 X - \gamma_1 Y & -\gamma_1 X \\ -\gamma_2 Y & \alpha_2 - 2\beta_2 Y - \gamma_2 X \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{5.25}$$

Since we know that $\alpha_1 - \beta_1 X - \gamma_1 Y = 0$ and $\alpha_2 - \beta_2 Y - \gamma_2 X = 0$, we can simplify the equation (5.25).

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\beta_1 X & -\gamma_1 X \\ -\gamma_2 Y & -\beta_2 Y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{5.26}$$

Eigenvalues can found by

$$\begin{vmatrix} -\beta_1 X - \lambda & -\gamma_1 X \\ -\gamma_2 Y & -\beta_2 Y - \lambda \end{vmatrix} = 0$$

Hence, we have

$$\lambda^2 + (\beta_1 X + \beta_2 Y)\lambda + (\beta_1\beta_2 - \gamma_1\gamma_2)XY = 0. \tag{5.27}$$

Thus

$$\lambda_{1,2} = \frac{-(\beta_1 X + \beta_2 Y) \pm \sqrt{(\beta_1 X + \beta_2 Y)^2 - 4(\beta_1\beta_2 - \gamma_1\gamma_2)XY}}{2} \tag{5.28}$$

We can discuss in two case.

1. If $(\beta_1\beta_2 - \gamma_1\gamma_2) < 0$, then $\sqrt{(\beta_1 X + \beta_2 Y)^2 - 4(\beta_1\beta_2 - \gamma_1\gamma_2)XY} > \beta_1 X + \beta_2 Y$. So eigenvalues are real and have opposite sign. Hence (X, Y) is an unstable saddle point.

2. If $(\beta_1\beta_2 - \gamma_1\gamma_2) > 0$, then $\sqrt{(\beta_1 X + \beta_2 Y)^2 - 4(\beta_1\beta_2 - \gamma_1\gamma_2)XY} < \beta_1 X + \beta_2 Y$. And also$\sqrt{(\beta_1 X + \beta_2 Y)^2 - 4(\beta_1\beta_2 - \gamma_1\gamma_2)XY} > 0$. Consequently, the eigenvalues are unequal negative real numbers. Thus the critical point is an asymptotically stable node. In this case, the sustained coexistence may occur.

The following graphs of various cases for competing species system confirm these conclusions. From the figure 1, S_1 is an unstable node, S_2 and S_3 are either saddle point or stable nodes, and S_4 is either a saddle point or a stable node.

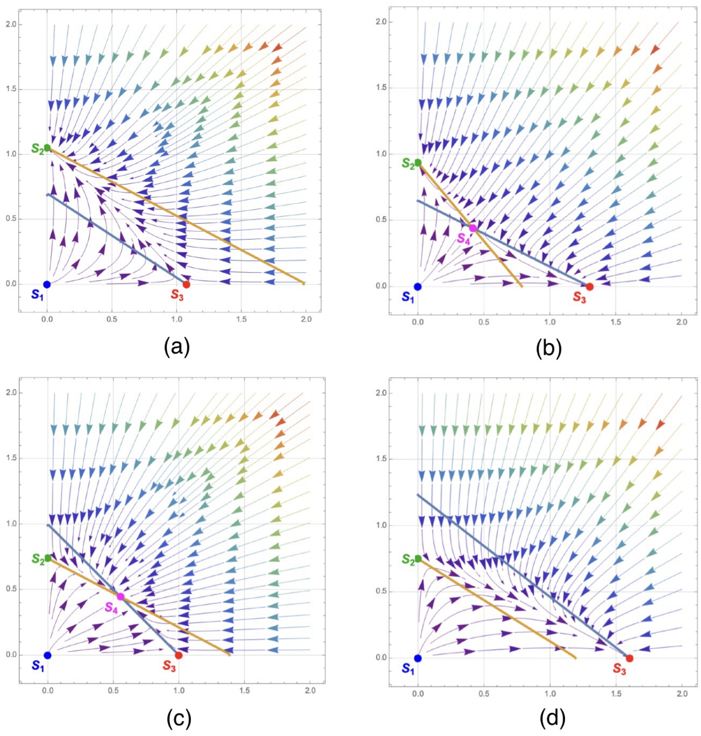

Figure 1: Four different cases of competing species

More generally, we can write the equations of the growth rate of two competing species as

$$x' = xM(x, y)$$
$$y' = yN(x, y)$$

where M, N represent the growth rate function with both variables.

Then several assumptions of growth rate function $M(x, y), N(x, y)$ can be concluded as followed :

1. Since two species are competing for available food apply, if the population of one species increase, then the growth rate of the other's population will go down.

2. If one specie's population is large enough, both populations decrease.

3. In the absence of either species, the other's growth rate will increase to a certain

number and then turn to negative beyond it.

The first condition implies that

$$M_y = \frac{\partial M}{\partial y} < 0$$
$$N_x = \frac{\partial N}{\partial x} < 0$$

By linearization at the equilibrium, the Jacobian matrix of linear system takes the form

$$\begin{pmatrix} xM_x & xM_y \\ yN_x & yN_y \end{pmatrix}$$

The trace of the matrix is $xM_x + yN_y < 0$ and the determinant is $xy(M_xN_y - M_yN_x)$. So $\Delta = (trA)^2 - 4\det A > 0$ in this case. Therefore, the equilibrium is either stale node or unstable saddle point.

Now we consider a specific situation of such models.

EXAMPLE 5.1 Consider the competition between bluegill x and redear y. Suppose that $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$ and $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$, and rewrite the equations in terms of the carrying capacity of the pond for bluegill ($B = \epsilon_1/\sigma_1$) in the absence of redear and its carrying capacity for redear ($R = \epsilon_2/\sigma_2$) in the absence of bluegill.

$$\begin{cases} \frac{dx}{dt} = \epsilon_1 x(1 - \frac{1}{B}x - \frac{\gamma_1}{B}y), \\ \frac{dy}{dt} = \epsilon_2 y(1 - \frac{1}{R}y - \frac{\gamma_2}{R}x) \end{cases}$$

where $\gamma_1 = \alpha_1/\sigma_1$ and $\gamma_2 = \alpha_2/\sigma_2$. Determine the coexistence equilibrium point (X, Y) in terms of B, R, γ_1, and γ_2, we have $(\frac{\gamma_1 R - B}{\gamma_1\gamma_2 - 1}, \frac{\gamma_2 B - R}{\gamma_1\gamma_2 - 1})$.

Suppose that an angler fishes only for bluegill with the effect that B is reduced. Given that $\epsilon_1/\alpha_1 > \epsilon_2/\sigma_2$ and $\epsilon_2/\alpha_2 > \epsilon_1/\sigma_1$, we know that $(\frac{\gamma_1 R - B}{\gamma_1\gamma_2 - 1}, \frac{\gamma_2 B - R}{\gamma_1\gamma_2 - 1})$ is an unstable saddle point. When B is reduced to a level that $\epsilon_1/\alpha_1 < \epsilon_2/\sigma_2$, the equilibrium points in this case turn out to be $(0,0), (B,0), (0,R)$, where the only equilibrium populations are either no fish or no redear or no bluegill. Therefore, it is possible to reduce the population of bluegill to such a level that they will die out by fishing.

5.2. Predator-Prey Model

Typically, the rate of change of a population is only dependent on the current size in some way. So, the equation for population is a typical autonomous system. Let us start with two interacting populations and the predator-prey model.

Italian mathematician Volterra came up with the predator-prey model (also called the Volterra-Lotka model) in the 1920s, which is one of the earliest models in mathematical ecology.

$$\frac{dx}{dt} = rx - axy \tag{5.29}$$

$$\frac{dy}{dt} = -dy + bxy \tag{5.30}$$

Adding the logistic term to this model, we have Logistic-Voterra model:

$$\frac{dx}{dt} = rx(1 - \frac{x}{N_1}) - axy \tag{5.31}$$

$$\frac{dy}{dt} = -dy(1 + \frac{y}{N_2}) + bxy \tag{5.32}$$

In these two models, a, b, r, d are positive real parameters describing the interaction of two species. x, y represent number of the prey and predator respectively; $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the instantaneous growth rates of the grow rates of the two populations; the parameter r represents the growth rate of prey in the absence of interaction with predators; the parameter a measures the impact of predation on $\frac{dx}{dt}$; the parameter d is the death rate of the predators in the absence of interaction with species x;

the term rx denotes the net rate of growth of the predator population in response to the prey population; the parameter N_1, N_2 represents the carrying capacity that the population size cannot exceed.

We assume that the prey population is the total food supply for predators. And when there are no predators, that is, when $y = 0$, the equation for prey becomes $x' = rx$, then we will have $x(t) = x_0 \exp(rt)$, so the prey population will increase in a rate proportional to the current population. Similarly, in the absence of prey, that is $x = 0$, the equation for predators is $y' = -dy$, then $y(t) = y_0 \exp(-dt)$ the population of predators decrease until becomes extinct.

The first step is to locate the equilibrium points. By letting the right-hand side of the equation (5.29),(5.30) to be zero, we have two critical points $(0, 0)$ and $(\frac{d}{b}, \frac{r}{a})$. The linearized system is

$$X' = \begin{pmatrix} r - ay & -ax \\ by & -d + bx \end{pmatrix} X$$

Usually we analyse the types of the critical points. First compute the Jacobian:

$$J = \begin{pmatrix} r - ay & -ax \\ by & -d + bx \end{pmatrix} \tag{5.33}$$

For the critical point (0,0), the Jacobian is $J(0, 0) = \begin{pmatrix} r & 0 \\ 0 & -d \end{pmatrix}$. In this case, we have a saddle point with eigenvalues r and $-d$. Hence the origin is unstable. And $y-$ and $x-$axes are stable and unstable curves respectively.

Evaluating J at $(\frac{d}{b}, \frac{r}{a})$, the Jacobian is $J(\frac{d}{b}, \frac{r}{a}) = \begin{pmatrix} 0 & -\frac{ad}{b} \\ \frac{rb}{a} & 0 \end{pmatrix}$.

We obtain the approximate linear system

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -ad/b \\ br/a & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

where $u = x - (d/b)$ and $v = y - (r/a)$. The eigenvalues of the above system are $\pm i\sqrt{rd}$, so the critical point is a stable center of the linear system. In order to find the trajectories of the system, we can divide the second equation by the first equation to obtain

$$\frac{dv}{du} = \frac{dv/dt}{du/dt} = -\frac{(br/a)u}{(ad/b)v}$$

or

$$b^2rudu + a^2dvdv = 0$$

Consequently,

$$b^2ru^2 + a^2dv^2 = k,$$

where k is a nonnegative constant of integration. Thus the trajectories of the linear system are ellipses.

EXAMPLE 5.2 Harvesting in a Predator-Prey Relationship. In a predator-prey situation, it may occurs that one or both species are valuable sources of food. Or, the prey species may be regarded as a pest, leading to efforts to reduce its numbers. In a constant-effort model of harvesting, we introduce a term E_1x in the prey equation and a term E_2y in the predator equation, where E_1 and E_2 are measures of the effort invested in harvesting the respective species. A constant-yield model of harvesting is obtained by including the term H_1 in the prey equation and the term H_2 in the predator equation. The constants E_1, E_2, H_1, and H_2 are always nonnegative.

Applying a constant-effort model of harvesting to the Predator–Prey equations, we obtain the system

$$x' = x(r - ay - E_1),$$
$$y' = y(-d + bx - E_2).$$

When there is no harvesting, the equilibrium solution is $(d/b, r/a)$.

Once one of the species are harvested, the responses may vary depending on the situation.

The equilibrium points are the solutions to the system of equations

$$0 = x(r - ay - E_1),$$
$$0 = y(-d + bx - E_2).$$

Then the equilibrium solutions are $(0, 0)$ and $(\frac{E_2+d}{b}, \frac{r-E_1}{a})$. We compare this to equilibrium solution with no harvesting.

When the prey is harvested, but not the predator $(E_1 > 0, E_2 = 0)$, we have

$$x' = x(r - ay - E_1),$$
$$y' = y(-d + bx).$$

The critical points are $(0, 0), (\frac{d}{b}, \frac{r-E_1}{a})$. So prey remain the same but the predators decrease.

When the predator is harvested, but not the prey $(E_1 = 0, E_2 > 0)$, the critical points are $(0, 0), (\frac{E_2+d}{b}, \frac{r}{a})$. Therefore preys increase and predators remain same. When both are harvested $(E_1 > 0, E_2 > 0)$, preys increase and predators decrease.

If we modify the Lotka–Volterra equations by including a self-limiting term

$-\sigma x^2$ in the prey equation, and then assume constant-effort harvesting, we obtain the equations

$$x' = x(r - \sigma x - ay - E_1),$$
$$y' = y(-d + bx - E2).$$

In the absence of harvesting, the equilibrium solution is $x = d/b, y = (r/a) - (\sigma d)/(ab)$.

The equilibrium of this nonlinear system are $(0, 0)$ and $(\frac{E_2+d}{b}, \frac{r}{a} - \frac{\sigma(E_2+d)}{ab} - \frac{E_1}{a})$.

When $E_1 > 0, E_2 = 0$, that is, the prey is harvested, but not the predator, then we have that prey remain same but predators decrease.

When $E_1 = 0, E_2 > 0$, that is, the predator is harvested, but not the prey, then we have that prey increase but predators decrease.

When $E_1 > 0, E_2 > 0$, that is, both are harvested, then we have that prey increase and predators decrease, too.

6. LIAPUNOV'S SECOND METHOD

The stability theory is an important branch of differential equations. However, strategies are limited to determine the stability of solutions of the system. Liapunov made some advancement by establishing two methods to deal with the stability of the solution:

Liapunov's first method analysed the stability by using explicit solution of differential equation. So it is an indirect method.

Liapunov's second method is also known as direct method because it can determine the stability directly by constructing a function without the knowledge of the solutions of the system of differential equations, which makes the "direct method" practical and powerful.

6.1. Liapunov Functions

Consider the autonomous system

$$\frac{dx}{dt} = F(x), \quad x \in \mathbb{R}^n \tag{6.34}$$

Suppose $F(x) = (F_1(x), \cdots, F_n(x))^T$ is Lipschitz continuous on $G = \{x \in R^n, \|x\| \leq K\}$ and $F(0) = 0$.

DEFINITION 6.1 The function $F(x)$ is Lipschitz continuous in x if there exists $K > 0$ such that

$$\|f(x_2) - f(x_1)\| \leq K\|x_2 - x_1\|.$$

for all x_1 and x_2.

DEFINITION 6.2 (Liapunov Function) Let X be a vector field on R^n, and let x_0 be an equilibrium point for X, that is $X(x_0) = 0$. A Liapunov function is a continuous function $V : U \to R$ defined on a neighborhood U of x_0, differentiable on $U \setminus \{x_0\}$, and satisfying the following conditions:

1. $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$, namely, V is positive definite.

2. $\dot{V} \leq 0$ for all $x \in U \setminus \{x_0\}$ and $t > 0$, where $\dot{V} = \frac{d}{dt}V(X(t)) = \sum_{i=1}^{n} V_{x_i} \cdot F_i(x)$

THEOREM 6.3 For system (6.34), if there exists a positive definite Liapunov function $V(x,t)$, whose derivative along the trajectories of the system $\frac{dV}{dt} \leq 0$, then the origin of the system is a stable critical point.

Proof. For any $\epsilon > 0$, let $\Gamma = \{x, \|x\| = \epsilon\}$, then according to the fact that $V(x)$ is positive definite and continuous, we have

$$b = \min_{x \in \Gamma} V(x) > 0.$$

Since $V(0) = 0$ and $V(x)$ is continuous, there exists a $\delta > 0$ ($\delta < \epsilon$) such that $V(x) < b$ when $\|x\| < \delta$. We claim that

$$x(t, t_0, x_0) < \epsilon, \ t \geq t_0. \tag{6.35}$$

Suppose that the inequality(6.35) does not hold. Then there exists $t_1 > t_0$ such that $x(t, t_0, x_0) < \epsilon$ for $t \in [t_0, t_1]$, and $x(t_1, t_0, x_0) = \epsilon$, then from the concept of b we have

$$V(x(t_1, t_0, x_0)) \geq b \tag{6.36}$$

From another aspect, since

$$\frac{dV}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} F_i(x) \leq 0,$$

thus

$$\frac{dV(x(t, t_0, x_0))}{dt} \leq 0 \quad \textit{holds true on } [t_0, t_1].$$

Hence $V(x(t, t_0, x_0)) \leq V(x_0) < b$ for $t \in [t_0, t_1]$. So $V(x(t_1, t_0, x_0)) < b$ which contradicts inequality (6.36). Then (6.35) is true and therefore the origin is a stable critical point of the system.

□

LEMMA 6.4 If $V(x)$ is positive definite Liapunov function with $\dot{V} < 0$ for $x \neq 0$ and for continuous bounded function $x(t)$ it satisfies

$$\lim_{t \to \infty} V(x(t)) = 0,$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose the trajectory $x(t)$ does not converge to zero. Note that $V(x(t))$ is decreasing in t since $\frac{d}{dt}V(x(t)) \leq 0$ and positive, so it converges to some ϵ as $t \to \infty$. Since $x(t)$ does not converge to 0, we have $\epsilon > 0$. Thus, $\epsilon \leq V(x(t)) \leq V(x(0))$ for $\forall t \geq 0$. The region $D = \{x \in \mathcal{R}^n : \epsilon \leq V(x) \leq V(x(0)\}$ is closed and bounded in $\mathcal{R}^n$, hence compact. So $\dot{V}(x(t))$ attains its supremum on D, suppose $\sup_{x(t)\in D} = -m < 0$. Since $\dot{V}(x(t)) \leq -m$ for all t, we have

$$V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t))dt \leq V(x(0)) - mT$$

Letting $T \to \infty$ leads to $V(x(0)) < 0$, a contradiction. Hence every trajectory $x(t)$ converges to 0, that is, $\lim_{t\to\infty} x(t) = 0$. □

V is called negative definite if $V(0) = 0$ and $V(x) < 0$ for $x \neq 0$.

THEOREM 6.5 If there exists a positive definite Liapunov function $V(x,t)$, whose derivative along the trajectories of the system is negative definite, then the origin of the system is an asymptotically stable critical point.

Proof. From Theorem 6.3 we know that the zero solution of system (6.34) is stable. Take $\bar{\epsilon}$ as the ϵ in the proof of Theorem 6.3. Then when $\|x\| \leq \bar{\epsilon}$, $V(x(t,t_0,x_0))$ is monotone decreasing. If $x_0 = 0$, according to the uniqueness, we have $x(t, t_0, x_0) \equiv 0$. So

$$\lim_{t \to +\infty} x(t, t_0, x_0) = 0$$

Let $x_0 \neq 0$. Due to the uniqueness of the solution of initial value problem, $x(t, t_0, x_0) \neq 0$ for any t. From the positive definiteness of $V(x)$, $V(x(t, t_0, x_0)) > 0$ always holds true. Then there exists $a \geq 0$ such that

$$\lim_{t \to +\infty} V(x(t, t_0, x_0)) = a.$$

Suppose $a > 0$, considering the monotonicity of $V(x(t, t_0, x_0))$, we have

$$a < V(x(t, t_0, x_0)) < V(x_0), \quad t > t_0.$$

Since $V(0) = 0$, there exists $h > 0$ such that when $t \geq t_0$

$$h < \|x(t, t_0, x_0)\| < \epsilon \tag{6.37}$$

Since

$$\frac{dV}{dt} \leq M = \max_{h \leq \|x\| \leq \epsilon} \frac{dV}{dt} < 0.$$

Thus

$$\frac{dV(x(t, t_0, x_0)}{dt} \leq M. \tag{6.38}$$

Hence, we have

$$V(x(t, t_0, x_0)) - V(x_0) \leq M(t - t_0),$$

which means

$$\lim_{t \to +\infty} V(x(t, t_0, x_0)) = -\infty.$$

The contradiction proves that $a = 0$, that is, $\lim_{t \to +\infty} V(x(t, t_0, x_0)) = 0$.

By Lemma 6.4, we have

$$\lim_{t \to +\infty} x(t, t_0, x_0) = 0.$$

The theorem is proved. □

THEOREM 6.6 Suppose that in every neighborhood of equilibrium $(0, 0)$ there is at least one point at which $V(x, t)$ is positive (negative). Assume further that the derivative along the trajectories of the system $\frac{dV}{dt} > 0 \quad (\frac{dV}{dt} < 0)$, then the origin of the system is an unstable critical point [1].

Namely, The Liapunov function can provide an another approach to analyse the stability of the equilibrium points of predator-prey systems. For the system (5.29)-(5.30) we look for a Liapunov function of the form

$$V(x, y) = F(x) + G(y).$$

We compute

$$\begin{aligned} \dot{V}(x, y) &= \frac{d}{dt} V(x(t), y(t)) \\ &= \frac{dF}{dx} x' + \frac{dG}{dy} y'. \end{aligned}$$

Then plug in $x' = rx - axy$ and $y' = -dy + bxy$, we have

$$\dot{V}(x, y) = \frac{dF}{dx}(rx - axy) + \frac{dG}{dy}(-dy + bxy).$$

Let $\dot{V} \equiv 0$, we have

$$x \frac{dF}{dx}(r - ay) \equiv y \frac{dG}{dy}(d - bx).$$

Hence

$$\frac{xdF/dx}{d-bx} \equiv \frac{ydG/dy}{r-ay}.$$

Since x, y are independent variables, this is possible if and only if both sides are equal to the same constant, that is

$$\frac{xdF/dx}{d-bx} = \frac{ydG/dy}{r-ay} = C.$$

Let $C = 1$, we have

$$\frac{dF}{dx} = -b + \frac{d}{x}$$
$$\frac{dG}{dy} = -a + \frac{r}{y}$$

Integrating both equations, we obtain

$$F(x) = -bx + d\log x$$

$$G(y) = -ay + r\log y$$

Hence the Liapunov function becomes

$$V(x, y) = -bx + d\log x - ay + r\log y$$

Next we consider the equilibrium point $E = (\frac{d}{b}, \frac{r}{a})$. We compute $V_x = -b + \frac{d}{x}$, $V_y = -a + \frac{r}{y}$, thus

$$V_{xx} = -\frac{d}{x^2},\ V_{yy} = -\frac{r}{y^2},\ V_{xy} = 0$$

Hence

$$
\begin{aligned}
D &= V_{xx}V_{yy} - V_{xy}^2 \\
&= \frac{dr}{x^2y^2} > 0
\end{aligned}
$$

Besides, $V_{xx} < 0$. By second partial derivative test, we can conclude that the equilibrium point $E = (\frac{d}{b}, \frac{r}{a})$ is an absolute minimum for V. So V, more precisely $V - V(E)$ is a Liapunov function of the system. Therefore E is a stable equilibrium point.

EXAMPLE 6.7 Consider nonlinear differential equations

$$
\begin{aligned}
x' &= -y + x^5 \\
y' &= x + 2y^3
\end{aligned}
$$

with the origin as an isolated critical point and the eigenvalues of the linearized system can be calculated by solving

$$
\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0
$$

So the eigenvalues are $\pm i$. The stability criterion cannot be applied based on eigenvalues alone. Then the Liapunov second method can rescue the situation. We search for a function $V(x, y) = ax^2 + by^2$ where $a, b > 0$.

$$
\begin{aligned}
\dot{V} &= 2ax(-y + x^5) + 2by(x + 2y^3) \\
&= 2xy(b - a) + 2ax^6 + 4by^4
\end{aligned}
$$

If we choose $a = b = 1$, it follows that $\dot{V} = 2ax^6 + 4by^4 > 0$ if $(x, y) \neq (0, 0)$. Thus, the origin is unstable.

EXAMPLE 6.8 Consider the nonlinear differential equations

$$x' = -y - x^5$$
$$y' = x - 2y^3$$

with the origin as an isolated critical point. Follow the same steps as Example 6.7, the eigenvalues are $\pm i$. The stability criterion gives us nothing about the stability. Then we search for a function $V(x, y) = ax^2 + by^2$ where $a, b > 0$.

$$\begin{aligned}\dot{V} &= 2ax(-y - x^5) + 2by(x - 2y^3)\\ &= 2xy(b - a) - 2ax^6 - 4by^4\end{aligned}$$

If we choose $a = b = 1$, in this case $\dot{V} = 2ax^6 + 4by^4 < 0$ if $(x, y) \neq (0, 0)$, making $\dot{V}$ negative definite. Thus, the origin is asymptotically stable.

From the preceding two examples, we can conclude that if the spectrum of the matrix lies on the imaginary axis, the stability of nonlinear differential equations at equilibrium point is uncertain, which is ultimately determined by the nonlinear terms.

EXAMPLE 6.9 Pendulum is a model in which the material point of mass m is suspended on a weightless and inextensible string of length l, see figure 2. α describes the angle of the mass from the straight down position, so the the position of the mass at time t is at $(l \sin \alpha(t), -l \cos \alpha(t))$. If the force of the gravity and friction are considered, then we can conclude that

(a) The component of the force of gravity tangent to the circle of the motion is $-mg \sin \alpha$.

(b) The force due to friction is proportional to velocity, which is $-kld\alpha/dt$ if we take the proportional constant $k > 0$. For a conservative system, $k = 0$.

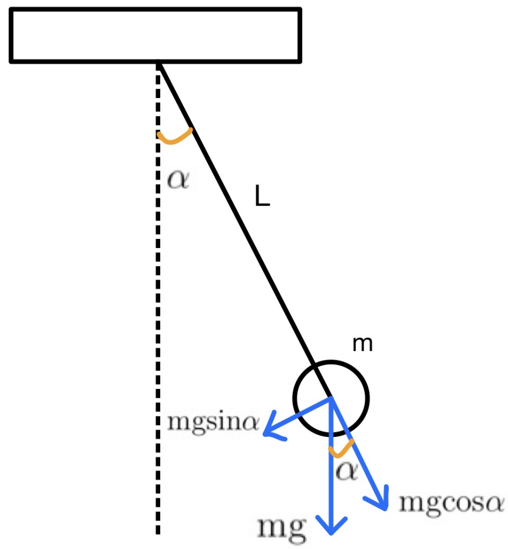

Figure 2: Graph of pendulum

According to Newton's second Law, we have the second-order equation for the pendulum

$$ml\frac{d^2\alpha}{dt^2} = -kl\frac{d\alpha}{dt} - mg\sin\alpha$$

Assume $m = g = l = 1$ for purpose of simplification, we have

$$\frac{d^2\alpha}{dt^2} = -k\frac{d\alpha}{dt} - \sin\alpha$$

Rewrite the equation as a system, we get

$$\begin{cases} \alpha' &= v \\ v' &= -kv - \sin\alpha \end{cases}$$

The critical points of this system are $v = 0, \alpha = \pi$ and $v = 0, \alpha = 0$. Intuitively, the upward position $v = 0, \alpha = \pi$ is an unstable equilibrium point. At the downward equilibrium point $v = 0, \alpha = 0$, the linearization of the system takes this form:

$$Y' = \begin{pmatrix} 0 & 1 \\ -1 & -k \end{pmatrix} Y$$

The eigenvalues of the matrix either have negative real parts (when $k > 0$) or are pure imaginary (when $b = 0$). So the downward equilibrium point $v = 0, \alpha = 0$ is (asymptotically) stable. Consider the total energy function of this system, which is the sum of the potential energy and the kinetic energy, we have

$$\begin{aligned} E(\alpha, v) &= \frac{1}{2}v^2 + 1 - \cos\alpha \\ \dot{E} &= vv' + \alpha' \sin\alpha \\ &= -kv^2 \end{aligned}$$

Since $k \geq 0$, we have $\dot{E} \leq 0$. Here the energy function E is a Liapunov function. For conservative system, we have $\dot{E} = 0$ since $k = 0$, hence E is a constant along any trajectory of the system.

The analysis of the pendulum system shows the power of the Liapunov direct method and that the energy function is often a candidate for the Liapunov function. Since in general it is difficult to come up with a Liapunov function, the following criteria can be very useful.

THEOREM 6.10 $V(x, y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $4ac - b^2 > 0$. Similarly, $V(x, y)$ is negative definite if and only if $a < 0$ and $4ac - b^2 > 0$.

Proof. If $a > 0$, the function $V(x, y) > 0$ only occurs $\Delta < 0$, that is $4ac - b^2 > 0$. Similarly, when $a < 0$, the function $V(x, y) < 0$ only occurs $\Delta < 0$, that is $b^2 - 4ac < 0$ as well. $\square$

The following example illustrates the usefulness of the criteria.

EXAMPLE 6.11 Consider undamped linear vibration equation $\ddot{x} + \omega^2 x = 0$. Determine the stability of the equilibrium position.

Proof. Write the linear vibration equation as

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 x$$

Then we construct a Liapunov function $V(x, y) = \frac{1}{2}(x^2 + \frac{1}{\omega^2}y^2)$. Then

$$\frac{dV}{dt} = x \cdot x' + \frac{1}{\omega^2} y \cdot y' = 0$$

So $V(x, y)$ is positive definite and $\frac{dV}{dt} = 0$. According to Theorem 6.3, the zero solution of this equation is stable, that is, the equilibrium position is stable. □

Now consider a 3-dimentional system with a parameter.

EXAMPLE 6.12 Consider the system of differential equation

$$\begin{cases} x' &= (\epsilon x + 2y)(z + 1) \\ y' &= (-x + \epsilon y)(z + 1) \\ z' &= -z^3 \end{cases}$$

where ϵ is a parameter. Obviously, $(0, 0, 0)$ is the only equilibrium point of the system. We can write the linearization in a matrix form at the origin:

$$Y' = \begin{pmatrix} \epsilon & 2 & 0 \\ -1 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} Y$$

or in shorthand, $Y' = AY$, where A has three eigenvalues: $\lambda = 0$ and $\lambda = \epsilon \pm \sqrt{2}i$. If $\epsilon > 0$, the origin is unstable. And it is asymptotically stable when $\epsilon < 0$. Take $V(x, y, z) = ax^2 + by^2 + cz^2$ as the Liapunov function for the origin, where $a, b, c > 0$. Then, we have

$$\dot{V} = 2(axx' + byy' + czz')$$

Plug in x', y' and z', we get

$$\begin{aligned}
\dot{V} &= 2ax[(\epsilon x + 2y)(z+1)] + 2by[(-x + \epsilon y)(z+1)] + 2cz(-z^3) \\
&= 2[a(\epsilon x^2 + 2xy)(z+1) + b(-xy + \epsilon y^2)(z+1) - cz^4] \\
&= 2(z+1)[a\epsilon x^2 + 2axy - bxy + b\epsilon y^2] - 2cz^4 \\
&= 2(z+1)[\epsilon(ax^2 + by^2) + (2a - b)xy] - 2cz^4
\end{aligned}$$

Take $a = 1, b = 2$ and $c = 1$ to get rid of the xy term in $\dot{V}$, so that

$$\dot{V} = 2\epsilon(z+1)(x^2 + 2y^2) - 2z^4$$

When $\epsilon = 0$, $\dot{V} = -z^4 \leq 0$, so by Theorem 6.2 the origin of the system is a stable equilibrium point. Also we can show that the origin is not asymptotically stable in this case. In fact, since $\epsilon = 0$ we have that

$$\begin{cases}
x' &= 2y(z+1) \\
y' &= -x(z+1) \\
z' &= -z^3
\end{cases}$$

Now we want to show that the origin is not asymptotically stable, which means that there exist some solutions that don't tend towards the origin as $t \to \infty$ but stay close to the origin.

Assume that $z = 0$, in which case we have the equations

$$\begin{cases} x' &= 2y \\ y' &= -x \\ z' &= 0 \end{cases}$$

This is a 2×2 system with matrix

$$\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

which has eigenvalues $\pm\sqrt{2}i$. Thus, we have solutions to this system of form

$$\begin{bmatrix} x \\ y \end{bmatrix} = C_1 \begin{bmatrix} \cos(\sqrt{2}t) \\ -\sin(\sqrt{2}t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \end{bmatrix}$$

Putting it together, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C_1 \begin{bmatrix} \cos(\sqrt{2}t) \\ -\sin(\sqrt{2}t) \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} \sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ 0 \end{bmatrix}$$

which are ellipses around the origin. These are stable but not asymptotically stable.

If $\epsilon < 0$, and $z > -1$, then $\dot{V} < 0$. We conclude that the origin is asymptotically stable in this case. In order to estimate the basin of attraction, we may use the LaSalle's Invariance Principle. We give the definition of basin of attraction first.

6.2. LaSalle's Invariance Principle

DEFINITION 6.13 [3]. The basin of attraction is the set of all initial conditions whose solutions approaches to the equilibrium point as time t tends to ∞ .

EXAMPLE 6.14 Consider the system

$$\frac{dx}{dt} = -(2+y)(x+y)$$
$$\frac{dy}{dt} = -y(1-x).$$

The critical points are $(0,0), (1,-1), (1,-2)$. From the figure 3, we notice that solutions near $(0,0)$ approach the origin therefore $(0,0)$ is stable. The trajectories around $(1,-1)$ move away $(1,-1)$, thus it is unstable. Also, we can see that the trajectories around (1,-2) generate several limit cycles, which we will talk about later. The stable limit cycles are examples of basin of attraction.

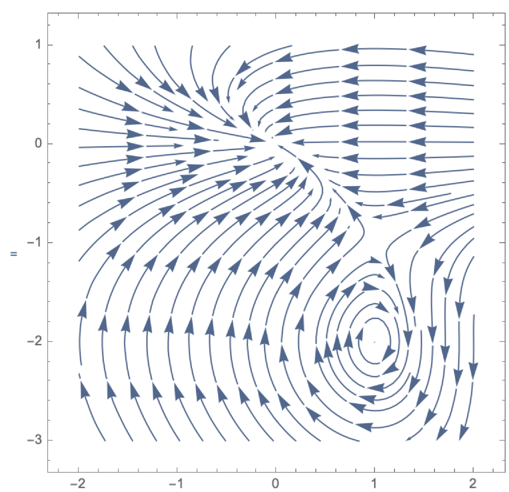

Figure 3: The basin of attraction

THEOREM 6.15 (LaSalle's Invariance Principle) Let X^* be an equilibrium point for $X' = F(X)$ and let L be a Liapunov function for X^* on a bounded open set U containing X^*. Let $P \subseteq U$ be a neighborhood of equilibrium point X^* that is closed and bounded. Suppose that P is positively invariant, and that there's no entire solution in $P \setminus \{X^*\}$ on which L is constant. Then X^* is asymptotically stable and P is contained in the basin of attraction of X^* [3].

For a proof of the theorem, see [3].

In Example 6.12, instead of considering $V = x^2 + 2y^2 + z^2$. It's much easier to consider z separately from (x, y). So the Liapunov function becomes:

$$V(t) = x^2 + 2y^2$$

$$\dot{V}(t) = -2|\epsilon|(z+1)(x^2 + 2y^2)$$

where $\epsilon < 0$. Since $z' = -z^3$, we know that $z(t)$ goes to zero as $t \to \infty$ and $\dot{z}|_{z=-1} = 1 > 0$. So z is increasing along the trajectory. Hence $P = \{(x, y, z) \in \mathbf{R^3} : z > -1\}$ is a positively invariant set. Then

$$\dot{V}(t) = -2|\epsilon|(z+1)V(t)$$

We can say that $\dot{V}$ can be bounded by $-cV$ with constant $c > 0$, so V tends exponentially fast to zero for $z(t) > -1$ as $t \to \infty$, which forces $x(t), y(t)$ to go to zero as well. Thus, any solution starting at $(x, y, z) \in P$ tends to zero. The basin of attraction of the origin contains the region $z > -1$.

We can take an another look at the damped pendulum problem that we discussed previously. The LaSalle's Invariance Principle can be applied to the equilibrium $X^* = (0, 0)$.

Recall that the Liapunov function of the damped oscillation is given by $E(\alpha, v) = \frac{1}{2}v^2 + 1 - \cos\alpha$ and $\dot{E} = -kv^2$. When $v = 0$, $\dot{E} = 0$, so the Liapunov function is not strict.

In order to determine the basin of attraction of the origin, fix a constant c with $0 < c < 2$, and define

$$P_c = \{(\alpha, v)|E(\alpha, v) \leq c \; and \; |\alpha| < \pi\}.$$

It is obvious that the equilibrium $(0,0) \in P_c$. We shall prove that P_c is contained in the basin of attraction of $(0,0)$.

First we want to show that the set P_c is positively invariant. Suppose that $(\alpha(t), v(t))$ is a solution of with $(\alpha(0), v(0) \in P_c$. We claim that $(\alpha(t), v(t)) \in P_c$ for all $t \geq 0$. Also we have $E(\alpha(t), v(t)) \leq c$ since $\dot{E} \leq 0$. If $|\alpha(t)| \geq \pi$, then there must exist a smallest time t_0 such that $\alpha(t_0) = \pm\pi$. However, then

$$\begin{aligned} E(\alpha(t_0), v(t_0)) &= E(\pm\pi, v(t_0)) \\ &= \frac{1}{2}v(t_0)^2 + 1 - \cos(\pm\pi) \\ &\geq 2 \end{aligned}$$

But, by the assumption we have

$$E(\alpha(t_0), v(t_0)) \leq c < 2.$$

It is a contradiction which shows that $\alpha(t_0) < \pi$, hence P_c is positively invariant.

Next we want to show that there is no entire solution in $P_c - X^*$ on which E is constant. Suppose that there is such a solution. Then along the solution we have $\dot{E} \equiv 0$ so $v \equiv 0$. Recall that

$$\begin{cases} \alpha' &= v \\ v' &= -kv - \sin\alpha \end{cases}$$

Then $v \equiv 0$ gives us $\alpha' = 0$, so α is a constant on the solution. Also we have $v' = -\sin\alpha = 0$ on the solution. Since $|\alpha| < \pi$, then it follows that $\alpha \equiv 0$. Thus in this case the only entire solution in P_c on which E is constant is the equilibrium point $(0,0)$, that is, there is no entire solution in $P_c - X^*$ (except the equilibrium solution) on which E is constant.

Also we can check that P_c is a closed set. Because if (α_0, v_0) is a limit point of the set P_c, then we will have $|\alpha_0| \leq \pi$ and $E(\alpha_0, v_0) \leq c$ by continuity of E. But as shown above $|\alpha_0| = \pi$ implies that $E(\alpha_0, v_0) > c$. Thus $|\alpha_0| < \pi$ and (α_0, v_0) does belong to P_c, so we prove that P_c is a closed set.

By applying the LaSalle's Invariance Theorem we conclude that the origin is asymptotically stable and P_c is contained in the basin of attraction of equilibrium $(0, 0)$ for each $0 < c < 2$. So when we take the union of P_c for each $0 < c < 2$, that is,

$$P = \bigcup \{P_c | 0 < c < 2\}$$

is also contained in the basin of attraction of the origin. We can rewrite the set as

$$P = \{(\alpha, v) | E(\alpha, v) < 2 \text{ and } |\alpha| < \pi\}.$$

7. PERIODIC SOLUTIONS

7.1. Periodicity

For the linear differential systems, we can say that their stability is determined by the property of the equilibrium points. When analysing nonlinear systems, so far we focused on analysing the critical points and how trajectories behave in the neighborhood of critical point. However, there also exist another important possibility that one of the trajectories traces out a closed curve Γ. In this case, the solution $X(t) = (x(t), y(t))$ will go around and around the curve Γ with a certain period T, which means that

$$x(t+T) = x(t)$$

$$y(t+T) = y(t)$$

for all t. For such a closed curve Γ, the trajectories near Γ may exhibit various properties such as staying close or away from the closed orbit Γ instead of an equilibrium point. Thus, the concept of orbital stability is emerged. These possibilities are illustrated below, but we prove a relevant lemma first.

LEMMA 7.1 For the autonomous system

$$\frac{dx}{dt} = F(x,y), \quad \frac{dy}{dt} = G(x,y), \tag{7.39}$$

suppose $x = \phi(t), y = \psi(t), -\infty < t < \infty$, is a solution of the system. If the trajectory generated by this solution is closed, then the solution is periodic.

Proof. Since the trajectory is closed, there exist at least one point (x_0, y_0) such that $\phi(t_0) = x_0, \psi(t_0) = y_0$ and $\phi(t_0+T) = x_0, \psi(t_0+T) = y_0$ for some t_0 and $T > 0$. We have

$$\frac{d\phi(t)}{dt} = F(\phi(t), \psi(t)), \quad \frac{d\psi(t)}{dt} = G(\phi(t), \psi(t)) \quad \text{for all t.}$$

Define $\Phi(t) = \phi(t+T), \Psi(t) = \psi(t+T)$, then we have

$$\frac{d\Phi}{dt}(t) = \frac{d\phi}{dt}(t+T) = F[\phi(t+T), \psi(t+T)] = F[\Phi(t), \Psi(t)],$$
$$\frac{d\Psi}{dt}(t) = \frac{d\psi}{dt}(t+T) = G[(\phi(t+T), \psi(t+T)] = G[\Phi(t), \Psi(t)],$$

Thus, $x = \Phi(t) = \phi(t+T), y = \Psi(t) = \psi(t+T)$ is a solution to the autonomous system (7.39). Also,

$$\Phi(t_0) = \phi(t_0+T) = \phi(t_0) = x_0$$
$$\Psi(t_0) = \psi(t_0+T) = \psi(t_0) = y_0.$$

By the existence and uniqueness theorem, this solution and the given solution must be the same, that is,

$$\phi(t) = \Phi(t) = \phi(t+T), \quad \psi(t) = \Psi(t) = \psi(t+T).$$

for any t. Hence the given solution is periodic. □

7.2. Limit Cycle

Limit cycle is a unique property of nonlinear systems.

DEFINITION 7.2 A isolated closed trajectory C is called a limit cycle if the nearby trajectories can either spiral in toward C or spiral away from C.

According to the manner of motion of the trajectory and whether the nearby curves spiral towards C, away from C, or both, there exist three types of the limit cycle:

- *Stable limit cycle:* All the trajectories around the limit cycle converge to it, see Figure 4(a).

• *Unstable limit cycle:* All the trajectories around it go away from it, see Figure 4(b).

•*Semi-stable limit cycle:* One part of the trajectory approached it, another part of the trajectory stay away from it, see Figure 4(c).

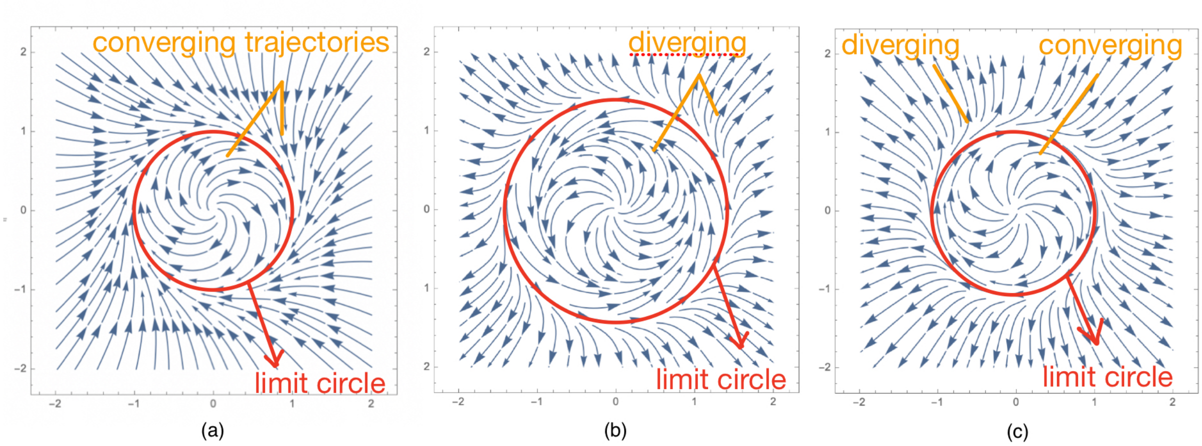

Figure 4: Three types of limit cycles

The stability mentioned exists as a limit of orbit, so it is called orbital stability. Stable limit cycle represents a stable periodic state, which has a significant meaning in nonlinear systems. A prominent method to determine the existence of limit cycles is the famous Poincare-Bendixson Theorem, see [3].

THEOREM 7.3 (Poincare-Bendixson Theorem) Suppose D is a finite region of the plane lying between two simple closed curves L_1 and L_2. Suppose that V is a vector field for the system such that

(a) At each point of L_1 and L_2, the field V points toward the interior of D,

(b) D contains no critical points.

Then the system has a closed trajectory lying inside D, namely, a periodic solution.

Recall that the equation of nonlinear pendulum takes the form

$$\begin{cases} \alpha' &= v \\ v' &= -kv - \sin\alpha, \end{cases}$$

EXAMPLE 7.4 We can further explore the nonlinear pendulum problem, if we add a constant torque to the pendulum in the counterclockwise direction, that is to say, the nonlinear system becomes

$$\alpha' = f(\alpha, v) = v$$

$$v' = g(\alpha, v) = -kv - \sin\alpha + b,$$

where the constant $b \geq 0$ and $0 < \alpha < 2\pi, \alpha = mod(2\pi)$. It can be considered that the system is defined on a cylinder $S^1 \times \mathbb{R}$, where S^1 denotes the unit circle, see figure 5. By letting right-hand side of the system equal 0, we have

$$v = 0 \quad and \quad \alpha = \sin^{-1}(b)$$

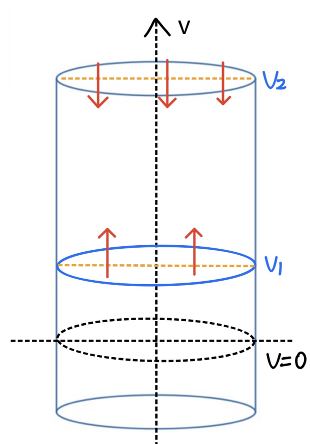

Figure 5: The cylinder $S^1 \times \mathbb{R}$

Since we have $\sin\alpha = b$, if $0 < b < 1$, there are two intersect points of the graph which induces two critical points. Similarly, if the $b = 1$, there will be only one equilibrium point. Next we mainly look at the third case where $b > 1$ and show the existence and uniqueness of the periodic solution for this system.

When $b > 1$, we notice that there's no equilibrium point for the system, which satisfies the second condition of Poincare-Bendixson Theorem. Thus, we may explore the possibility of the existence of the periodic solution. Since $\frac{b-\sin\alpha}{k}$ is positive and bounded, there exist $v_2 > v_1 > 0$ such that

$$0 < v_1 < \frac{b - \sin\alpha}{k} < v_2$$

Let $I = \{y \in \mathcal{R} : v_1 \leq v \leq v_2\} = [v_1, v_2] \in \mathcal{R}$ and $\Omega = S^1 \times I$. Note that

$$\begin{aligned} v' &= -kv - \sin\alpha + b \\ &= k(\frac{b - \sin\alpha}{k} - v) \end{aligned}$$

Hence, we have $v' > 0$ for $0 < v < v_1$ and $v' < 0$ for $v > v_2$. Therefore, $v = v(t)$ is increasing on the cylinder $S^1 \times I$ when v is near v_1 and decreasing on the cylinder $S^1 \times I$ when v is near v_2. Hence, the vector field out of the system is invariant on $\Omega = S^1 \times I = S^1 \times [v_1, v_2]$. It is obvious that Ω is homomorphic to the annulus $D = \{(x, y) \in \mathcal{R}^2 : v_1 \leq \sqrt{x^2 + y^2} \leq v_2\}$, as shown in figure 6.

Therefore, the vector field at each points on the curves $v = v_1$ and $v = v_2$ are approaching toward the interior of the region D, which meets the first condition of the Poincare-Bendixson Theorem. Then we can apply the Poincare-Bendixson Theorem to the system on Ω to conclude that the nonlinear pendulum problem with a constant torque has a periodic solution when $b > 1$. Thus we proved the existence of the periodic solution for this system. Next, we show the uniqueness of this periodic solution for the system.

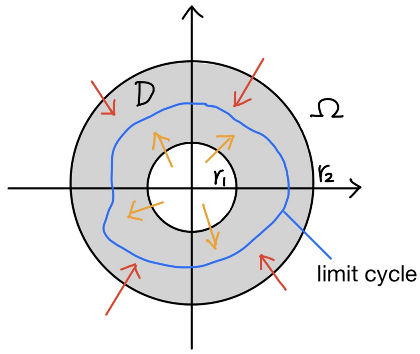

Figure 6: The annulus D

Let (α, v) be one periodic solution guaranteed by Poincare-Bendixson Theorem. Recall that the energy function of pendulum problem is

$$E(\alpha, v) = \frac{1}{2}v^2 + 1 - \cos\alpha$$

Fix $b > 1$. Since the net change of energy along any periodic solution is zero, we have

$$\int_0^{2\pi} \frac{dE}{d\alpha} d\alpha = 0$$

So,

$$\begin{aligned}
\frac{dE}{d\alpha} &= v\frac{dv}{d\alpha} + \sin\alpha \\
&= v\frac{\frac{dv}{dt}}{\frac{d\alpha}{dt}} + \sin\alpha \\
&= v\frac{-kv - \sin\alpha + b}{v} + \sin\alpha \\
&= -kv - \sin\alpha + b + \sin\alpha \\
&= -kv + b
\end{aligned}$$

Then the integral along the periodic solution should be zero since the energy is conserved. If v represents the velocity where $p(v) = v$, then

$$\begin{aligned}\int_0^{2\pi} \frac{dE}{d\alpha} d\alpha &= \int_0^{2\pi} (-kv + b) d\alpha \\ &= -k \int_0^{2\pi} v d\alpha + b\alpha|_0^{2\pi} \\ &= -k \int_0^{2\pi} v d\alpha + 2\pi b \\ &= 0\end{aligned}$$

$$\Rightarrow \int_0^{2\pi} v d\alpha = \frac{2\pi b}{k},$$

which shows that for any periodic solution (α, v), $\int_0^{2\pi} v d\alpha$ remains the same constant $\frac{2\pi b}{k}$. We prove the uniqueness by contradiction. Suppose that there exists another periodic solution (α, v^*). Since the trajectories cannot cross each other in the vector field, due to the uniqueness of the solution satifying an initial condition and that the system is autonomous, we must have either $v(\alpha) < v^*(\alpha)$ for all α or $v(\alpha) > v^*(\alpha)$ for all α.

So by the monotonicity of the integral, we either have

$$\begin{aligned} &\int_0^{2\pi} v(\alpha) d\alpha < \int_0^{2\pi} v^*(\alpha) d\alpha \\ \textit{or} \quad &\int_0^{2\pi} v(\alpha) d\alpha > \int_0^{2\pi} v^*(\alpha) d\alpha \end{aligned}$$

In either of these two cases, $\int_0^{2\pi} v^* d\alpha$ will never be equal to $\frac{2\pi b}{k}$, which contradicts to the fact that any periodic solution satisfies $\int_0^{2\pi} v d\alpha = \frac{2\pi b}{k}$. Hence (α, v^*) cannot be another periodic solution. Therefore, there is a unique periodic solution for this system when $b > 1$.

So far we illustrated that the existence of the limit cycle may be determined by Poincare-Bendixson Theorem. And there are two theorems which can sometimes be used to show that a limit cycle does not exist.

THEOREM 7.5 (Bendixson's Criterion) If f_x and g_x are continuous on a region D

which is simply-connected and

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \neq 0$$

at any point of D, then the system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

has no closed trajectories inside D.

Proof. Prove by contradiction. Suppose there exists a closed trajectory Γ. Apply Green's Theorem, we have

$$\oint_\Gamma (-g dx + f dy) = \int\int_\Omega (f_x + g_y) dx dy$$

where Ω is in the interior of Γ. Suppose that the closed trajectory is $x = x(t), y = y(t), \alpha \leq t \leq \beta$. Then the left-hand side is

$$\oint_\Gamma (-g dx + f dy) = \int_\alpha^\beta (-g(x(t), y(t))x'(t) + f(x(t), y(t))y'(t)) dt = 0$$

since it satisfies the differential equation. However, the integrand of right-hand side is always positive or negative, which results in the integral never equal 0. This contradiction shows that there is no closed orbit lying in D. □

As an obvious consequence of Theorem 7.5, we have

THEOREM 7.6 A closed trajectory has a critical point in its interior.

See book [1] we note the distinction between this theorem, which says that limit cycles enclose regions which do contain critical points, and the Poincare-Bendixson theorem, which seems to imply that limit cycles tend to lie in regions which don't contain critical points. The difference is that while the region D_2 used in Theorem 7.6, must contain a critical point, the region D, used in Poincare-Bendixson Theorem, always contain a hole in which all the critical points lie. See the following graphic illustration, figure 7, where D_1 is a hole. Therefore, the two circumstances where these two theorems are valid are in fact completely consistent.

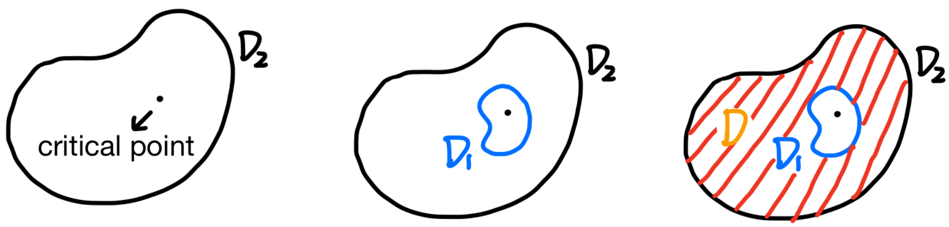

Figure 7: The graphs for theorems

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