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On the Chromatic Numbers of Subgroup Lattices

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ON THE CHROMATIC NUMBERS OF SUBGROUP LATTICES

A Master's Thesis

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In partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

In this thesis we investigate the chromatic number of the Hasse diagram of a subgroup lattice. We combine results of Bollobás and Tůma to show that there exist infinite groups whose subgroup lattices have arbitarily high chromatic numbers. We show that finite supersolvable groups have bipartite subgroup lattices but that CLT and non-solvable groups may not have bipartite subgroup lattices. Lastly, we give a preliminary argument suggesting that there are an infinite number of non-solvable groups whose subgroup lattices are bipartite.

KEYWORDS: subgroup lattice, chromatic number, supersolvable, projective special linear groups, lattice, Hasse diagram

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1. INTRODUCTION: BASICS OF LATTICE THEORY AND GRAPH THEORY

Much of pure mathematics deals with highly abstract structures, and because of this abstractness, it is often a slow and difficult process for one to gain intuition about the structure. A common method of gaining intuition is to represent the abstract structure visually as a two-dimensional drawing known as a **graph**. Graphs are drawn using only points and edges (that is, a line segment or a curve), and thus they are a simple and familiar visualization. Representing an abstract structure as a graph provides insight into the relationships between members of the structure. But perhaps of equal importance, the graph itself suggests numerous other questions. One of these questions is to find the **chromatic number** of a graph, which is, in short, the smallest number of colors needed to color a graph so that no two points connected by an edge have the same color. There is wide variation in the chromatic numbers of certain types of graphs, and finding the chromatic number of a given graph is a notoriously difficult problem. In this thesis we will restrict our attention to finding the chromatic numbers of the category of graphs known as **subgroup lattices**.

The subgroup lattice of a group is a powerful tool used to visualize a group; it can be used for classification of the group and for determining normality of a subgroup, amongst other things. Subgroup lattices, especially those of groups of large order, can become quite complex. However, all of the groups considered in an introductory group theory course have small chromatic numbers - only two or three - which suggests that subgroup lattices may (or may not) have a great deal more structure than other types of graphs. This observation provides the motivation for the entirety of our work.

Before examining subgroup lattices, it is first necessary to introduce some basic concepts in lattice theory and graph theory.

1.1 Lattice Theory

Definition 1.1. A set S is a **partially ordered set**, or **poset**, if S is a nonempty set equipped with a relation $\leq$ that satisfies the following properties. For all $x, y, z \in S$,

1) (Reflexivity) $x \leq x$;

2) (Antisymmetry) if $x \leq y$ and $y \leq x$, then $x = y$; and

3) (Transitivity) if $x \leq y$ and $y \leq z$, then $x \leq z$.

We write $x < y$ if $x \leq y$ but $x \neq y$.

Definition 1.2. Let S be a poset and $A \subseteq S$. An **upper bound** of A in an element $u \in S$ with the property that for all $a \in A$, then $a \leq u$. Similarly, an element $l \in S$ is a **lower bound** of A if for all $a \in A$, then $l \leq a$. We call $u' \in S$ a **least upper bound**, or **supremum**, of A if u' is an upper bound of A and $u' \leq u$ for all upper bounds u of A. Similarly, we call $l' \in S$ a **greatest lower bound**, or **infimum**, of A if l' is a lower bound of A and $l \leq l'$ for all lower bounds l of A.

For a poset S, it is possible that $A \subseteq S$ has no upper or lower bounds and hence no supremum or infimum. However, if A does have a supremum or infimum, it follows immediately by antisymmetry that the supremum or infimum is unique.

Definition 1.3. A poset L is called a **lattice** if every pair of elements $x, y \in L$ has a supremum and infimum in L. The supremum of x and y is often called the **join** of x and y and is denoted $x \vee y$. The infimum of x and y is often called the

meet of x and y and is denoted $x \wedge y$. For any subset $A = \cup_{i \in I} a_i \subseteq L$, if A has a join or meet we will denote it as $\vee_{i \in I} a_i$ or $\wedge_{i \in I} a_i$. We say L is **complete** if every nonempty subset of L has both a join and a meet.

With this notation we make a simple observation about operations in a lattice L that follows immediately from the definition of supremum: if $a, b \in L$ and $a \vee b = x$, then if y is any element such that $a \leq y$ and $b \leq y$, then $x \leq y$. An analogous property holds for infimums, and we shall use these properties in the next proposition.

Proposition 1.4. Join and meet operations $\vee$ and $\wedge$ in a lattice are commutative and associative.

Proof. We show that these properties hold for the join operation $\vee$; the proof is similar for the meet operation $\wedge$. To show commutativity, let $a, b \in L$. Suppose $x = a \vee b$ and $y = b \vee a$. So $a \leq x$, $b \leq x$, and $x \leq z$ for any $z \in L$ with $a \leq z$ and $b \leq z$. Similarly $a \leq y$, $b \leq y$, and $y \leq z$ for any z with $a \leq z$ and $b \leq z$. Thus $x \leq y$ and $y \leq x$, and by antisymmetry $x = y$. To show associativity, suppose $(a \vee b) \vee c = x$ and $a \vee (b \vee c) = y$. Since $a \vee b \leq x$, by transitivity we have $a \leq x$ and $b \leq x$. Since we also have $c \leq x$, it follows that $b \vee c \leq x$. Now we have $a \leq x$ and $b \vee c \leq x$ so that $y = a \vee (b \vee c) \leq x$. An analogous argument shows that $x \leq y$ so that by antisymmetry $x = y$. $\square$

Definition 1.5. A lattice L is **bounded** if there exist elements 0 and 1 (not necessarily denoting the integers 0 and 1) in L such that for every $x \in L$, $0 \leq x \leq 1$. We call 0 and 1 the **minimal** and **maximal** elements, respectively, of L.

Proposition 1.6. Every finite lattice L is bounded.

Proof. We prove the contrapositive. Suppose L is not bounded, and without loss of generality assume L has no minimal element. Thus no element $l \in L$ has the property that $l \leq x$ for all $x \in L$. So every element $l \in L$ has the property that $l > x$ for some $x \in L$. Thus there exists a sequence $l_1 > l_2 > l_3 > \ldots$ of elements in L so that L is not finite. $\square$

Example 1.7. We include here some examples and non-examples of lattices.

1) The set $\mathbb{Z}$ of integers under the normal ordering $\leq$ is a lattice. For $x, y \in \mathbb{Z}$, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$.

2) The set $\mathbb{Z}^+$ of positive integers is a lattice under the ordering relation of divisibility. It is easy to see that $\mathbb{Z}$ is a poset, and furthermore, $x \vee y = \operatorname{lcm}(x, y)$, and $x \wedge y = \gcd(x, y)$. Clearly $\mathbb{Z}^+$ is not bounded since there is no maximal element.

3) The set of all positive divisors of 24, namely $\{1, 2, 3, 4, 6, 8, 12, 24\}$, is also a lattice under the ordering relation of divisibility since the greatest common divisors and least common multiples of all pairs of elements are contained in this set. However, if the element 24 is removed, the remaining set $\{1, 2, 3, 4, 6, 8, 12\}$ is not a lattice since 6 and 8 have no supremum in this set; in fact, these elements do not have any upper bounds.

4) Let G be a group, and let $\mathcal{P}$ be the set of all subgroups of G. Suppose $H, K \in \mathcal{P}$. When $H \leq G$ means, as usual, that H is a subgroup of G, then $\mathcal{P}$ is a lattice with $H \vee K = \langle H, K \rangle$ and $H \wedge K = H \cap K$. The trivial subgroup 1

is the minimal element, and the group G is the maximal element so that $\mathcal{P}$ is bounded. We refer to this lattice as the **lattice of subgroups**, or **subgroup lattice**, of G. This lattice is the primary lattice with which we will be concerned in Section 2.

There are many properties of lattices that are worthwhile of study, but we shall focus on giving visual representations of lattices. The most common way to visualize a lattice L is via the **Hasse diagram**, of L. In order to understand Hasse diagrams and the questions we will ask about them, we introduce some basic concepts in graph theory.

1.2 Graph Theory

Definition 1.8. Let V be a nonempty set, and let E be a set containing unordered pairs of elements of V. Together V and E form a **graph** G. The set V is called the **vertex set** of G, and the set E is called the **edge set** of G. An element $v \in V$ is called a **vertex**, and an element $\{v, v'\} \in E$ is called an **edge**. We say that v is **adjacent** to v' if $\{v, v'\} \in E$.

Note that since elements of E are *unordered* pairs, the relationship of adjacency is symmetric: that is, if v is adjacent to v', then v' is adjacent to v. Thus we are justified in simply saying that v and v' are adjacent.

The geometric language of *graph*, *vertex*, and *adjacent* is far from a coincidence. Indeed, we may imagine each vertex of a graph G to be drawn as a point in the plane, and each pair of adjacent elements v and v' as connected by an edge, drawn as a line segment (or arc) with endpoints v and v'. Visualizing G in this way tremendously aids our intuition while preserving all necessary mathematical rigor, and hence we shall often analyze G solely by its geometric visualization and without explicit references to V or E.

Example 1.9. We give an example in which we produce the geometric visualization, as described above, of an abstractly defined graph G. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ and

$$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}\}.$$

Representing each element of V as a vertex and connecting the two elements of each unordered pair in E by a line segment or arc gives the graph G in Figure 1.

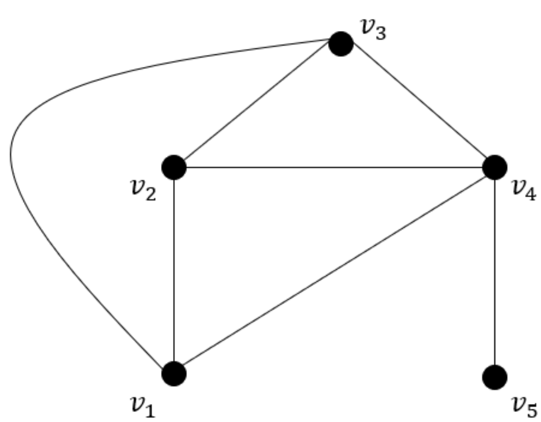

Figure 1: The geometric visualization of the graph G from Example 1.8.

Definition 1.10. The **order** of G is the cardinality of V, and the **size** of G is the cardinality of E. The **complete graph on** r **vertices**, denoted K_r, is the unique graph of order r and size $\frac{r(r-1)}{2}$. That is, K_r is the graph in which the vertices v_i and v_j are adjacent for all $i \neq j$, where $1 \leq i, j \leq r$.

Intuitively, the complete graph K_r can be constructed by drawing r vertices and connecting every pair of vertices. Although we have not given a formal notion of what it means for two graphs to be identical, this construction makes it clear that there is only one complete graph on r vertices.

Definition 1.11. Let G be a graph with vertices u and v. A **path** from u to v is a sequence $P = \{u = v_1, v_2, v_3, ..., v_n = v\}$ of distinct, adjacent vertices of G. The **length** of the path P is $n - 1$, the number of edges used to connect consecutive vertices in P. A **cycle** is a sequence $C = \{v = v_1, v_2, v_3, ..., v_m = x\}$ of at least 3 adjacent vertices in G where each vertex in C is distinct other than the first and last elements $v = v_1$ and $v = v_m$. Since a cycle has the same beginning and ending element, we simply say that C is a cycle beginning at v, and we say that C has length $m - 1$. The smallest cycle length of any cycle in G is called the **girth** of G and is denoted $g(G)$. If no cycles exist in G, then we write $g(G) = \infty$.

The condition that a cycle must contain at least 3 distinct vertices is to ensure that a path of the form $\{v_1, v_2, v_1\}$ is not considered a cycle. Thus every cycle must have length at least 3 so that for any graph G, $g(G) \geq 3$ or $g(G) = \infty$.

Definition 1.12. A **coloring** of a graph G is an assignment of colors to the vertices of G so that each vertex is assigned exactly one color and no two adjacent vertices are assigned the same color. An ***n*-coloring** of G is a coloring of G using n colors (where it is possible that some of these n colors may not be assigned to any vertices). The smallest number p for which there exists a p-coloring of G is called the **chromatic number** of G. We write $\chi(G) = p$ for the chromatic number of G. In the special case that $\chi(G) = 2$, we say that G is **bipartite**.

We note that when coloring a graph, it is convenient to denote the colors simply by positive integers. We shall follow this custom.

Note also that by the definition of a k-coloring, if there is a k-coloring of G but there is not a $(k-1)$-coloring of G, then $\chi(G) = k$.

Example 1.13. The graph G from Example 1.9 has a path $\{v_1, v_2, v_4, v_5\}$ from v_1 to v_5 of length 3. The sequences $\{v_1, v_4, v_3, v_2, v_1\}$ and $\{v_4, v_2, v_3, v_4\}$ are cycles beginning at v_1 and v_4 and having lengths 4 and 3, respectively. Since there exists a cycle of length 3 and no cycle can have a length smaller than 3, we conclude that $g(G) = 3$.

We show that for the graph G from Example 1.9, $\chi(G) = 4$ by giving a 4-coloring of G and showing that no 3-coloring exists. A 4-coloring of G is exhibited in Figure 2. To show that no 3-coloring exists, we note that the four vertices v_1, v_2, v_3, and v_4 are mutually adjacent (that is, there is a copy of K_4 within G), and hence we must assign four different colors to these vertices. Thus no 3-coloring of G exists, and $\chi(G) = 4$.

Unfortunately, determining the chromatic number of a graph G is, in general, quite difficult, especially when G is of large order. However, as in Example 1.13, if we can find a graph of chromatic number greater than or equal to k contained in G, then it must be that $\chi(G) \geq k$. This simple tool will be used frequently and is formalized in Proposition 1.15.

Definition 1.14. A **subgraph** H of a graph G is a graph in which every vertex of H is a vertex of G and every pair of adjacent elements in H is also adjacent in G.

Proposition 1.15. If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof. Suppose $\chi(G) = p$. Such a p-coloring of G automatically gives a p-coloring of H so that $\chi(H) \leq \chi(G)$. □

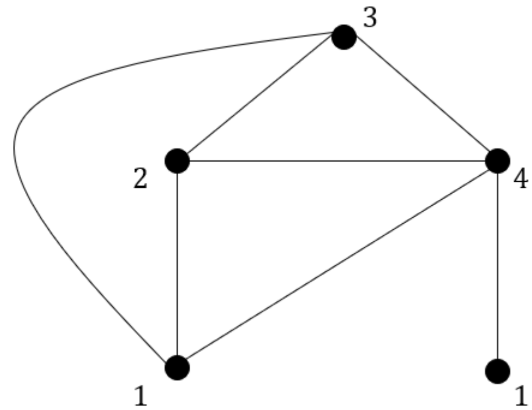

Figure 2: A 4-coloring of the graph G in Example 1.13.

Definition 1.16. A graph G is said to be **triangle-free** is there are no triangles when G is drawn. That is, there are no cycles of length 3 in G.

We are now ready to return to the Hasse diagram of a lattice and present fundamental results about it.

1.3 Results on Hasse Diagrams

Definition 1.17. The **Hasse diagram**, or **covering graph**, of a lattice L is the graph H formed by drawing each element $x \in L$ as a vertex of H and forming edges between vertices as follows: the elements $x_1, x_2 \in L$ are adjacent in H if and only if $x_1 < x_2$ and there is no $x_3 \in L$ with $x_1 < x_3 < x_2$. We denote the Hasse diagram of L by $G(L)$.

We note that in order to form the Hasse diagram of L, it is not necessary that L be a lattice. Rather, L need only be a poset, for the formation of the Hasse diagram does not at all depend on joins and meets. However, we shall form the Hasse diagram almost exclusively for lattices.

The next proposition follows immediately from the definition of the Hasse diagram.

Proposition 1.18. The Hasse diagram of a lattice L is triangle-free.

Proof. Suppose to the contrary that there exists a triangle in $G(L)$, that is, there exists a cycle $C = \{x, y, z, x\}$ of length 3 in $G(L)$ with distinct elements x, y, and z. We may assume without loss of generality that $x < y < z$. But then $x \leq y \leq z \leq x$ so that by antisymmetry and transitivity we have $x = y = z$, contradicting the assumption that these elements are distinct. □

The graph K_r has chromatic number r, and thus it is simple to create a graph of high chromatic number. However, K_r has many triangles for $r \geq 3$, so the fact that the Hasse diagram of a lattice is triangle-free may, at first glance, seem to limit the chromatic number. The following result from Béla Bollobás [1] quickly puts this notion to rest. In our proof we use the same argument as Bollobás but give extra detail and clarity.

Theorem 1.19 (Bollobás)**.** Given a natural number k there is a lattice L whose Hasse diagram has chromatic number greater than k.

Proof. Let H be a graph of finite order and $\chi(H) = k + 1$, and choose H so that $g(H) > 4k$ (such graphs exist via Paul Erdős [3]). We construct a lattice L containing H as a subgraph so that by Proposition 1.15, $\chi(G(L)) \geq \chi(H)$. Let $V_i = \{x \in H \,|\, x \text{ is colored with } i \text{ in } H\}$, $1 \leq i \leq k + 1$. Define a partial order on $V = \cup_{i=1}^{k+1} V_i$ as follows. Put $x \leq y$ if $x = y$ or there exists a path of increasing color classes from x to y, that is, if there exists a sequence $S = \{x = x_0, x_1, x_2, ..., x_n = y\}$ with the property that for all $x_i, x_j \in S$, if $i < j$ and $x_i \in V_{i'}$ and $x_j \in V_{j'}$, then

$i' < j'$. It is straightforward to show that this ordering on V makes V into a poset. We now show that if any pair of elements in V have an upper or lower bound, then they have a supremum or infimum. Let $x_1, x_2 \in V$, and suppose there exist distinct elements $y_1, y_2 \in V$ such that y_1 and y_2 are both minimum elements greater than both x_1 and x_2. Thus for $1 \leq i \leq 2$ and $1 \leq j \leq 2$, x_i is adjacent to y_j in H so that there exists a path of increasing color classes from x_i to y_j. Since each such path is of increasing color classes and there are $k+1$ color classes, each of these 4 paths has length no more than k. But then there exists a cycle beginning at x_1 with length no more than $4k$, contradicting the fact that $g(H) > 4k$. Thus any pair of elements in V with an upper bound has a supremum, and an analogous argument shows that any pair of elements in V with a lower bound has an infimum. Add the two elements 0 and 1 to V to form $L = V \cup \{0, 1\}$. Now every pair of elements in L has a supremum or infimum so that L is a lattice. We now show that H is a subgraph of L. To do this, we show that every vertex of H is in L and that two adjacent vertices in H are adjacent in $G(L)$. First, $L = H \cup \{0, 1\}$, so every vertex of H is in L. Now suppose a and b are adjacent in H. Thus a and b belong to different color classes, say $a \in V_i$ and $b \in V_j$ with $i < j$. Thus there is a path of length 1 of increasing color classes from a to b in H so that $a < b$ in L. Now a and b are adjacent in $G(L)$ if and only if there does not exist $c \in L$ with $a < c < b$. We show by contradiction that no such c exists. If there exists $c \in L$ with $a < c < b$, then c cannot be 0 or 1. Thus $c \in H$. Since $a < c$ and $c < b$, there exist paths of increasing color classes from a to c and from c to b in H, and these paths both have length at most k since there are $k+1$ distinct color classes. Thus there is a cycle in H beginning at

a with length at most $2k+1$, contradicting the fact that $g(H) > 4k$. Thus no such $c \in L$ exists, and a and b are adjacent in $G(L)$. □

Bollobás's theorem affirms that lattices may have arbitrarily high chromatic numbers. It is natural to ask if this theorem can be strengthened further to *subgroup* lattices: that is, can subgroup lattices have arbitrarily chromatic numbers? While the answer to this question is unknown for finite groups, we will show that infinite groups may have subgroup lattices of arbitrarily high chromatic number. We must first consider the notion of an **algebraic** lattice.

Definition 1.20. An element x in a lattice L is called **compact** if $x \leq \vee_{i \in I} y_i$ implies that there is a finite subset $J \subseteq I$ such that $x \leq \vee_{j \in J} y_j$. A complete lattice is called **algebraic** if every element $z \in L$ is a join of compact elements in L.

Proposition 1.21. A finite lattice L is algebraic.

Proof. We will show that L is complete and that every element of L is compact so that, trivially, every element of $x \in L$ can be represented as the join $x = x \vee 0$. Let $A \subseteq L$ with $A \neq \emptyset$. Since L is finite, so is A. We induct on the order of A. Since L is a lattice, the result holds when $|A| = 2$. Now assume $|A| = n$ and that the result holds for all natural numbers $k < n$. Write $A = \cup_{i=1}^{n} a_i$. By the induction hypothesis the set $\{a_1, a_2, ..., a_{n-1}\}$ has a join $b \in L$, and by associativity the join of A can be written as $b \vee a_n$, which is an element of L since it is the join of 2 members of L. An analogous argument shows that every subset of L also has a meet in L so that L is complete. Showing that every $x \in L$ is compact is trivial: for if $x = \cup_{i \in I} y_i$ for a subset $I \subseteq L$, then x is immediately seen to be compact since any subset of L is finite. Now $x = x \vee 0$ is the join of compact elements in L so that L is algebraic. □

Definition 1.22. For a poset S and a subset $I \subseteq S$, I is said to be an **interval** of S if for all $x, y \in I$ and any $z \in L$ with $x \leq z \leq y$, then $z \in I$.

The next theorem is from Jiří Tůma in [9].

Theorem 1.23 (Tůma)**.** Every algebraic lattice is isomorphic to an interval in the subgroup lattice of an infinite group.

Combining Tůma's result with that of Bollobás, we obtain an interesting theorem.

Theorem 1.24. There exist infinite groups whose subgroup lattices have arbitrarily high chromatic number.

Proof. From Bollobás we obtain a finite lattice L such that $G(L)$ is of arbitrarily high chromatic number. We know that L is algebraic by Proposition 1.21. By Tůma an isomorphic copy of L can be found as an interval in the subgroup lattice of an infinite group G. Thus the subgroup lattice of G contains $G(L)$ as a subgraph so that by Proposition 1.15, the subgroup lattice of G has a chromatic number at least as large as that of $G(L)$. □

Having seen that infinite groups can have subgroup lattices of arbitrarily high chromatic number, it is desirable to examine the chromatic numbers of the subgroup lattices of finite groups. Indeed, this is the focus of Section 2. Before then, we consider a few more important results relating to the chromatic number of the Hasse diagram of a lattice.

Definition 1.25. Let L be a finite lattice and $x, y \in L$. A **chain** from x to y is a sequence $\{x = x_0, x_1, x_2, ..., y = x_n\}$ of elements in L where $x = x_0 < x_1 < x_2 < ... < y = x_n$. The **length** of a chain is n, that is, one less than the number of elements in the chain. A chain is said to be **maximal** if for all $1 \leq i \leq n$, x_{i-1} and x_i are adjacent in the Hasse diagram of L. If for any two elements $x, y \in L$, the length of every maximal chain from x to y is the same, we say that L satisfies the **Jordan-Dedekind chain condition**, or **JDCC**. If the length of every maximal chain from x to y has the same parity, we say that L satisfies the ***mod* 2 JDCC**.

It is clear that every finite lattice satisfying the JDCC also satisfies the *mod* 2 JDCC. Thus Propositions 1.26, 1.27, and 1.28 below apply to lattices which satisfy the JDCC.

Proposition 1.26. If a finite lattice L satisfies the *mod* 2 JDCC, then $G(L)$ is bipartite.

Proof. Since L is finite, it is bounded by Proposition 1.6. Thus it has a minimal element 0. For $x \in L$, color x as follows. Choose a maximal chain from 0 to x of length n. If n is odd, color x with 1; if n is even, color x with 2. Since L satisfies the *mod* 2 JDCC, the length of every maximal chain from 0 to x has the same parity so that the coloring of x is independent of the chosen maximal chain. We now show that no two adjacent elements have the same color. Suppose x and y are adjacent and without loss of generality that $x < y$. Choose a maximal chain $C = \{0 = x_0, x_1, ..., x_m = x\}$ from 0 to x of length m. Since x and y are adjacent, the maximal chain $D = \{0 = x_0, x_1, ..., x_m = x, x_{m+1} = y\}$ from 0 to y containing C has length $m + 1$, which is not of the same parity as m. Now again using the *mod* 2 JDCC, every maximal chain from 0 to y has the same parity as $m + 1$. Thus x and y are colored differently, and $G(L)$ is bipartite. $\square$

The next proposition says that in order to show that a finite lattice satisfies the *mod* 2 JDCC, it suffices only to show that the length of every maximal chain from 0 to 1 has the same parity.

Proposition 1.27. For a finite lattice L, if the length of every maximal chain from 0 to 1 has the same parity, then L satisfies the *mod* 2 JDCC.

Proof. We prove the contrapositive. Suppose L does not satisfy the *mod* 2 JDCC, that is, suppose that for $x < y$ there exist maximal chains $B = \{x = b_0, b_1, ..., b_m = y\}$ and $B' = \{x = b'_0, b'_1, ..., b'_n = y\}$ from x to y with lengths m and n of opposite parities. Choose a maximal chain $A = \{0 = a_0, a_1, ..., a_j = x\}$ from 0 to x and a maximal chain $C = \{y = c_0, c_1, ..., c_k = 1\}$ from y to 1. Now consider the two chains D and D' from 0 to 1 given by

$$D = \{0 = a_0, a_1, ..., a_j = x = b_0, b_1, ..., b_m = y = c_0, c_1, ..., c_k = 1\}$$

$$D' = \{0 = a_0, a_1, ..., a_j = x = b'_0, b'_1, ..., b'_n = y = c_0, c_1, ..., c_k = 1\}.$$

It is clear that D and D' are both maximal chains from 0 to 1, and since m and n are of opposite parity, D and D' have lengths of opposite parity. Thus not all maximal chains from 0 to 1 have lengths of the same parity. □

We conclude this section by giving a complete characterization of lattices whose Hasse diagrams are bipartite.

Proposition 1.28. Let L be a finite lattice. Then $G(L)$ is bipartite if and only if L satisfies the *mod* 2 JDCC.

Proof. Proposition 1.26 establishes that if L satisfies the *mod* 2 JDCC, then $G(L)$

is bipartite. To show the converse, suppose to the contrary that L does not satisfy the *mod* 2 JDCC but $G(L)$ is bipartite. So there exist maximal chains $A = \{x = x_0, x_1, ..., x_m = y\}$ and $A' = \{x = x'_0, x'_1, ..., x'_n = y\}$ from x to y of lengths m and n, respectively, of opposite parity. Suppose without loss of generality that x is colored with 1 and m is odd (so that n is even). Since A is maximal and $G(L)$ is bipartite, x_i and x_{i+1} are colored oppositely for all $0 \leq i \leq m-1$; in particular, y is colored with 2. But since A' is maximal, x'_i and x'_{i+1} are colored oppositely for all $0 \leq i \leq n-1$; in particular, x'_{n-1} is colored with 2. But then x'_{n-1} and y are adjacent elements both colored with 2, contradicting the fact that $G(L)$ is bipartite.

□

We are now ready to begin our primary investigation, which is to determine the chromatic numbers of certain subgroup lattices.

2. CHROMATIC NUMBERS OF SUBGROUP LATTICES

In this section we will study the relationship between certain classes of groups and the chromatic numbers of the Hasse diagrams of their respective subgroup lattices, providing many proofs and counterexamples. We assume a basic knowledge of group theory and introduce some deeper group theoretic concepts throughout the section.

A word on notation: we use C_n to denote the cyclic group of order n as opposed to Z_n. Furthermore, for a group G, for brevity we shall say "the subgroup lattice of G has chromatic number k" to mean that the *Hasse diagram* of the subgroup lattice of G has chromatic number k.

2.1 Supersolvable Groups

Definition 2.1. A group G with $G \neq 1$ is called **simple** if the only normal subgroups of G are 1 and G.

Definition 2.2. For a group G, a sequence of subgroups

$$1 = G_0 \leq G_1 \leq G_2 \leq ... \leq G_k = G,$$

where $G_i \trianglelefteq G_{i+1}$ and G_{i+1}/G_i is simple for all $0 \leq i \leq k-1$, is called a **composition series** of G. The quotients G_{i+1}/G_i are called **composition factors** of G.

Proposition 2.3. Every simple abelian group G is isomorphic to C_p for some prime p.

Proof. Since G is abelian, every subgroup of G is normal in G. But since G is simple, it has only two normal subgroups, namely 1 and G, and it follows that G has only these two subgroups. In particular, for every nonidentity element $x \in G$, we have that $\langle x \rangle = G$ so that G is cyclic. It cannot be the case that G is the infinite cyclic group since this group has an infinite number of distinct subgroups. Thus G is a finite cyclic group. For every prime p dividing $|G|$, Cauchy's theorem gives that G has a subgroup of order p. But since the only nontrivial subgroup of G is G itself, it follows that G has prime order and is hence isomorphic to C_p for some prime p. $\square$

Example 2.4. Consider the group $G = C_{12} = \langle t \rangle$ and its subgroup lattice pictured in Figure 3. Since G is abelian, every subgroup $H \leq G$ is normal within any subgroup containing H. Thus we obtain the following three composition series of G:

$$\{1\} \trianglelefteq \langle t^4 \rangle \trianglelefteq \langle t^2 \rangle \trianglelefteq G$$

$$\{1\} \trianglelefteq \langle t^6 \rangle \trianglelefteq \langle t^2 \rangle \trianglelefteq G$$

$$\{1\} \trianglelefteq \langle t^6 \rangle \trianglelefteq \langle t^3 \rangle \trianglelefteq G.$$

We note that these are the only composition series of G. For example, the series $\{1\} \trianglelefteq \langle t^4 \rangle \trianglelefteq G$ is not a composition series of G since the quotient $K = G/\langle t^4 \rangle$ is not simple: K is abelian since G is abelian, and by Lagrange's theorem $|K|$=12/3=4. But by Proposition 2.3, if K were simple, it would have prime order, so we conclude that K is not simple and the series is not a composition series. Furthermore,

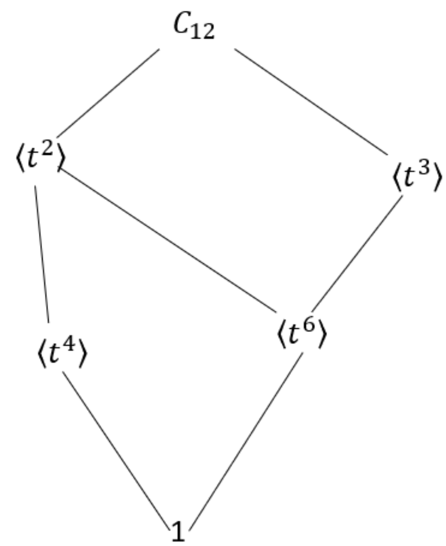

Figure 3: The subgroup lattice of C_{12}.

note that each of the composition series of G, when viewed as a chain from 1 to G, has length 3; and that in each composition series there are two factors of order 2 and one factor of order 3. These observations anticipate the next theorem, the proof of which is omitted and the reader referred to [2, § 3.4].

Theorem 2.5 (Jordan–Hölder)**.** Let G be a finite group with $G \neq 1$. Then

1) G has a composition series and

2) The composition factors in a composition series are unique, that is, if there are two composition series

$$1 = N_0 \leq N_1 \leq ... \leq N_r = G, \quad 1 = M_0 \leq M_1 \leq ... \leq M_s = G$$

of G, then $r = s$ and there exists a permutation π of $\{0, 1, ..., r-1\}$ such that $M_{\pi(i)+1}/M_{\pi(i)} \cong N_{i+1}/N_i$ for all $0 \leq i \leq r-1$.

The Jordan-Hölder theorem is analgous to the Fundamental Theorem of Arithmetic: the composition series of a finite group plays the role of the prime factorization of an integer, and the composition factors play the role of the prime factors. However, an important break in the analogy is worth mentioning. If one is given the numbers $p_1^{a_1}$, $p_2^{a_2}, \ldots, p_n^{a_n}$, then certainly the product $\prod_{i=1}^{n} p_i^{a_i}$ is a unique integer. On the other hand, a given list of composition factors may not, up to isomorphism, determine a unique group. To see this, observe that the nonisomorphic groups C_4 and V_4 both have exactly the same composition factors, namely two C_2's.

For us, the Jordan-Hölder theorem will serve as an important tool in analyzing the Hasse diagram of the subgroup lattice of a group. Its use is illustrated in Proposition 2.7.

It is worthwhile to mention that a maximal chain in a subgroup lattice does not necessarily correspond to a composition series of the group. To demonstrate this, consider the group A_4. This group has a maximal chain $1 \leq \langle(1\,2\,3)\rangle \leq A_4$, but $\langle(1\,2\,3)\rangle$ is not normal in A_4 since

$$[(1\,2)(3\,4)](1\,2\,3)[(1\,2)(3\,4)]^{-1} = (2\,1\,4) \notin \langle(1\,2\,3)\rangle.$$

(See the subgroup lattice of A_4 in Figure 4, Section 2.2.) Hence this maximal chain is not a composition series.

Definition 2.6. A group finite G is said to be **solvable** if there exists a sequence of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G$$

where for all $0 \leq i \leq k-1$, the quotient group G_{i+1}/G_i is abelian and $G_i \trianglelefteq G$ for all $0 \leq i \leq k$.

It is known that there are 18 infinite families of finite simple groups as well as 26 other finite simple groups not belonging to any of these families [2, § 3.4]. As shown in Proposition 2.3, there is only one family of simple abelian groups, namely the cyclic groups of prime order. The next proposition shows that finite solvable groups are exactly the groups whose composition factors consist only of cyclic groups of prime order.

Proposition 2.7. A finite group G is solvable if and only if all of its composition factors are of prime order.

Proof. If all of the composition factors of the finite group G are of prime order, then any composition series of G (which exists by the Jordan–Hölder theorem) has cyclic, and hence abelian, factors. Thus G is solvable. For the converse, suppose G is solvable. So there exists a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ \dots \ \trianglelefteq G_r = G \tag{2.1}$$

with G_{i+1}/G_i abelian for all $0 \leq i \leq r-1$. We show that we can refine this series such that each factor is isomorphic to a C_p. If G_{i+1}/G_i is simple, then by Proposition 2.3 it is isomorphic to a C_p, and no refinement is needed. If G_{i+1}/G_i is not simple, then by the Jordan–Hölder theorem there is a series

$$\overline{1} = G_i/G_i = \overline{K_0} \trianglelefteq \overline{K_1} \trianglelefteq \ \dots \ \trianglelefteq \overline{K_j} = G_{i+1}/G_i$$

with each $\overline{K}_{i+1}/\overline{K_i}$ simple for all $0 \leq i \leq j-1$. Since G_{i+1}/G_i is abelian, so is each $\overline{K_i}$ and hence each $\overline{K}_{i+1}/\overline{K_i}$. Thus $\overline{K}_{i+1}/\overline{K_i}$ is simple abelian and therefore isomorphic to a C_p. Now by the Lattice Isomorphism theorem, there is a series

$$G_i = K_0 \trianglelefteq K_1 \trianglelefteq \ \dots \ \trianglelefteq G_{i+1} = K_j, \tag{2.2}$$

and by the Third Isomorphism theorem $K_{i+1}/K_1 \cong \overline{K}_{i+1}/\overline{K_i}$. Thus K_{i+1}/K_i is isomorphic to a C_p. Now inserting equation (2) into equation (1) for each G_{i+1}/G_i that is not a C_p gives a composition series of G in which each factor has prime order. □

A finite solvable group has a chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_k = G$$

where for all $0 \leq i \leq k-1$, the quotient group G_{i+1}/G_i is *abelian* and $G_i \trianglelefteq G$ for all $0 \leq i \leq k$. By Proposition 2.7 we may refine this chain further so that each quotient G_{i+1i} is *cyclic*. However, when doing such a refinement there is no guarantee that each subgroup remains normal in G. Finite solvable groups that possess a chain in which each quotient is cyclic *and* each subgroup in the chain is normal in the original group are given a special name.

Definition 2.8. A group finite G is said to be **supersolvable** if there exists a sequence

$$H_0 \trianglelefteq H_1 \trianglelefteq \ \dots \ \trianglelefteq H_s = G$$

with H_{i+1}/H_i cyclic and $H_i \trianglelefteq G$ for all $0 \leq i \leq s-1$.

We will show that every finite supersolvable group has a bipartite subgroup lattice. To do this we will mostly follow the development of supersolvable groups by Marshall Hall in [§ 10.5 [3]], restricting our attention only to finite supersolvable groups.

Proposition 2.9. A finite supersolvable group G has a series

$$1 = B_0 \trianglelefteq B_1 \trianglelefteq \dots \trianglelefteq B_k = G$$

with the factor group B_{i+1}/B_i isomorphic to a C_p for all $0 \leq i \leq k-1$.

Proof. The result follows immediately from the observation that every finite supersolvable group is solvable and the application of Proposition 2.11. □

Proposition 2.10. Any subgroup of a supersolvable group is supersolvable.

Proof. Suppose G is supersolvable and $H \leq G$. So there exists a series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$$

with G_{i+1}/G_i cyclic and $G_i \trianglelefteq G$ for all $0 \leq i \leq n-1$. For all $0 \leq i \leq n$, let $H_i = H \cap G_i$. Since $G_i \subseteq G_{i+1}$, $H_i \subseteq H_{i+1}$; furthermore, for any $a \in H_i$ and $h \in H$, $hah^{-1} \in H_i$ since $G_i \trianglelefteq G$. Thus $H_i \trianglelefteq H$. Hence we can show that H is supersolvable if in the series

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_n = H$$

each factor group is cyclic. Note that we may assume without loss of generality that $H_i \neq H_j$ whenever $i \neq j$. Now

$$H_{i+1}/H_i = (H \cap G_{i+1})/(H \cap G_{i+1}) \cap G_i \cong (H \cap G_{i+1})G_i/G_i$$

by the Diamond Isomorphism theorem, and the latter group is a subgroup of the cyclic group G_{i+1}/G_i since $(H \cap G_{i+1})G_i \leq G_{i+1}$. Thus H_{i+1}/H_i is isomorphic to a subgroup of a cyclic group and is therefore cyclic. $\square$

Recall that if $H \leq G$ and $K \leq G$, then the set HK is defined as $HK = \{hk \,|\, h \in H, k \in K\}$. We mention here a couple of well-known results about HK that will be useful for our next theorem. The reader is referred to [2, § 3.2] for proofs of these results.

Proposition 2.11. Let H and K be subgroups of a group G. Then

1) If $H \in N_G(K)$ or $K \in N_G(H)$, then $HK \leq G$.

2) The order of the set HK is $|HK| = \frac{|H||K|}{|H \cap K|}$.

Proposition 2.12. In a finite supersolvable group G, any chain of distinct subgroups

$$1 = M_0 \subseteq M_1 \subseteq \; ... \; \subseteq M_s = G$$

may be refined by the insertion of further subgroups:

$$M_i = M_{i,0} \subseteq M_{i,1} \subseteq \; ... \; \subseteq M_{i,t} = M_{i+1},$$

where $t = t(i)$, $0 \leq i \leq s-1$ such that $M_{i,j}$ is of prime index in $M_{i,j+1}$.

Proof. We show that such a refinement can be made between M_{s-1} and $M_s = G$. Since M_{s-1} is also supersolvable by Proposition 2.10, the same argument may then be repeated to make refinements between M_{s-2} and M_{s-1}, M_{s-3} and M_{s-2}, etc. Since G is supersolvable, there exists a series

$$1 = A_0 \trianglelefteq A_1 \trianglelefteq \ ... \ \trianglelefteq A_r = G$$

with A_{i+1}/A_i cyclic, and by Proposition 2.9 we may assume that each factor has prime order. Since $1 \subseteq M_{s-1}$ and $M_{s-1} \neq A_r = G$, there exists an index i such that $A_i \subseteq M_{s-1}$ but $A_{i+1} \nsubseteq M_{s-1}$. If $M_{s-1} = A_i$, then since $A_i = M_{s-1}$ has prime index in A_{i+1}, the chain

$$M_{s-1} \trianglelefteq A_{i+1} \trianglelefteq \ ... \ \trianglelefteq A_r = M_s = G$$

is the desired refinement. So assume $M_{s-1} \neq A_i$. It cannot be that $M_{s-1} = A_{i+1}$, for this would contradict the fact that $A_{i+1} \nsubseteq M_{s-1}$. Thus $M_{s-1} \neq A_{i+1}$, and it follows that $M_{s-1} \cap A_{i+1}$ is a proper subset of A_{i+1} containing A_i. Since A_i has prime index in A_{i+1}, Lagrange's theorem gives that $M_{s-1} \cap A_{i+1} = A_i$. Similarly, it cannot be that $M_{s-1} \subseteq A_{i+1}$, for the only proper subset of A_{i+1} containing A_i is A_i. Let $M^* = M_{s-1}A_{i+1}$. From Proposition 2.11 part 1), M^* is a group, and $M^* \neq M_{s-1}$ and $M^* \neq A_{i+1}$ since neither M_{s-1} nor A_{i+1} are subsets of the other. Now by Lagrange's theorem and Proposition 2.11 part 2), the index of M_{s-1} in M^* is

$$[M^* : M_{s-1}] = \frac{|M^*|}{|M_{s-1}|} = \frac{|M_{s-1}||A_{i+1}|}{|A_i||M_{s-1}|} = \frac{|A_{i+1}|}{|A_i|} = [A_{i+1} : A_i],$$

and since A_i has prime index in A_{i+1}, we conclude that M_{s-1} has prime index in M^*. Now we have the series

$$M_{s-1} \subseteq M^* \subseteq M_s = G,$$

and if M^* does not have prime index in $M_s = G$, we may repeat the same construction to obtain a series

$$M_{s-1} \subseteq M^* \subseteq \ldots \subseteq M_s = G$$

where each group in this chain has prime index in the group immediately following. Recalling finally that we may repeat this argument to make refinements between M_{s-2} and M_{s-1}, M_{s-3} and M_{s-2}, etc., the proof is complete. □

We are ready to prove our first major result, which follows quickly from Proposition 2.12.

Theorem 2.13. The subgroup lattice of any finite supersolvable group is bipartite.

Proof. Let G be a finite supersolvable group of order r. We show that every maximal chain from 1 to G in the subgroup lattice of G has the same length so that Propositions 1.27 and 1.26 establish the result. Let

$$1 = M_0 \subseteq \ M_1 \subseteq \ldots \subseteq M_k = G$$

and

$$1 = M'_0 \subseteq \ M'_1 \subseteq \ldots \subseteq M'_l = G$$

be maximal chains of length k and l, respectively, in the subgroup lattice of G. We show that $k = l$. Proposition 2.12 guarantees that $[M_{i+1} : M_i]$ is prime for $0 \leq i \leq k-1$ and that $[M'_{j+1} : M'_j]$ is prime for $0 \leq j \leq l-1$. By Lagrange's theorem, we have

$$P = \prod_{i=0}^{k-1}[M_{i+1} : M_i] = r = \prod_{j=0}^{l-1}[M'_{j+1} : M'_j] = P'.$$

Now P and P' are both composed only of prime factors that multiply to r, and by the Fundamental Theorem of Arithmetic, P and P' must be identical up to the arrangement of their factors. Thus $k = l$ so that every maximal chain from 1 to G has the same length. □

2.2 CLT Groups

Definition 2.14. If a finite group G satisfies the converse to Lagrange's theorem, then G is said to be **CLT**. That is, G is CLT if G has a subgroup of order d for every positive integer d which divides $|G|$.

From, for example, [5, § 5.2], it is known that every finite supersolvable group is CLT and that every CLT group is solvable; furthermore, these containments are strict. Since the subgroup lattices of all finite supersolvable groups are bipartite, it is natural to ask whether or not the subgroup lattices of all CLT groups are bipartite. Proposition 2.16 shows that the answer to this question is negative. Before proving this, we record some helpful results, the first of which is intuitive but foundational for our study.

Proposition 2.15. If H' and H are isomorphic groups and $H \leq G$, then the subgroup lattice of H' is contained in the subgroup lattice of G as a subgraph.

Proof. Since $H \cong H'$, their subgroup lattices are identical. Thus we need only show that the subgroup lattice of H is contained in the subgroup lattice of G as a subgraph. Clearly every vertex in the subgroup lattice of H is a vertex of the subgroup lattice of G. Now suppose H_1 and H_2 are adjacent in the subgroup lattice of H with $H_1 \leq H_2$. Clearly $H_1 \leq H_2$ in the subgroup lattice of G, and suppose to the contrary there exists $K \leq G$ properly contained between H_1 and H_2 in the subgroup lattice of G. Then K is also properly contained between H_1 and H_2 in the subgroup lattice of H so that H_1 and H_2 are not adjacent in the subgroup lattice of H, a contradiction. So H_1 and H_2 are adjacent in the subgroup lattice of G, and the subgroup lattice of H is a subgraph of the subgroup lattice of G as desired. □

Lemma 2.16. The subgroup lattice of A_4 is not bipartite.

Proof. The maximal chains

$$1 \leq \langle (1\,2)(3\,4) \rangle \leq \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \leq A_4$$

and

$$1 \leq \langle (1\,2\,3) \rangle \leq A_4$$

have lengths of opposite parity so that the subgroup lattice of A_4 is not bipartite by Proposition 1.28. □

The proof of Lemma 2.16 is made clear by Figure 4.

Proposition 2.17. There exist CLT groups whose subgroup lattices are not bipartite.

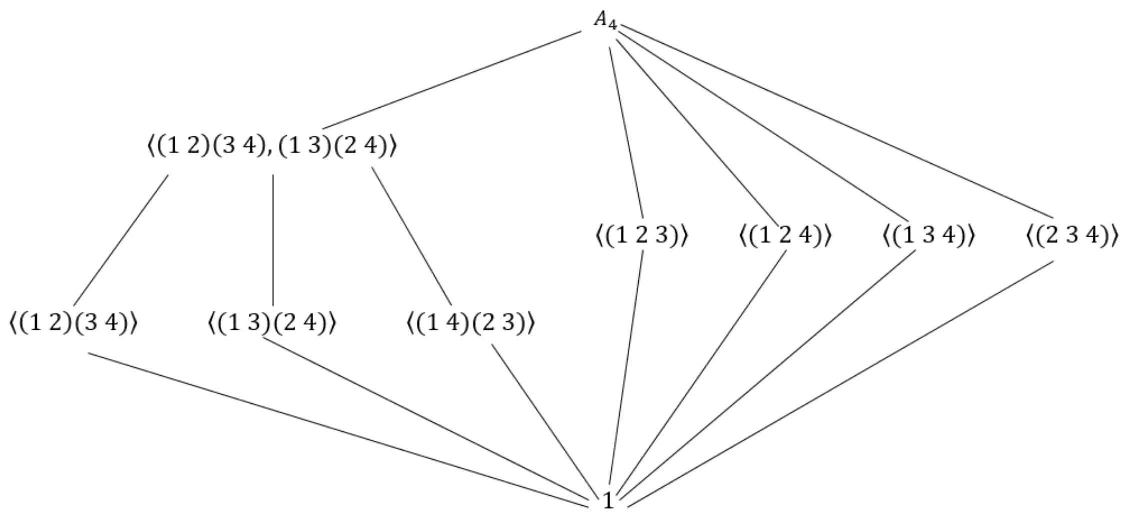

Figure 4: The subgroup lattice of A_4 does not satisfy the *mod* 2 JDCC and hence is not bipartite by Proposition 1.28.

Proof. We show that the group $G = A_4 \times C_2$ is CLT but the subgroup lattice of G is not bipartite. Clearly G contains an isomorphic copy of A_4 as a subgroup so that the subgroup lattice of G is not bipartite by Proposition 2.15, Lemma 2.16, and Proposition 1.15. We now show that G is CLT. It is known that A_4 has subgroups of orders 1, 2, 3, 4, and 12 and that the subgroup of order 4 is isomorphic to $C_2 \times C_2$. So G has a subgroup of order 6 that is isomorphic to $C_3 \times C_2$, and G has a subgroup of order 8 that is isomorphic to $C_2 \times C_2 \times C_2$. Clearly G has subgroups of orders 1, 2, 3, 4, 12, and 24. Thus G has subgroups of order d for every positive integer d dividing $|G| = 24$, so G is CLT. □

2.3 Non-solvable Groups

We have shown that solvable groups may or may not have bipartite subgroup lattices. It is then natural to consider the chromatic numbers of subgroup lattices of groups that are not solvable. We will show that for a group that is not solvable, the subgroup lattice may or may not be bipartite.

The next proposition generalizes Proposition 2.3.

Proposition 2.18. A group G is simple and solvable if and only if G is isomorphic to C_p for some prime p.

Proof. Clearly C_p is simple and solvable for each prime p. For the converse, suppose G is simple and solvable. Since G is solvable, there exists a sequence of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq ... \trianglelefteq G_k = G,$$

where each quotient G_i/G_{i-1} is abelian. However, since G is simple, this sequence must be simply $1 \trianglelefteq G$ so that $G/1 \cong G$ is abelian. Since G is a simple abelian group, by Proposition 2.3 G is isomorphic to C_p for some prime p. $\square$

Our main result is to exhibit an infinite family of finite, non-solvable groups whose subgroup lattices are not bipartite. One way to show this is to show that $A_4 \leq G$. Then since the subgroup lattice of A_4 is not bipartite (Lemma 2.16), Propositions 1.15 and 2.15 guarantee that the subgroup lattice of G is not bipartite. To show that $A_4 \leq G$, we will show that A_4 has a non-normal Sylow 3-subgroup and a normal Sylow 2-subgroup isomorphic to $C_2 \times C_2$. Soon we will show that A_4 is the only such group of order 12, and in doing so we will come across a useful presentation of A_4.

We recall some computational facts about S_n and A_n. The proofs of these results can be found in [2, § 3.5 and § 4.3].

Proposition 2.19.

1) A permutation $\sigma \in S_n$ is even if and only if the number of cycles of even

length in its cycle decomposition is even.

2) Two elements are conjugate in S_n if and only if they have the same cycle type.

Proposition 2.20. The group A_4 has a non-normal Sylow 3-subgroup isomorphic to C_3 and a normal Sylow 2-subgroup isomorphic to $C_2 \times C_2$.

Proof. By Proposition 2.19 part 1), the elements $\sigma_1 = (1\,2\,3)$ and $\sigma_2 = (1\,2\,4)$ are in A_4. It is straightforward to verify that $|\sigma_1| = |\sigma_2| = 3$, and computations show that

$$[(1\,2)(3\,4)](1\,2\,3)[(1\,2)(3\,4)]^{-1} = (2\,1\,4) \notin \langle(1\,2\,3)\rangle.$$

Thus the Sylow 3-subgroup $\langle(1\,2\,3)\rangle \cong C_3$ is not normal in A_4. Similarly, the elements $\sigma_3 = (1\,2)\,(3\,4)$ and $\sigma_4 = (1\,3)\,(2\,4)$ are in A_4, and it is straightforward to compute that $|\sigma_3| = |\sigma_4| = 2$ and $\sigma_3\sigma_4 = \sigma_4\sigma_3 = (1\,4)\,(2\,3)$. Thus the group $H = \langle\sigma_3, \sigma_4\rangle$ is a Sylow 2-subgroup isomorphic to $C_2 \times C_2$. To show that $H \trianglelefteq A_4$, let $g \in A_4$. Then by Proposition 2.19 part 2) the elements $g\sigma_3 g^{-1}$ and $g\sigma_4 g^{-1}$ are of the same cycle type as σ_3 and σ_4. However, there are only 3 elements of the cycle type $(a\,b)\,(c\,d)$, namely σ_3, σ_4, and $\sigma_3\sigma_4$, all of which are in H. Thus $H \trianglelefteq A_4$ as desired. $\square$

To show that A_4 is the only group of order 12 with the conditions mentioned in Proposition 2.20, we introduce a method of constructing a larger group from two arbitrary groups. This construction, known as the **semidirect product**, is what we will now develop.

Definition 2.21. An **automorphism** of a group G is an isomorphism from G to itself. The set of all automorphisms of G is denoted by $\text{Aut}(G)$.

It is straightforward to verify that $\text{Aut}(G)$ is a group under composition. We record the following well-known results on the automorphism groups of certain groups.

Proposition 2.22.

1) If G is cyclic of order n, then $\text{Aut}(G)$ is isomorphic to $\mathbb{Z}_n^\times$, the group of units in the ring $\mathbb{Z}_n$.

2) Suppose p is a prime and V is an abelian group (written additively) with $|V| = p^n$. If V has the property that $pv = 0$ for all $v \in V$, then $\text{Aut}(V) \cong GL_n(\mathbb{F}_p)$. In particular, if $V = C_2 \times C_2$, then $\text{Aut}(V) \cong GL_2(\mathbb{F}_2) \cong S_3$.

Proof. To prove 1), suppose that x is a generator of the cyclic group G of order n. Then any homomorphism ϕ from G to itself is determined completely by where it maps x. Thus ϕ can be represented by the homomorphism ϕ_a, where $\phi_a(x) = x^a$, for some $0 \leq a \leq n-1$. Now ϕ_a is an automorphism if and only if $|x^a| = n$, which happens if and only if a and n are relatively prime. Thus the function $\Psi : \text{Aut}(G) \longrightarrow \mathbb{Z}_n^\times$ defined by $\Psi(\phi_a) = a(\text{mod } n)$ is surjective, and clearly Ψ is injective. We now show Ψ is a homomorphism. For ϕ_a, $\phi_b \in \text{Aut}(G)$, we have

$$\phi_a \circ \phi_b(x) = \phi_a(x^b) = \phi_a(x)^b = (x^a)^b = x^{ab} = \phi_{ab}(x)$$

so that

$$\Psi(\phi_a \circ \phi_b) = \Psi(\phi_{ab}) = ab(\text{mod } n) = \Psi(\phi_a)\Psi(\phi_b).$$

To prove 2), we need only note that since $pv = 0$ for all $v \in V$, V is a vector space over $\mathbb{F}_p$; this is straightforward to verify. Then the automorphisms of V are precisely the invertible linear transformations from V to itself, that is, $\text{Aut}(V) \cong GL_n(\mathbb{F}_p)$. □

It is noteworthy that the groups in Proposition 2.22 part (2) are determined uniquely by p and n; in fact, they are isomorphic to C_p^n.

The development and examples of semidirect products below largely follows [2, § 5.5].

Proposition 2.23. Let H and K be finite groups and let ϕ be a homomorphism from K to $\text{Aut}(H)$. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$. Define a multiplication on G by $(h_1, k_1)(h_2, k_2) = (h_1\phi(k_1)(h_2), k_1k_2)$. Then

1) G is a group of order $|H||K|$ and

2) there exist subgroups $H' \leq G$ and $K' \leq G$ with $H \cong H'$ and $K \cong K'$. Identifying H' as H and K' as K gives

3) $H \trianglelefteq G$

4) $H \cap K = 1$ and

5) for all $h \in H$ and $k \in K$, $khk^{-1} = \phi(k)(h)$.

Definition 2.24. The group G in Proposition 2.22 is called the **semidirect product** of H and K and is denoted $H \rtimes_\phi K$. When the homomorphism is clear from context, we shall simply write $H \rtimes K$.

It is immediate from the definition that to compute in $H \rtimes_\phi K$, we must know 1) how to multiply in H, 2) how to multiply in K, and 3) how to conjugate elements of H by elements of K (which is dependent on the choice of homomorphism ϕ).

Example 2.25. Let $H = C_n = \langle x \rangle$ and $K = C_2 = \langle y \rangle$. Let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be the map that sends y to the automorphism of inversion on H, that is, for $h \in H$ we have $\phi(y)(h) = yhy^{-1} = h^{-1}$. It is straightforward to see that inversion is an automorphism of H and that ϕ is a homomorphism. Thus a presentation of $G = H \rtimes K$ is

$$G = \{x, y \,|\, x^n = y^2 = 1,\, yxy^{-1} = x^{-1}\},$$

and from this it is clear that $G \cong D_{2n}$.

The next proposition is a key result in allowing us to recognize a group G as a semidirect product of two of its subgroups.

Proposition 2.26. Suppose G is a group with subgroups H and K such that $H \trianglelefteq G$ and $H \cap K = 1$. Let $\phi : K \longrightarrow \mathrm{Aut}(H)$ be defined by mapping $k \in K$ to the automorphism of left conjugation on H by k. Then $HK \cong H \rtimes_\phi K$. In particular, if $G = HK$ with $H \trianglelefteq G$ and $H \cap K = 1$, then $G \cong H \rtimes_\phi K$.

Proof. Proposition 2.11 establishes that $HK \leq G$ and that $|HK| = |H||K|$. Thus every element in HK can be written uniquely as a product hk, and so the map $f : HK \longrightarrow H \rtimes \phi K$ given by $f(hk) = (h, k)$ is a bijection. To show that f is a homomorphism, let h_1k_1 and h_2k_2 be elements of HK. Then

$$f(h_1k_1h_2k_2) = f(h_1k_1h_2k_1^{-1}k_1k_2) = (h_1\phi(k_1)h_2), k_1k_2)$$

$$= (h_1, k_1)(h_2, k_2) = f(h_1k_1)f(h_2k_2).$$

$\square$

Proposition 2.26 allows us to show that every group of order 12 is isomorphic to a semidirect product of two of its subgroups.

Proposition 2.27. Every group G of order 12 is isomorphic to a semidirect product of two of its subgroups.

Proof. By Sylow's theorem, G has a Sylow 2-subgroup H and a Sylow 3-subgroup K. Since $|H| = 4$ and $|K| = 3$ are relatively prime, $H \cap K = 1$. Furthermore, Proposition 2.11 part 2) gives that $|HK| = 12$. So HK is a 12-element subset of G and hence $G = HK$. Sylow's theorem also gives that $n_2 = 1$ or 3 and $n_3 = 1$ or 4. We cannot have $n_2 = 3$ and $n_3 = 4$: three such Sylow 2-subgroups have a minimum of 8 distinct elements, and four such Sylow 3-subgroups have a minimum of 7 distinct elements, none of which except the identity element are also in a Sylow 2-subgroup. This gives G a minimum of $8+6 = 14$ elements, which is impossible. So either $H \trianglelefteq G$ or $K \trianglelefteq G$, and thus by Proposition 2.26, G is a semidirect product of H and K. $\square$

We are finally ready to show that A_4 is the only group of order 12 with a non-normal Sylow 3-subgroup and a normal Sylow 2-subgroup isomorphic to $C_2 \times C_2$.

Proposition 2.28. If a group G is of order 12, has a non-normal Sylow 3-subgroup, and has a normal Sylow 2-subgroup isomorphic to $C_2 \times C_2$, then $G \cong A_4$.

Proof. Let $H \trianglelefteq G$ be isomorphic to $C_2 \times C_2$ and $K = \langle x \rangle \leq G$ be a non-normal

Sylow 3-subgroup of G. By Proposition 2.27 $G \cong H \rtimes_\phi K$ for some homomorphism $\phi : K \longrightarrow \mathrm{Aut}(H)$. By Proposition 2.22 part 2), $\mathrm{Aut}(H) \cong S_3$, and hence $\mathrm{Aut}(H)$ has a unique subgroup $\langle y \rangle$ of order 3. Now ϕ is a homomorphism if and only if $\phi(x)$ is an element of order 1 or 3. Thus there are 3 possible homomorphisms ϕ_i given by $\phi_i(x) = y^i$ for $i = 0$, 1, or 2. The homomorphism ϕ_0 maps x to the identity automorphism and thus maps every element of K to the identity automorphism. Hence for all $h_1, h_2 \in H$ and $k_1, k_2 \in K$, we have

$$(h_1, k_1)(h_2, k_2) = (h_1\phi_0(k_1)(h_2), k_1k_2) = (h_1h_2, k_1k_2)$$

so that $H \rtimes_{\phi_0} K$ is isomorphic to $H \times K$, which is not the case since G is not abelian. Thus we must have $i = 1$ or $i = 2$, but the homomorphisms $H \rtimes_{\phi_1} K$ and $H \rtimes_{\phi_2} K$ are isomorphic since they differ only in a choice of generator for K. Thus there is only one group meeting the criteria in the hypothesis, and since A_4 satisfies these criteria by Proposition 2.20, we conclude that $G \cong A_4$. $\square$

The proof of Proposition 2.28 gives an important presentation of A_4, which we express as a corollary.

Corollary 2.29. The group A_4 can be presented as

$$A_4 = \{a,\, b,\, c \,|\, a^2 = b^2 = c^3 = 1, ab = ba, cac^{-1} = b, cbc^{-1} = ab\}.$$

Proof. The proof of Proposition 2.28 makes all of the listed relations clear except for the last two. Using the notation from this proof, we have that ϕ_1 and ϕ_2 must map the nonidentity elements of H to automorphisms of order 3 in $\mathrm{Aut}(H)$. It is straightforward to check that the only such automorphisms are of the form $(a\,b\,ab)$, and thus, without loss of generality, we have $cac^{-1} = b$ and $cbc^{-1} = ab$. $\square$

Now it will take only a couple of short lemmas for us to exhibit an infinite family of finite non-solvable groups whose subgroup lattices are not bipartite.

Lemma 2.30. Let $q = p^k$ for some prime p and natural number k. The field $\mathbb{F}_q$ has exactly $\frac{q+1}{2}$ perfect squares.

Proof. The set $\mathbb{F}_q - \{0\}$ has $q - 1$ elements. Partition $\mathbb{F}_q - \{0\}$ into subsets of the form $R_x = \{x, -x\}$ for each $x \in \mathbb{F}_q - \{0\}$. It is clear that there are $\frac{q-1}{2}$ distinct subsets of this form. Let $R = \{R_x \,|\, x \in \mathbb{F}_q - \{0\}\}$, that is, R is the collection of sets R_x. We show that there is a bijection between R and $\mathbb{F}_q^2 - \{0\}$, the set of nonzero perfect squares in $\mathbb{F}_q$. Let $f : R \longrightarrow \mathbb{F}_q^2$ be given by $f(R_x) = x^2$ for some $x \in R_x$. Since $(-x)^2 = (x)^2$, the function f is independent of the choice of representative for R_x and hence is well-defined. If $c \in \mathbb{F}_q^2 - \{0\}$, then $c = a^2$ for some $a \in \mathbb{F}_q - \{0\}$ so that $f(R_a) = c$. So f is surjective. If $f(R_a) = f(R_b)$, then $a^2 = b^2$ so that $a^2 - b^2 = (a - b)(a + b) = 0$. Since $\mathbb{F}_q$ is a field, this means that $a - b = 0$ or $a + b = 0$. Thus $b = \pm a$ so that $R_a = R_b$, and f is injective. Thus there are $\frac{q-1}{2}$ nonzero perfect squares in $\mathbb{F}_q$, and since $0 = 0^2$ is a perfect square, there are exactly $\frac{q-1}{2} + 1 = \frac{q+1}{2}$ perfect squares in $\mathbb{F}_q$. $\square$

Lemma 2.31. Let a and b be nonzero elements of $\mathbb{F}_q$ and let c be any element in $\mathbb{F}_q$. Then the equation $ax^2 + by^2 = c$ has a solution in $\mathbb{F}_q$.

Proof. It is sufficient to show that any equation of the form $x^2 + dy^2 = e$, where

$d, e \in \mathbb{F}_q$ and $d \neq 0$, has a solution in $\mathbb{F}_q$: for if so, then the equation $x^2 + \frac{b}{a}y^2 = \frac{c}{a}$ has a solution (x_1, y_1) for the given a and b, and it is clear that (x_1, y_1) is also a solution to $ax^2 + by^2 = c$. Consider the function $f : \mathbb{F}_q \longrightarrow \mathbb{F}_q$ given by $f(u) = \frac{e-u}{d}$. It is straightforward to show that f is a bijection, and it follows from Lemma 2.17 that $f(\mathbb{F}_q^2)$ has $\frac{q+1}{2}$ elements. But there are only $q - \frac{q+1}{2} = \frac{q-1}{2}$ nonsquares in $\mathbb{F}_q$, so at least one element of $\mathbb{F}_q^2$ must map to another square. That is, there exists an element $x^2 \in \mathbb{F}_q^2$ such that $f(x^2) = y^2$. So $f(x^2) = \frac{e-x^2}{d} = y^2$, and it follows that $x^2 + dy^2 = e$ as desired. $\square$

We are now ready for our next major result.

Theorem 2.32. There exists an infinite family of finite non-solvable groups whose subgroup lattices are not bipartite. Specifically, any simple group from the family $PSL(2, q)$, where $q = p^n$ for an odd prime p and natural number n, is not solvable and has a subgroup lattice that is not bipartite.

Before proving this theorem, we give a definition and quick discussion of the **projective special linear**, or **PSL** groups.

Definition 2.33. The **special linear group** $SL(n, q)$ is the group of $n \times n$ matrices with determinant 1 and entries in the finite field $\mathbb{F}_q$. The **projective special linear group** $PSL(n, q)$ is the group $SL(n, q)/Z(SL(n, q))$.

Definition 2.34. A subgroup M of a group G is called a **maximal subgroup** of G if there are no proper subgroups $H \leq G$ with $M \leq H \leq G$.

It is useful to record a couple of facts about the special linear and projective special linear groups, one of which deals with the maximal subgroups of $PSL(2, q)$.

Proposition 2.35.

1) When q is even, the maximal subgroups of $PSL(2,q)$ are dihedral groups of orders $2(q-1)$ and $2(q+1)$, a subgroup of order $q(q-1)$, and $PSL(2,2) \cong S_3$. [7, Corollary 2.2]

2) The center of $SL(n,q)$ consists of all multiples of the identity matrix xI with $x^n = 1$. In particular, the center of $SL(2,q)$ is $\{\pm I\}$. [6, Theorem 14.3]

Proposition 2.35 part 2) is useful for calculations in $PSL(2,q)$: we can simply work with matrices in $SL(2,q)$, keeping in mind that cosets $\overline{A}$ and $\overline{B}$ in $PSL(2,q)$ with representatives A and B are equal in $PSL(2,q)$ whenever $A = \pm B$ in $SL(2,q)$.

We are now ready to prove Theorem 2.32.

Proof. (Theorem 2.32) It is well known that every group of the form $PSL(2,q)$, where $q = p^n$ for a prime p and natural number n, is simple except for when $q = 2$ and $q = 3$. Thus assume $q \neq 2$ and $q \neq 3$ so that by Proposition 2.18, $PSL(2,q)$ is not solvable. We first consider the case when $p = 3$ so that $q = 3^n$ with $n \geq 2$. We refer to [8] to show that $PSL(2,3) \cong A_4$. We now show that $PSL(2,3) \leq PSL(2,q)$, and since A_4 is not bipartite but has an isomorphic copy contained in $PSL(2,q)$, the latter group is not bipartite by Propositions 1.15 and 2.15. Observe that $SL(2,3) \leq SL(2,q)$. Furthermore, by Proposition 2.35 part 2), these groups have the same center $Z = \{I, -I\}$: for $x \in \mathbb{F}_q$, $x^2 = 1$ exactly if $x = \pm 1$ since $\mathbb{F}_q$ is a field. Now by the Lattice Isomorphism theorem $PSL(2,3) = \frac{SL(2,3)}{Z} \leq \frac{SL(2,q)}{Z} = PSL(2,q)$ as desired.

We now consider the case when $q = p^n$ with $p \geq 5$. We again show that the subgroup lattice of G is not bipartite by showing that $A_4 \leq PSL(2,q)$. To show that $A_4 \leq PSL(2,q)$, we exhibit elements A and B in $SL(2,q)$ so that $\overline{A}$ and

$\overline{B}$ have order 2 in $PSL(2,q)$ with $AB + BA = 0$ (so that $\overline{A}$ and $\overline{B}$ commute in $PSL(2,q)$) and $C \in SL(2,q)$ with $\overline{C}$ of order 3 in $PSL(2,q)$; additionally, we need $\overline{CAC^{-1}} = \overline{B}$ and $\overline{CBC^{-1}} = \overline{AB}$. Then by Corollary 2.29, it follows that $A_4 \leq PSL(2,q)$. Familiar properties of determinants can be used to verify that each of A, B, and C are indeed in $SL(2,q)$. It is straightforward to verify that $\overline{C}$, where C is the matrix $C = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, has order 3 in $PSL(2,q)$. Let A be any matrix of the form $A = \begin{bmatrix} a & b \\ \frac{-1-a^2}{b} & -a \end{bmatrix}$ with $b \neq 0$. It is also straightforward to verify that $A^2 = -I$ so that $\overline{A}$ has order 2 in $PSL(2,q)$. Let B be defined by $B = CAC^{-1}$. Since $A \neq \pm I$, it follows that $B \neq \pm I$; and since B is a conjugate of A, $\overline{B}$ also has order 2. One computes that $AB + BA = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$, where

$$\alpha = \frac{-(a^2 - ab + b^2 + b + 1)(a^2 - ab + b^2 - b + 1)}{b^2}.$$

Thus $\overline{AB}$ and $\overline{BA}$ commute in $PSL(2,q)$ if we can choose a and b such that either of the factors in the numerator of α are 0. Let (x,y) be a solution to $3x^2 + y^2 = -8$ in $\mathbb{F}_q$. The existence of a solution is guaranteed by Lemma 2.31. Now let $b = \frac{y+2}{3}$ and $a = \frac{x+b}{2}$. Then

$$3(2a - b)^2 + (3b - 2)^2 + 8 = 0$$

$$\Rightarrow 12a^2 - 12ab + 12b^2 - 12b + 12 = 0$$

$$\Rightarrow a^2 - ab + b^2 - b + 1 = 0 \Rightarrow \alpha = 0.$$

So $\overline{A}$ and $\overline{B}$ commute in $PSL(2, q)$. Finally, one computes that

$$CBC^{-1} - AB = \beta \begin{bmatrix} \frac{a+b}{b} & \frac{1-a}{b} \\ \frac{1+a^2-b-ab}{b^2} & \frac{1+a^2-b}{b^2} \end{bmatrix},$$

where $\beta = a^2 - ab + b^2 - b + 1$. But we have chosen a and b such that $\beta = 0$ so that $CBC^{-1} - AB = 0$, and thus $\overline{CBC^{-1}} = \overline{AB}$, completing the proof. $\square$

We note that the argument used when $p \geq 5$ does not work when $p = 3$ because defining the element b to be $\frac{y+2}{3}$ is not valid in $\mathbb{F}_q$ if $q = 3^n$: in this case, the characteristic of $\mathbb{F}_q$ is three so that $\mathbb{F}_3$ is a subfield of $\mathbb{F}_q$, but $3 = 0$ in $\mathbb{F}_3$.

3. MORE RESULTS AND FURTHER QUESTIONS

Having examined the chromatic number of the subgroup lattice of $PSL(2, q)$ when q is the power of an odd prime, it is natural to consider the other case, namely when $q = 2^n$ for a natural number n. Preliminary investigations strongly suggest that there is an infinite family of these groups whose subgroup lattices are bipartite. We state this as a conjecture and discuss reasons to support it.

Conjecture 3.1. There exists an infinite family of finite non-solvable groups whose subgroup lattices are bipartite. Specifically, there are an infinite number of members from the family $PSL(2, q)$, where $q = 2^n$ for a natural number n, that are not solvable and whose subgroup lattices are bipartite.

As mentioned in the proof of Theorem 2.32, $PSL(2, q)$ is simple except for when $q = 2$ or $q = 3$. So if $q \neq 2$ and $q \neq 3$, then $PSL(2, q)$ is not solvable. Proposition 2.35 part 1) indicates that the maximal subgroups of $PSL(2, q)$ consist of two dihedral groups, a subgroup of order $q(q-1)$, and S_3 when q is even. Furthermore, it is known that the group of order $q(q-1)$ is a semidirect product in the form $C_2^k \rtimes C_{2^k-1}$, where $q = 2^k$. It is straightforward to verify that dihedral groups are supersolvable. Now the proof of Theorem 2.13 shows that the length of a maximal chain in D_{2m} is dependent solely on the prime factorization of $2m$. In particular, Proposition 2.35 part 1) shows that the dihedral groups are of order $2(q-1)$ and $2(q+1)$. Furthermore, it is quite certain that the maximal subgroup of the form $C_2^k \rtimes C_{2^k-1}$ has maximal chains of length $k+1$ and length 2. Finally, it is clear that maximal chains in S_3 are of length 2. Heuristically there should be an infinite number of values of q such that maximal chains passing through any of these three maximal subgroups have a length of even parity so that $PSL(2, q)$ is bipartite by Proposition 1.28.

Proposition 2.17 exhibited one group that is CLT but not bipartite. It is quite likely that this example could be generalized to find an infinite class of groups that are CLT but not bipartite, but we are yet to examine this question in detail.

Since determining the chromatic number of a given subgroup lattice can be quite difficult in general, it is of interest to find a simpler graph that gives useful bounds on the chromatic number of the given subgroup lattice. One possible technique to do this is through the **conjugacy class poset** described below.

Proposition 3.2. Let G be a finite group and $H, K \leq G$. We write $[H]$ to denote the conjugacy class of H and $\mathcal{C}_G$ to denote the set of all conjugacy classes of subgroups of G. Define a partial order $\preceq$ on $\mathcal{C}_G$ by $[H] \preceq [K]$ whenever $H \leq gKg^{-1}$ for some $g \in G$, that is, whenever H is contained in a conjugate of K. Then $\mathcal{C}_G$ is a poset under $\preceq$.

Proof. Since $H \leq 1H1^{-1}$, we have $[H] \preceq [H]$. Thus $\preceq$ is reflexive. If $[H] \preceq [K]$ and $[K] \preceq [H]$, then $H \leq g_1Kg_1^{-1}$ and $K \leq g_2Hg_2^{-1}$ for some $g_1, g_2 \in G$. So

$$H \leq g_1Kg_1^{-1} \leq g_1g_2Hg_2^{-1}g_1^{-1} = g_1g_2H(g_1g_2)^{-1}.$$

Now each conjugate of a given group has the same order as the group, so by Lagrange's theorem we have that $|H| \mid |K| \mid |H|$. Thus $|H| = |K| = |gKg^{-1}|$ so that $H = gKg^{-1}$. Hence $H \in [K]$, and since conjugate subgroups have the same conjugacy class, we have $[H] = [K]$. So $\preceq$ is antisymmetric. Finally, if $[H] \leq [K] \leq [L]$, then $H \leq g_1Kg_1^{-1}$ and $K \leq g_2Lg_2^{-1}$. So

$$H \leq g_1g_2Lg_2^{-1}g_1^{-1} = (g_1g_2)L(g_1g_2)^{-1},$$

and $[H] \leq [L]$ so that $\preceq$ is transitive. $\square$

Definition 3.3. For a finite group G, the set $\mathcal{C}_G$ under the partial order $\preceq$ described in Proposition 3.2 is called the **conjugacy class poset** of G.

Since $\mathcal{C}_G$ is a poset, we may draw its Hasse diagram $C(\mathcal{C}_G)$. Preliminary investigations suggest 1) that $C(\mathcal{C}_G)$ is often (though not always) a subgraph of the subgroup lattice of G and 2) that a coloring of $C(\mathcal{C}_G))$ induces a valid coloring of the subgroup lattice of G by assigning all $H \in [H]$ the same color as $[H]$ in $C(\mathcal{C}_G)$. Whenever 1) is true, $\chi(C(\mathcal{C}_G))$ is a lower bound on the chromatic number of the subgroup lattice of G by Proposition 1.15, and whenever 2) is true, $\chi(C(\mathcal{C}_G))$ is a lower bound on the chromatic number of the subgroup lattice of G. Determining the conditions under which 1) and 2) are true would make these into valuable tools for further study. Specifically, finding conditions that satisfy both 1) and 2) would equate the chromatic number of the subgroup lattice of G and $\chi(C(\mathcal{C}_G))$, which could considerably reduce the difficulty of finding the chromatic numbers of certain subgroup lattices.

Lastly, it is still of interest whether or not one can find a subgroup lattice of an arbitrarily high chromatic number. Theorem 2.32 gives a family of groups whose subgroup lattices are not bipartite, but it is unknown if these groups have chromatic number 3.

4. CONCLUSION

In this thesis we investigated properties of lattices, represented them graphically via the Hasse diagram, and considered their chromatic numbers. We saw that lattices may have arbitrarily high chromatic numbers and proved that subgroup lattices of infinite groups may also have arbitrarily high chromatic numbers. We considered the subgroup lattices of finite groups, showing that supersolvable groups have bipartite subgroup lattices and that CLT groups may or may not have bipartite subgroup lattices. Finally, we exhibited an infinite family of non-solvable groups with subgroup lattices that are not bipartite and provided evidence of a similar family of non-solvable groups whose subgroup lattices are likely to be bipartite. We hope that our work will lead to the determination of whether or not finite groups may have subgroup lattices of arbitrarily high chromatic number.

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