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Understanding and Advancing College Students' Mathematical Reasoning Using Collaborative Argumentation

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UNDERSTANDING AND ADVANCING COLLEGE STUDENTS' MATHEMATICAL REASONING USING COLLABORATIVE ARGUMENTATION

A Master's Thesis

Presented to

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In Partial Fulfillment

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Master of Science in Education, Secondary Education

By

Rachel Kay Heili

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ABSTRACT

This study explored students' mathematical reasoning skills and offered supports to advance them through a collaborative argumentation framework in a college intermediate algebra class. The goals of this study were to make observations about student reasoning, identify specific actions to address those observations, and document student growth in reasoning as a result of those actions. An iterative analysis, mixed method study was conducted in which the researcher engaged students in responding to questions that required conceptual understandings using a collaborative argumentation framework as a tool to identify and code components of their responses—claim, evidence, and reasoning. After coding and analyzing students' responses and evaluating themes from the researcher's observations, the results increased the researcher's understandings of students' reasoning and indicated an advancement in conceptual understandings, pattern exploration strategies, and written mathematical arguments. By following the instructional guidelines and workshops of the collaborative argumentation framework, the frequency of reasoning and quality of evidence in students' responses increased. Additionally, observations indicated advancements from procedural to conceptual understandings, recognizing and systematically exploring patterns, and communicating arguments in a cohesive manner. The researcher also identified challenges of getting students to use generalized reasoning strategies and creating enough class time for effective and thorough feedback and reflection.

KEYWORDS: collaborative argumentation, claim, evidence, reasoning, conceptual understanding

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I dedicate this thesis to my future students. May you feel empowered by math, realize you have the tools to problem-solve and overcome any challenge, and believe in yourselves as much as I believe in you. We are all mathematicians.

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CHAPTER I: INTRODUCTION

Students often ask math teachers, "when will I use this in the future?" As teachers, we must ponder: Are we teaching math in such a way that students can use it in the future? That is, are the understandings developed in today's math classrooms deep enough and of a nature that will allow students to call on those understandings and adapt them in future math courses and in life outside the classroom? If a student learns now that "keep change flip" is how we divide fractions, how else can they use that knowledge later? Unless they are given a non-contextual problem that explicitly asks them to divide fractions, that's not very helpful. Now consider all the possibilities for later connections created if students understand that dividing by some number is the same as multiplying by its reciprocal. This idea can be called on in a multitude of situations, being especially helpful in the manipulation of equations and functions throughout algebra.

The notion of developing mathematical understandings that can be transferred to other situations (mathematical or otherwise) later is often addressed through discussion of and research regarding conceptual understanding in mathematics (Nachowitz, 2019; Russell et al., 2020; DeCaro, 2016; Smith et al., 2018; National Council for Teachers of Mathematics [NCTM], 2014). In fact, "build procedural fluency from conceptual understanding" is one of the eight effective teaching practices outlined by NCTM (2014, p. 42). The existing research explores the development of conceptual understanding through the lens of written response, progression of problem complexity, class discussion, and one-on-one coaching (Nachowitz, 2019; Russell et al., 2020; DeCaro, 2016; Smith et al., 2018; NCTM, 2014). These studies also involve requiring students to communicate their understanding either verbally or in writing. A component of such communication is mathematical reasoning.

Creating opportunities for students to reason mathematically has also been recommended by NCTM (2014) as an effective teaching practice. When conceptual understandings and reasoning collide in the math classroom, students are given the tools to problem solve, make claims, and support those claims. In my experiences as a math teacher, making claims and supporting them does not come naturally to all students. In answering math questions, specifically those that ask for explanation, students can be stumped by the math, but they can also be stumped by the task of clearly communicating their thinking or understanding why their own reasoning led them to the correct answer. For example, in a high school geometry class, almost all my students could accurately tell me that two triangles were similar due to side-angle-side similarity. However, when asked to create a flow chart explaining this statement, few could do so, and many rejected the task as "tedious and dumb." I realized that up to that point, my students had not been taught how to organize their thoughts in a mathematical context. So often "showing your work" looks like a series of disjoint procedural executions with little labeling or description of why this work supports the final answer or claim. I needed a framework to coach my students to be able to effectively communicate their mathematical thinking. This idea led to an exploration of argumentation in the classroom.

Rationale for the Study

According to the National Council of Teachers in Mathematics (NCTM), using mathematical argumentation in the classroom is an evidence-based practice essential to developing conceptual understanding and mathematical proficiency (Rumsey & Langrall, 2016).

Research has suggested that critical thinking skills benefit from argumentation practices, and such processes lead students to create multiple connections between mathematical concepts (Wagner et al., 2014). Furthermore, there is a positive correlation between argumentation and knowledge construction (Hershkowitz et al., 2001). Several researchers have introduced teaching frameworks for the use of argumentation in the classroom (Toulmin, 2003; Ervin-Kassab, Roddick, Vickery, & Tapper, 2020; Zwiers, 2019).

However, from my experience using Toulmin's (2003) Claim-Evidence-Warrant model in a previous study, students had a difficult time internalizing the framework due to confusing language. Ervin-Kassab, et al.'s (2020) Claim-Rule-Connection framework had success, but the article gave few recommendations for implementation. Zwier's (2019) Collaborative Argumentation model was developed as a generalized framework for all subjects, not just mathematics. My plan was to combine these frameworks and develop teaching materials and guidelines with respect to argumentation. Then, I wanted to implement the framework and guidelines to improve student reasoning.

Purpose of the Study

The purpose of this iterative analysis study was to explore how algebra students' reasoning changes over the course of a semester when using a collaborative argumentation framework with specific instructional guidelines motivated by student need. The collaborative argumentation model was defined as a combination of Toulmin's model (2003), Claim-Rule-Connection (Ervin-Kasaab, et al., 2020), and Zwier's Collaborative Argumentation steps (2019). Students were asked to argue open-ended questions—those in which students must draw on multiple algebra concepts and may take multiple pathways to draw a conclusion for which they must provide justification. Key indicators of student responses observed included correct claims, use of mathematical rules or theorems, and a statement connecting or comparing concepts or ideas.

Research Questions

The existing research on argumentation models in mathematics classrooms and previous teaching experiences led to the research questions:

1. What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?
2. In what ways does the use of such a framework advance student mathematical reasoning?

The goals of answering this question were to

a. use suggested framework to make observations about student reasoning
b. identify specific actions to address those observations
c. document the progression of growth in student reasoning over the course of the study

At the end of each four-week period of instruction (iteration), the researcher analyzed data and observations through the lens of the following three questions: (1) What are the data telling me? (2) What is it I want to know? (3) What is the dialectical relationship between what the data are telling me and what I want to know? Through this reflection, the researcher identified a new focus for each iteration. The first iteration focus was developing need for and adopting the collaborative argumentation framework. The second iteration focus was breaking old reasoning patterns and giving students confidence. The third iteration focus was reflecting, revising, and questioning against the collaborative argumentation framework. The fourth iteration focus was to make observations about student internalization of the framework and reasoning advancements.

Research Design

This study followed the iterative analysis for qualitative research framework as outlined in the International Journal of Qualitative Methods. This is a method in which the researcher recognizes that many patterns and themes emerge as data is collected and new, deeper understandings of the material develop. Thus, iterative analysis outlines a reflexive process. Reflexivity in qualitative research allows for the researcher to reflect on their own experiences in relation to the data collection and observations to be able to separate themselves from the research and make necessary changes (Srivastava & Hopwood, 2009). Iterative analysis starts with a research question, then answers the following three questions outlined by Srivastava and Hopwood (2009) with each iteration:

> Q1: What are the data telling me? (Explicitly engaging with theoretical, subjective, ontological, epistemological, and field understandings)
> Q2: What is it I want to know? (According to research objectives, questions, and theoretical points of interest)
> Q3: What is the dialectical relationship between what the data are telling me and what I want to know? (Refining the focus and linking back to research questions) (p. 78)

The researcher chose iterative analysis for this study because of the nature of research involving education practices. Since teaching mathematics is already such a reflexive process—that is, the teacher reflects on lessons, gauges student understandings, and adjusts accordingly—it was fitting to choose a reflexive research process. Collecting data in iterations also gave the researcher flexibility to adjust the argumentative framework supports throughout the semester and focus the research questions and observations with respect to such adjustments. By systematically documenting this development, the research is better suited for use by other educators.

All iterations were done using two intermediate algebra classes of 30 students each, taught by the researcher at a midwestern 4-year university during one fall semester. The undergraduate course catalog describes the course as recommended for students who have not yet mastered the algebra concepts necessary for success in college algebra. Students in this class had most likely been introduced to the topics of the class in high school algebra courses but based on their ACT scores and/or university mathematics placement test results, they did not qualify for enrollment in a college algebra course at the university.

In the first iteration, students participated in two weeks of instruction during which they worked in groups to solve problems and explain concepts. There was at least one question requiring a conceptual understanding (CEQ) each class period and on each assignment. This was followed by the unit 1 test of CEQs. During the next two weeks of instruction, students participated in Collaborative Argumentation Workshop I, during which the collaborative argumentation framework was introduced. Then, students took the unit 2 test. Responses on unit 2 assignments and tests were coded based on the collaborative argumentation framework developed for this study and compared to the nature and quality of the responses from unit 1. During this iteration, the research focus was to introduce and create a need for the collaborative argumentation framework. The researcher reflected on observations about student engagement, specific teaching moves, areas of student growth, and areas that needed improvement. The researcher then used these observations to develop a new Collaborative Argumentation Workshop, amend instructional guidelines, and identify the research focus, marking the beginning of the second iteration. This pattern continued for a total of four iterations, each with a new research focus and updated workshops and guidelines. These are described in detail in Chapter III.

Significance of the Study

This study holds significance for secondary and post-secondary mathematics teachers. Due to the unique nature of the course used to collect data—that is, post-secondary students being retaught secondary topics—this study offers insight into gaps in high school knowledge and misunderstandings. It aimed to identify missing pieces in student reasoning and argumentation skills as well as suggest a teaching framework to help improve such responses.

The researcher identified very specific instructional guidelines and collaborative argumentation framework supports, made observations about their effectiveness, and offered suggestions for improvement. This provides an outline for other teachers to use when wanting to implement collaborative argumentation in a mathematics classroom.

Assumptions

In conducting this study, it was assumed that:

1. The researcher employed the curriculum through argumentation models consistently.
2. Students participating in the study maintained some level of engagement during class and with course materials.

Limitations

1. All student participants in this study were college students enrolled at the same midwestern university, limiting the generalizability of these results.
2. Data collection spanned only one semester, so observations about knowledge retention were not as long as high school courses may require.
3. Students' previous knowledge or experiences with mathematics could not be controlled by the researcher and may have affected their response to the teaching materials and engagement in the course.

Definition of Terms

The following terms are defined as such for the entirety of this paper.

1. Collaborative Argumentation: derived from Zwiers (2019) and Conner (2013), students working together to build up multiple arguments, evaluate evidence, and draw their own conclusions
2. Reasoning: according to Lannin, Ellis, and Elliott (2011), "an evolving process of conjecturing, generalizing, investigating why, and developing and evaluating arguments" (p. 12)
3. Conceptual Understandings: based on the ideas of Hiebert (1986), "knowledge that is rich in relationships" (p. 3) and that focuses on why mathematics works and how it works together
4. Connections: according to NCTM (2000), the ability to "recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics outside of mathematics." (p. 64)
5. Scaffolding: according to Wood, Bruner, and Ross (1976), "an interactive system of exchange in which the tutor operates with an implicit theory of the learner's acts in order to recruit his attention, reduces degrees of freedom in the task to manageable limits, maintains 'direction' in the problem solving, marks critical features, controls frustration and demonstrates solutions when the learner can recognize them." (p. 99), Bakker, Smit, and Wegerif (2015) note that it involves adapting teaching based on student actions
6. Shared Understanding: term used by the researcher to reflect when ideas or expectations are developed as a class, e.g., students know when a question asks them to "justify their answer," the teacher is expecting a written argument with all the pieces outlined in a framework developed in class through exemplars, card sorts, practice, and discussion
7. Questions requiring conceptual explanations (CEQs): term used by the researcher to reference those that require students to draw on multiple conceptual understandings or have more than one procedural pathway, often embedded in real-world context with varying correct reasoning approaches

Summary

Intentionally implementing argumentation in the mathematics classroom under specific framework and considerations has support from the National Council for Teachers of Mathematics (Rumsey & Langrall, 2016), but existing research offers a variety of models to choose from and says little about argumentation as a class structure rather than a way to approach certain tasks. This study explored how algebra students' reasoning changes over the course of a semester when using a collaborative argumentation framework and specific instructional guidelines motivated by student need and existing research.

The researcher conducted a mixed method, iterative analysis to answer the research questions "What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?" and "In what ways does the use of such a framework advance student mathematical reasoning?" The study spanned four iterations, characterized by four weeks of instruction, a Collaborative Argumentation Workshop, and two tests. During instruction, students worked in groups to answer questions that required conceptual understandings (CEQs) and evaluate their responses against the framework defining claim, evidence, and reasoning as the components of a solid mathematical argument. The focus of the first iteration was to develop a need for and introduce the collaborative argumentation framework. The focus of the second iteration was advancing students' prior reasoning strategies. The focus of the third iteration was reflection and revision against the framework. The focus of the fourth iteration was internalization of the framework. To address each iteration focus, the researcher coded sample student responses to track the frequency and quality of argument components, kept a reflective journal with observations about student engagement and

interactions, and updated a running list of instructional guidelines to follow based on student need and existing research.

This study holds significance for secondary and post-secondary mathematics teachers that wish to understand and advance the reasoning of their own students. It provides a detailed account of implementing instructional guidelines and a collaborative argumentation framework in an algebra classroom. The researcher identifies specific areas of student growth and areas in which students need more support.

CHAPTER II: LITERATURE REVIEW

Introduction

In a push for reform in mathematics education during the 1990s, it was noted that when a class is structured in an "I do, we do, you do" format, the students come to believe that there is one way to solve each problem and there is no need to reflect on the reasonableness of their work (Schoenfeld, 1992; Stein et al., 1996). Beginning in 2010, organizations such as the National Council for Teachers of Mathematics (NCTM) and the Common Core State Standards Initiative (CCSSI) have published recommendations and goals to engage in practices that encourage more creativity and reflection among students. One such recommendation is to use mathematical argumentation to develop conceptual understanding and mathematical proficiency (Rumsey & Langrall, 2016; CCSSI, 2010).

According to the CCSSI (2010), mathematical argumentation involves making conjectures or conclusions and justifying them with definitions, logical reasoning, counterexamples, or previously established results. The researcher broke down mathematical argumentation as a lens for teaching into three questions: What prompts a mathematical argument? What does a solid mathematical argument look like in the classroom? What support do students need to participate in mathematical argumentation? The existing literature addressing these questions points to four themes: (1) Asking questions that require conceptual explanations to create reason for (2) collaborative argumentation, through which students start with current understandings and are urged to practice (3) mathematical reasoning by using appropriate (4) language and developing shared understandings.

Questions that require conceptual explanations prompt a mathematical argument because they require students to develop and test ideas (Klavir & Hershkovitz, 2008), and then justify those ideas with explanations (Yee, 2002). However, students need support in developing solid mathematical arguments (Ervin-Kassab, Roddick, Vickery, & Tapper, 2020). One method of support has been described and refined using variations on collaborative argumentation models (Ervin-Kassab et al., 2020; Wagner et al., 2014; Zwiers, 2019). Within collaborative argumentation, students are required to reason mathematically (Zager, 2016; Conner, 2013; Yackel, 2001). However, when students lack such skills, they need support through specific language and developing shared understandings (NCTM, 2009; Brendefur & Frykholm, 2000; Conner et al., 2014; Weber et al., 2008).

Questions Requiring Conceptual Explanations (CEQs)

Across researchers, teachers, and organizations, there is a consensus that purposeful or "good" questions are essential in the mathematics classroom, and a common theme among definitions of such questions is that they require conceptual understandings (NCTM, 2014; Smith et al., 2018; Bingölbali & Bingölbali, 2021; Rahayuningsih et al., 2021; Kwon et al., 2006; Aziza, 2021; Fatah et al., 2016; Sholihah et al., 2020; Hancock, 1995). That is, they require students to draw on multiple conceptual understandings or have more than one procedural pathway, and they are often embedded in real-world context with varying reasoning approaches. For example, the task in Figure 1 is a CEQ because it requires students to understand the conceptual underpinnings of integers, exponents, and equivalence, and there are many approaches they can take to solving as well as varying explanations they can provide.

Max and Phoebe have a disagreement about the meaning of the expression 2^{4^2}.

- Max thinks it means $(2^4)^2$
- Phoebe thinks it means $2^{(4^2)}$

Find an integer that satisfies the following equation if Phoebe interpreted the left side and Max interpreted the right side.

$$\overset{\text{Phoebe}}{3n^{3^2}} = \overset{\text{Max}}{81n^{3^2}}$$

Explain the difference in Max's and Phoebe's perspectives. Who do you agree with?

Figure 1. CEQ, modified from (Pelfrey, 2000)

Many argue that CEQs are a way to both develop and assess students' mathematical creativity (Fatah et al., 2016; Kwon et al., 2006; Rahayuningsih et al., 2021), confidence (Fatah et al., 2016; Masitoh & Fitriyani, 2018), and critical thinking skills (Smith et al., 2018). Additionally, using CEQs regularly in the classroom shows students that while procedures are often useful in mathematics, understanding and explanation are equally if not more important (Yee, 2002). Thus, CEQs are acting as a motivator for students to engage in mathematical thinking and learning by showing them that there is value in the ability to develop and test ideas (Klavir & Hershkovitz, 2008) and by revealing gaps in their knowledge. After building and implementing an applied mathematics curriculum around five CEQs, Cline (2005) noted that "because these open-ended problems lacked the normal cues that tell students how to get started, they were very effective at revealing student misconceptions and illuminating weaknesses in their understanding." (p. 274).

If the purpose of using CEQs in the mathematics classroom is to encourage creativity and connections between concepts (Klavir & Hershkovitz, 2008), it is important that such questions do not become just an opportunity for students to regurgitate procedures with memorized definitions or rules. For example, in a study exploring the nature of mathematical questions in relation to mathematical thinking and reasoning, 53% of the CEQs meant to produce meaningful connections led students to only carry out procedures. Researchers noted that the difference between question intention and answer production showed a lack of student ability to engage in mathematical thinking (Stein et al., 1996). To clarify, the teacher may present students with the following CEQ.

> A student claims that the expression $(2x + 3)^2$ is equivalent to $4x^2 + 9$. The student's reasoning is, 'I just applied the distributive property.' Provide a mathematical argument either supporting or refuting the student's reasoning.

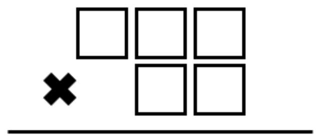
The teacher's intention with the question is to get students to think deeply about the meaning and application of the distributive property and how it does not apply to exponentiation over addition, but it could be used to expand the original expression $(2x + 3)^2 = (2x + 3)(2x + 3) = 4x^2 + 12x + 9$. However, if most student responses focus on showing that $(2x + 3)^2 \neq 4x^2 + 9$ with a counterexample, then there is a lack of student inclination to engage in the mathematical thinking that was asked of them. There may also be a lack of teacher support or structures in place to assist students with such thinking. That is, were students given tools to help them answer that question in a way that satisfies the teacher's intent? CEQs can be used to encourage students to think critically as well as realize gaps in their conceptual understandings, but the success of such a teaching strategy may have some dependence on the level and nature of teacher support.

Collaborative Argumentation

One way to support students through explanations and problem solving required when answering CEQs is through collaborative argumentation (Wagner et al., 2014; Rumsey &

Langrall, 2016; Hershkowitz et al., 2001). Collaborative argumentation, as derived from Zwiers (2019) and Conner (2013), is students working together to build up multiple arguments, evaluate evidence, and draw their own conclusions. If the teacher develops supportive actions and responses before implementation, collaborative argumentation can allow students to draw their own mathematical conclusions and internalize their understandings (Conner, 2013). However, asking students to participate in mathematical argumentation is asking them to provide more than their typically produced procedural explanations (Nachowitz, 2019). That is, students are pushed past outlining a series of steps as an explanation and are asked to provide a strong mathematical argument. Although different authors use different terminology, the commonly mentioned attributes of a strong mathematical argument can fit into the categories of claim/answer, evidence supporting that claim (such as a computation, diagram, definition, and/or a rule), and a connection or reason why the evidence supports the claim (Toulmin, 2003; Ervin-Kassab, et al., 2020; Zwiers, 2019). For example, in response to the CEQ shown in Figure 2, a group of students may present the argument shown in Figure 3. The response has been coded with the attributes of a solid mathematical argument. This is an example of a desired response during collaborative argumentation because the students have clearly explained their thinking and incorporated all three pieces of an argument. The claim is addressed in their direct response to the question, their evidence is shown in a description of, and hopefully visible, systematic testing of number combinations, and they explain the reasoning behind their approach by making connections between place value and multiplication. Following the collaborative argumentation model, if another group of students had questions or a countering argument, the two would be evaluated side-by-side and the evidence strengthened (Zwiers, 2019).

Place the digits 1, 3, 5, 7, and 9 in the proper boxes so that when multiplied they will produce the maximum product.



Generate a rule(s) that will enable you to properly place any five digits in a problem of a 3-digit number multiplied by a 2-digit number that will result in the maximum product.

Figure 2. CEQ two

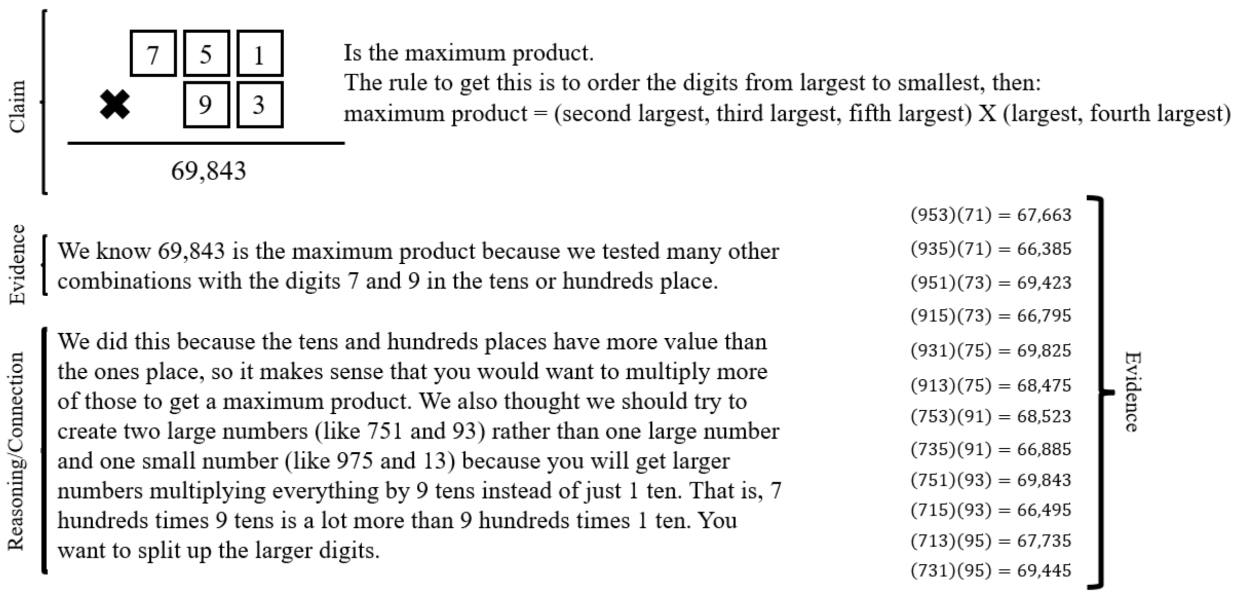


Figure 3. Mathematical argument exemplar

The collaborative portion of collaborative argumentation is supported by the notion that discourse among students strengthens their abilities to problem solve, consider counterarguments, and synthesize information (Vogel et al., 2016). That is, when students are working together to form or critique arguments, they are exposed to multiple perspectives and approaches as well as challenged to communicate their own thinking. In a study analyzing the interactions between students in an inquiry/argument culture, researchers found that students built and checked for consensus and took over the traditional teacher's role of validating mathematical ideas (Woods et al., 2006). Furthermore, a study regarding science education showed that collaborative argumentation helped students develop accurate conceptual understandings over time (Li, Li, & Wang, 2021).

Additionally, using collaborative argumentation routinely in the classroom supports the Common Core Math Practice 3: Construct viable arguments and critique the reasoning of others. As described on the Common Core (2010) website,

> Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. (p. 6)

Research has supported these claims by revealing that pushing students to form or discuss mathematical arguments in class leads to deeper discussions (Bostiga et al., 2016), and supports the completion of instructional and learning goals (Forman et al., 1998). Krummheuer's (1995) argumentation approach combines collaborative argumentation and Toulmin's (2003) model and has been used to show that emphasizing explanations through argumentation in the classroom centers mathematics learning around reasoning (Yackel, 2001), which is the second part of Math Practice 3.

Mathematical Reasoning

Within collaborative argumentation, students are required to use reasoning skills and strategies (Zager, 2016; Conner, 2013; Yackel, 2001; Hoffman et al., 2009). Reasoning is defined as “an evolving process of conjecturing, generalizing, investigating why, and developing and evaluating arguments” (Lannin et al., 2011). Student ability to reason is a foundational skill necessary for success in mathematics (Zager, 2016; Alawiya & Prasetyo, 2018; NCTM, 2000), and both critical thinking and problem solving involve mathematical reasoning (Mansi, 2003). Students who are adept at reasoning mathematically also possess highly transferrable skills. That is, students who can consistently reason through a problem demonstrate mastery of a way of thinking and analyzing situations that will allow them to problem solve, make connections, and communicate arguments outside of mathematics (Wenger, 2019; Papageorgiou et al., 2016). However, the repeated use of only examples to teach mathematics has resulted in students placing little value on investigating why an example or statement is true (Bieda & Lepak, 2014; Rowland, 2008), which is a part of reasoning. Many students can do mathematics without using reasoning skills, but this demonstrates only surface level knowledge (Mansi, 2003) which is inconsistent with the current goals of mathematics education (NCTM, 2014).

Furthermore, only about 6% of problems in textbooks and curriculums in the U.S. give students opportunities to reason about questions that require conceptual understandings (Thompson et al., 2012), which is “knowledge that is rich in relationships” (p. 3) and that focuses on why mathematics works and how it works together (Hiebert, 1986). Through analyzing different curricula, Thompson, Senk, and Johnson developed a list of different task requirements that lead to opportunities for students to reason. These include the requirement to make or investigate a conjecture, develop or evaluate an argument, find a counterexample, identify and correct a mistake, or identify pieces of an argument. Here, developing an argument is listed as a specific way to incorporate reasoning in the classroom. In fact, all these requirements are satisfied when students participate in collaborative argumentation. Studies that used tasks with these requirements to engage students in reasoning revealed some patterns in the nature of student reasoning prior to more directed practice. Specifically, they showed that the reasoning skills students often bring to the mathematics classroom lean more toward inductive—deciding their claim is true because they tried something that worked, necessary for beginning a mathematical argument—as opposed to deductive—using conceptual understanding to determine their claim is true in general (Edwards, 1997), which is necessary for finishing a mathematical argument. That is, students must be able to recognize patterns, provide rationale for procedures, and make connections between mathematical concepts in order to use deductive reasoning to develop an argument.

NCTM (2000) defines making connections as the ability to "recognize and use connections among mathematical ideas; understand how mathematical ideas interconnect and build on one another to produce a coherent whole; recognize and apply mathematics outside of mathematics." (p. 64). Thus, making connections, which is a large part of the reasoning process, is a determining factor in student ability to transfer mathematical knowledge. Kaur and Toh (2012) citing Coxford (1995) note "Teachers must provide students with opportunities to experience connections in the mathematics they learn. This is possible through links between conceptual and procedural knowledge, connections among mathematical topics and equivalent representations of the same concept." (p. 6). In order to form a solid mathematical argument, students must use reasoning skills and make connections to demonstrate a conceptual understanding of mathematics topics.

Promoting reasoning in the mathematics classroom makes mathematics a more cohesive, useful, and interesting subject for students (Bartels, 1995; NCTM, 2009). Research supports using iterative action/reflection cycles and generalizing using deductive reasoning (Ellis, 2007), as well as making predictions and reflecting on those predictions (Kasmer & Kim, 2011) to improve students' mathematical experiences as they relate to the current goals and recommendations of NCTM. Each of these pieces of support are reflected in the components of collaborative argumentation as outlined for this study—making and reflecting on predictions happens during the claim making process, generalizing using deductive reasoning is part of the reasoning component, and the iterative action/reflection cycle is incorporated through evaluating and editing peers' arguments during collaboration.

Language and Shared Understandings

One of the biggest qualities that students look for when evaluating mathematical arguments is clear communication of ideas (Bieda & Lepak, 2014). Research also claims that good reasoning skills follow good communication skills (Brendefur & Frykholm, 2000). Both of these claims support emphasizing good communication among students and teachers to allow collaborative argumentation to take place. For collaborative argumentation to be effective in the classroom, teachers and students must have a shared understanding of what constitutes a solid mathematical argument and the language that will be used in questions and responses (Yackel & Cobb, 1996; Ervin-Kassab, et al., 2020; Woods et al., 2006). Several frameworks that involve developing a shared understanding of expectations and classroom language for mathematical argumentation and reasoning have been developed and tested in the classroom (Hoffman et al., 2009; Nordin & Boistrup, 2018; Weber et al., 2008; Conner et al., 2014), leading to research-based suggestions for successful implementation of collaborative argumentation.

The first of such suggestions is that the responsibility of determining whether an argument is acceptable during class discussion should be given to the students (Weber et al., 2008). This encourages them to share partial ideas without fear of being corrected by the teacher and makes them stakeholders in the development of classroom knowledge and shared understandings. This is also supported by Hoffman's, et al. (2009) use of the Math Talk Learning Community Framework (Hufferd-Ackles et al., 2004) to implement argumentation in the classroom. This framework describes levels of classroom characteristics to foster discourse. The teacher roles in the highest level are described as follows:

> The teacher follows along closely to student descriptions of their thinking, encouraging students to make their explanations more complete…The teacher expects students to be responsible for co-evaluation of everyone's work and thinking. She supports students as they help one another sort out misconceptions. (Hufferd-Ackles et al., 2004, p. 90)

The second suggestion is that the teacher should pay attention to all modes of communication by students both during the development and presentation of an argument (drawings, scratch work, gestures, speech, writing, etc.). The teacher can then support students by showing them how to redirect all of these modes into one meaningful argument (Nordin & Boistrup, 2018; Conner et al., 2014).

The third suggestion is that students and teachers need to know why or why not a specific reasoning approach is appropriate. First, everyone involved in the classroom needs to be able to identify if reasoning is present. Second, students should collaborate in evaluating the strength of the reasoning. Inductive reasoning should prompt the class to work toward developing a backing that uses deductive reasoning. In doing so, they will either produce a solid mathematical argument, or they may determine that the reasoning is not mathematically sound at all (Weber et al., 2008).

Each of these suggestions centers around the class developing a shared understanding of expectations and language regarding argumentation. Shared understanding is developed through small group answer construction and representative tools (Michalchik et al., 2008), gestures (Alibali et al., 2013), shared experiences (Cobb et al., 1992), and whole class reflection/comparison of student responses as well as intentional positive reinforcement of desired response characteristics (Yackel & Cobb, 1996).

Summary

As teachers work to break the cycle of “I do, we do, you do” teaching in mathematics, one area to be explored is mathematical argumentation as a lens for teaching. Existing research on using mathematical argumentation in the classroom begins to address three questions: What prompts a mathematical argument? What does a solid mathematical argument look like in the classroom? What support do students need to participate in mathematical argumentation?

CEQs that require students to draw on multiple conceptual understandings or have more than one procedural pathway and are embedded in real-world context with varying reasoning approaches can act as a motivator for students to engage in argumentation. These types of questions do not offer the same cues that students are used to and instead assess students’ mathematical creativity, confidence, and gaps in understanding. With CEQs, students must use critical thinking and reasoning skills to get started and focus on patterns and concepts rather than procedures.

A synthesis of various argumentation frameworks used in the classroom classifies a solid mathematical argument as one including a claim, evidence to support that claim, and reasoning about how the evidence fits with the claim. Additionally, effective argumentation in the mathematics classroom involves collaboration in which students compare and critique arguments and develop shared understandings.

Students need a framework to work from when developing an argument, but ultimately, they need to be part of the experience of determining whether an argument is satisfactory. Students need to be involved in the process of identifying and evaluating the claim, evidence, and reasoning within an argument. Furthermore, students need flexibility and variety with modes of communication. It should be acceptable to communicate thinking verbally, through writing, gestures, and images or diagrams.

Using the above information, the focus of this study is refining the support students receive to make mathematical argumentation a more integral part of lessons and more accessible for all students and teachers.

CHAPTER III: METHODS

The purpose of this iterative analysis study was to explore how algebra students' reasoning changes over the course of a semester when using a collaborative argumentation framework with specific instructional guidelines motivated by student need. Presented in this methodology will be: (a) instrumentation and design, (b) site and subjects of the study, (c) data collection, and (d) analysis of data.

Instrumentation and Design

This was a mixed method, iterative analysis study, following the iterative analysis for qualitative research framework outlined in the International Journal of Qualitative Methods (Srivastava & Hopwood, 2009) and comparing coded student responses across two units after each iteration. Iterative analysis allowed the researcher to recognize patterns and themes as they emerged during data collection, which lent itself to the nature of teaching and learning in the mathematics classroom. The ability to reflect and make adjustments during the data collection process helped prevent the researcher from continuing to give students ineffective or inappropriate materials and lessons as soon as data collection indicated they needed adjustments.

Iterations were defined by a four-week period of instruction, during which the researcher implemented practices designed to support students with collaborative argumentation, and at the end of which the researcher made adjustments to these practices based on observations. During all four weeks of each iteration, students participated in instruction of two units of material by the researcher. This consisted of two lecture classes and two lab classes each week. In lecture classes, students worked in groups of three or four at whiteboards while the teacher facilitated discussion through scaffolded problems and questions. In lab classes, students worked in the same groups to complete assignments of open-ended problems related to the lecture content from the previous day. Students were given an 8-10 question test every two weeks (2 tests per iteration). During the second unit of each iteration, students participated in a Collaborative Argumentation Workshop designed based off existing research and study observations to support students with collaborative argumentation. At the end of each iteration, the researcher coded assignment and test responses from the two units and recorded observations about student engagement, specific teaching moves, areas of student growth within the collaborative argumentation framework, and areas that needed improvement. These qualitative and quantitative data were analyzed through the lens of the three iterative analysis framework questions from Srivastava and Hopwood (2009).

> Q1: What are the data telling me? (Explicitly engaging with theoretical, subjective, ontological, epistemological, and field understandings)
> Q2: What is it I want to know? (According to research objectives, questions, and theoretical points of interest)
> Q3: What is the dialectical relationship between what the data are telling me and what I want to know? (Refining the focus and linking back to research questions) (p. 78)

After analysis, the researcher made adjustments to the research focus and course materials as suggested by the data, developed the next Collaborative Argumentation Workshop, and repeated the process over the next four weeks. The specific research foci and Collaborative Argumentation Workshops are outlined in the data collection section of this methodology.

The collaborative argumentation framework referenced in this study is a theoretical framework adapted from a synthesis of tested frameworks (Toulmin, 2003; Ervin-Kassab, et al., 2020; Zwiers, 2019). It was used to support students in developing mathematical arguments.

Toulmin's model (2003) is a general framework designed to help students develop arguments in any subject. It breaks down an argument into three pieces: claim, evidence, and warrant. The claim is a statement of some mathematical solution, usually the direct answer to a problem or question. The evidence consists of data or work that supports the claim. The warrant explains why the data is valid and makes sense with the claim (Toulmin, 2003). Singletary and Conner (2015) analyzed the use of Toulmin's (2003) framework in a math classroom, noting the importance of the "warrant" piece as an avenue for addressing the "why?"

The Claim-Rule-Connection model (Ervin-Kassab, et al., 2020) was developed by middle school math teacher Alison Vickery, specifically to teach students how to justify their reasoning. "The 'claim' was the answer or response to the question; the 'rule' was the theorem, fact, or proof; and the 'connection' was an explanation of how the rule applied or connected to the claim." (p. 1004). Vickery (2020) found the language of this framework to be accessible to students, applicable across mathematical and nonmathematical arguments, and found that students eventually began approaching problems from the Claim-Rule-Connection perspective without being explicitly asked to.

The Collaborative Argumentation model (Zwiers, 2019) outlines six steps in the argumentation process, designed to be used in any classroom. In this framework, students work together to (1) make claims and state initial evidence, (2) build up all sides of the argument with more evidence, (3) poke holes in evidence, (4) decide on criteria to evaluate the evidence, (5) use criteria to evaluate the evidence and compare claims, (6) choose an argument and communicate why it is best. Although this specific model is not referenced, "Authentic Argumentation with Prospective Secondary Teachers: The Case of 0.999…" written by high school math teacher AnnaMarie Conner (2013) discusses a task in which the collaborative argumentation steps were followed as students debated whether 0.999… = 0. Conner (2013) notes that the keys to this successful implementation were choosing appropriate tasks/questions, clearly outlining classroom norms/expectations, and developing supportive responses/actions prior to class. This last key was imperative to ensure students came to their own conclusions through argumentation.

Each of these three frameworks address a different issue when it comes to argumentation in math. In Toulmin's (2003) model, the use of a warrant helps illuminate the "why?" in mathematics. That is, conceptual understandings become imperative for students when using this model. However, the Claim-Rule-Connection (2020) model offers a less nuanced and more classroom friendly language for students to work with. Then, the Collaborative Argumentation (2019) model focuses on students editing and developing arguments together as well as taking ownership of new knowledge. Together, these frameworks led to the researchers Claim-Evidence-Reasoning framework. The components of this framework are described in Figure 4. Similar to the pyramid structure, if the reasoning component is removed from an argument, the argument falls apart.

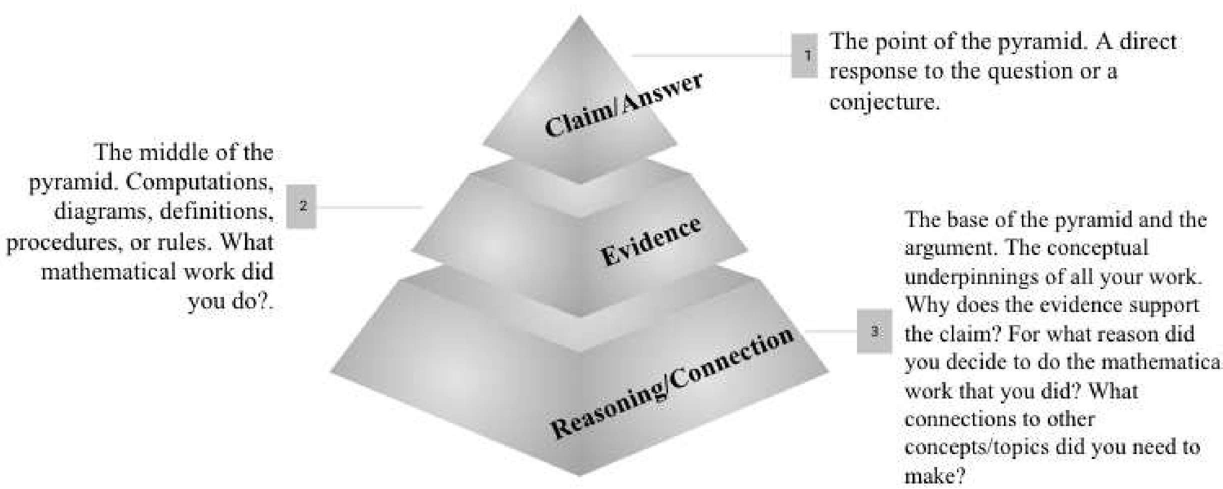


Figure 4. Collaborative argumentation framework

Student written and oral arguments were coded using these three characteristics of a strong mathematical argument suggested by the framework. This coding took place on written unit tests and in-class group work, as well as on responses during argumentation workshops. Additionally, the researcher kept a reflective record of classroom observations and themes emerging during each iteration.

Site and Subjects of the Study

This study took place at a midwestern state university with an enrollment of about 24,000. Students participating in the study were enrolled in the university's Intermediate Algebra course. Enrollment in this course is recommended by the university to students who have not mastered algebra concepts needed for college algebra. This is typically determined by the student's mathematics ACT score and/or the university mathematics placement exam. If placed in this course, students must receive a "C" or better to advance to other mathematics classes. Topics covered include linear and quadratic equations, absolute value equations and inequalities, linear and nonlinear inequalities, properties of exponents, rectangular coordinate systems, lines, circles, parabolas, systems of equations, polynomials and rational expressions, and functions. The course consists of 3 lecture contact hours and 2 lab contact hours.

The two sections of the Intermediate Algebra course used for the study took place on Mondays and Wednesdays from 1:00 – 2:15 pm or 2:30 – 3:45 pm and Tuesdays and Thursdays from 1:00 – 1:50 pm or 2:30 – 3:20 pm. Students also had the opportunity to attend office hours offered twice a week or visit the free tutoring center at the university library.

Although the whole class was receiving the same instruction, participation in the study was entirely voluntary. The researcher described the study and gave students consent forms at the beginning of the semester. 51 out of 61 students participated in this study. Participants completed an anonymous demographics and mathematics experience survey. Responses indicated that students in the course ranged from age 18 to 30 with over 75% between the ages of 18 and 20. Of those participants, 21 were first generation college students, five were English second language (ESL) and international students, and 10 were retaking the course. About 70% of the students reported they had passed Algebra II or a higher mathematics course in high school prior to enrolling in the Intermediate Algebra course at the university. 42% said they felt successful in past mathematics courses.

Classrooms had multiple whiteboards on each wall for students to work on and a projector at the front of the room for the teacher to display problems and questions. Students completed tests and assignments on paper and turned in test papers but submitted pictures of assignments on the classes Blackboard website. Answer keys with thorough examples of written arguments/explanations were posted on Blackboard for every assignment and test. Additionally, course materials such as PowerPoints, instructional videos, handouts, and supplemental materials were posted on Blackboard for student access at any time.

Data Collection

Permission to conduct this study was obtained from the International Review Board and Missouri State University with IRB number IRB-FY2022-530 on April 7, 2022 as seen in Appendix A. Subjects provided consent through signed forms.

First Iteration: Developing Need and Adopting the Framework. Under the research questions, “What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?” and “In what ways does the use of such a framework advance student mathematical reasoning?" the first iteration focus was developing need and adopting the framework. Through this focus, the researcher wanted to observe student response and receptiveness to the collaborative argumentation framework and identify students' initial capacity to engage with the framework. This iteration spanned units 1 and 2, written by the researcher and the research supervisor, covering integer subtraction, equal exchanges, operations with fractions, ratios, and percentages.

To address the first iteration focus, the researcher used the suggestions of previous studies on implementing collaborative argumentation (Hoffman et al., 2009; Nordin & Boistrup, 2018; Weber et al., 2008; Conner et al., 2014) and developing need and a shared understanding (Michalchik et al., 2008; Alibali et al., 2013; Cobb et al., 1992; Yackel & Cobb, 1996), to introduce content through the following guidelines.

1. Questions requiring conceptual explanations (CEQs) to disrupt students' current understandings
2. Collaboration (small groups, partners, whole class) to expose students to communicating mathematical thinking
3. Focus on the difference in students' current understandings and desired understandings
4. Use what motivates students to create a need for the collaborative argumentation framework
5. Collaborative Argumentation Workshop I (Appendix B) – introduction to the framework

Unit 1 instruction took place in this manner prior to students being introduced to the collaborative argumentation framework developed for this study. After exposure to CEQs and receiving feedback and scores on the unit 1 test, students participated in Collaborative Argumentation Workshop I, in which they were introduced to the collaborative argumentation framework. Then, students participated in unit 2 instruction and took the unit 2 test. The researcher kept a record of observations about this iteration's focus and coded student responses to CEQs from units 1 and 2 (Appendix C).

Second Iteration: Breaking Old Habits and Boosting Confidence. Now that steps were taken to create need for and introduce the collaborative argumentation framework and the researcher had identified students' current capacity for engaging with the framework, the second iteration focus was defined. Through instruction of units 3 and 4 and Collaborative Argumentation Workshop II (Appendix D), the researcher focused on breaking old reasoning patterns and giving students confidence in their ability to utilize the framework.

In addition to the instructional guidelines used for the first iteration, observations from the first iteration and results from an educational study that found self-grading to drastically improve test scores among middle school science students (Sadler & Good, 2006) led to the development of new guidelines.

1. No ideas are shut down by the teacher (instead, collaboration and questioning are used to move students to recognize faulty evidence or reasoning)
2. Students view exemplar arguments and practice coding them against the framework
3. Focus on difference between concepts and procedures
4. Collaborative Argumentation Workshop II (Appendix D) – practice coding an exemplar mathematical argument

This iteration followed the same four-week format as the first iteration—two weeks of instruction by the guidelines, unit 3 test, Collaborative Argumentation Workshop II and two more weeks of instruction, unit 4 test. Topics covered include maximizing area, pattern recognition, writing equations, exponents, doing & undoing (solving equations), and functions. The researcher kept a record of observations about this iteration's focus and coded student responses to CEQs from units 3 and 4 (Appendix E).

Third Iteration: Reflecting, Revising, and Questioning. For the third iteration, observations that student responses were still frequently lacking the reasoning component when coding and existing research suggesting that cues and corrective feedback can have valuable effects on learning (Hattie & Timperley, 2007) led to the development of the focus on reflecting, revising, and questioning against the collaborative argumentation framework. The following guidelines were added to the instructional practices.

1. Whole class reflection and revision of arguments (teacher models language and provides positive reinforcement)
2. Focus on components as cohesive rather than disjoint
3. Collaborative Argumentation Workshop III (Appendix F) – real-time self-reflection and grading by students as well as the teacher modeling grading and feedback.

Following the four-week format of the first and second iterations, students participated in two weeks of instruction per the guidelines, the unit 5 test, Collaborative Argumentation Workshop III and two more weeks of instruction, then the unit 6 test. Topics covered include the distributive property, difference of squares, rate of change, and linear and exponential relationships. The researcher kept a record of observations about this iteration's focus and coded student responses to CEQs from units 5 and 6 (Appendix G).

Fourth Iteration: Internalization and Advancements. Based on the record of observations from previous iterations, no additional instructional guidelines were added for the fourth iteration aside from Collaborative Argumentation Workshop IV (Appendix H), which asked students to explain the framework and use it to edit previous arguments. The focus of this iteration was to make observations about student internalization of the collaborative argumentation framework by no longer consistently prompting them to use the framework and to continue advancing the rigor of varying reasoning strategies using the instructional guidelines from prior iterations.

Due to the semester lasting 15 weeks, this iteration spanned three weeks instead of four. There were three weeks of unit 7 instruction and Collaborative Argumentation Workshop IV, followed by a final argumentation evaluation and a comprehensive course final. Topics covered in unit 7 include quadratic relationships, more linear and exponential relationships, area, and volume. The final argumentation evaluation consisted of five CEQs pulled from previous unit tests. Students were to choose three of these problems to complete and craft their best mathematical arguments. The researcher kept a record of observations about this iteration's focus and coded student responses to CEQs from unit 7 (Appendix I) and the final argumentation evaluation (Appendix J). Figure 5 shows an overview of the four iterations including the topics covered, research focus, and important characteristics of each Workshop.

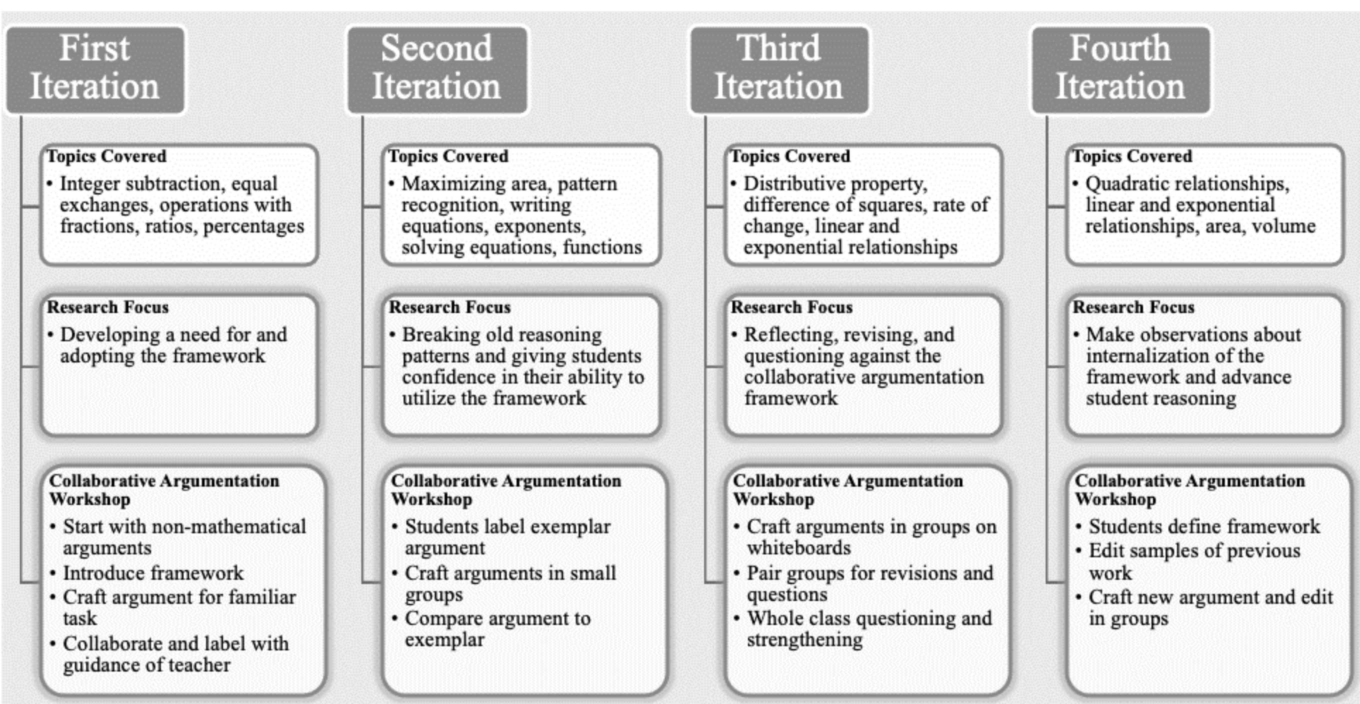


Figure 5. Iteration overview

Analysis of Data

Both quantitative and qualitative data were collected over the course of the study. As student responses were collected during each iteration, the researcher coded and tabulated the arguments according to the framework described in the Instrumentation and Design section. This coding labeled the components of "claim," "evidence," and "reasoning" if they were present. Additionally, each component was labeled with a plus sign to indicate "complete/correct" or a minus sign to indicate "incomplete/incorrect." The coding for each iteration was tabulated to create counts and percentages indicating the frequency of claims, evidence, and reasoning components throughout each iteration and the quality (plus/minus) of the existing components. For reference, Table 1 provides examples of each coding.

Table 1. Examples of framework response coding

Claim	
+ Terrel ate more pizza.	- Carlos ate more pizza.
Evidence	
+ It takes 8 eighths to make a whole and 6 sixths to make a whole. $8 - 7 = 1$ piece left for Terrell and $6 - 5 = 1$ slice left for Carlos. Eighths are smaller than sixths, so Terrell has less left over, which means he ate more initially.	- $\frac{7}{8} = \frac{42}{48}$ $\frac{5}{6} = \frac{40}{48}$ *(does not show how they produced these equal exchanges)*
Reasoning	
+ The more of a certain unit it takes to make 1 whole, the smaller the size of the unit. To compare amounts, we either need the quantity or unit size the same. If we look at what they have left over, the quantity is 1 for each of them, so we compare the size of eighths and sixths.	- $\frac{7}{8}$ is split into more pieces than $\frac{5}{6}$. *(does not describe why this is relevant or how they know)*

Argument components are in response to the CEQ, "Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell ate 7 eighths of the other pizza. Who ate more pizza? Provide a solid mathematical argument for your answer."

Along with response coding, the researcher kept a reflective record of themes and observations. This record was updated in regard to student engagement, specific teaching moves, areas of student growth within the collaborative argumentation framework, and areas that needed improvement. These themes and observations were the driving force behind the creation of each argumentation workshop and any changes made to research foci or reflection techniques between each iteration.

Summary

This study was a mixed method, iterative analysis, allowing the researcher to reflect and make adjustments during the data collection process. There were four iterations defined as four-week periods of instruction including one Collaborative Argumentation Workshop. During each iteration, the researcher kept a journal of observations about student engagement, specific teaching moves, areas of student growth within the collaborative argumentation framework, and areas that needed improvement. Additionally, they coded student responses to CEQs according to the collaborative argumentation framework developed for the study. This framework identified three components of a solid mathematical argument—claim, evidence, reasoning.

Participants in the study were enrolled in an intermediate algebra class taught by the researcher at a midwestern state university. These classes took place four days a week and consisted of group work and class discussions. Topics covered include linear and quadratic equations, absolute value equations and inequalities, linear and nonlinear inequalities, properties of exponents, rectangular coordinate systems, lines, circles, parabolas, systems of equations, polynomials and rational expressions, and functions. During each iteration, students completed two CEQ assignments per week and one test every two weeks.

Permission to conduct this study was obtained from the International Review Board and Missouri State University. The research questions were "What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?" and "In what ways does the use of such a framework advance student mathematical reasoning?" The focus in the first iteration was developing need for and adopting the collaborative argumentation framework. The second iteration focused on breaking old reasoning patterns and giving students confidence in their ability to utilize the framework. Reflecting, revising, and questioning against the collaborative argumentation framework was the focus that guided the third iteration. Finally, the fourth iteration focused on making observations about student internalization of the framework and continuing to advance student reasoning strategies.

Over the course of the four iterations, the following list of teaching guidelines was used when introducing content:

1. Questions requiring conceptual explanations (CEQs) to disrupt students' current understandings
2. Collaboration (small groups, partners, whole class) to expose students to communicating mathematical thinking
3. Focus on the difference in students' current understandings and desired understandings
4. Use what motivates students to create a need for the collaborative argumentation framework
5. Collaborative Argumentation Workshop I – introduction to the framework
6. No ideas are shut down by the teacher (instead, collaboration and questioning are used to move students to recognize faulty evidence or reasoning)
7. Students view exemplar arguments and practice coding them against the framework
8. Focus on difference between concepts and procedures
9. Collaborative Argumentation Workshop II – practice coding an exemplar mathematical argument

10. Whole class reflection and revision of arguments (teacher models language and provides positive reinforcement)
11. Focus on components as cohesive rather than disjoint
12. Collaborative Argumentation Workshop III – real-time self-reflection
13. Collaborative Argumentation Workshop IV – internalize framework and edit arguments

These guidelines were updated and amended with each iteration according to the previous iteration's data collection themes. These themes were identified through coding and comparing student responses to CEQs and synthesizing journal observations. For an example of coded responses, see Table 1 in the Analysis of Data section.

CHAPTER IV: RESULTS

These results include quantitative data consisting of response coding tabulations and qualitative data consisting of a synthesis of themes and observations from the researcher's reflective journal. Some sample student responses that encapsulate overall student responses are included to show how responses changed throughout the iterations. This will cover (a) frequency of reasoning component in student responses, (b) student interaction with the framework, (c) areas of student growth, (d) areas that still need addressed.

Frequency of Reasoning Component in Student Responses

Argumentative responses to open-ended questions were collected and coded throughout each iteration to track the frequency and quality of each framework component. Students responded to in-class group tasks, tests, and argumentation workshops. Each response was coded first by "claim", "evidence", and "reasoning" components (frequency), then each component was coded with a plus to denote "complete/correct" or a minus to denote "incomplete/incorrect" (quality). Overall, frequency of a claim component remained above 90% for each iteration, with more than 70% of those claims being coded with a plus. Coding tabulations revealed growth in the evidence and reasoning components. Table 2 shows the percent of responses including each component within each iteration. Additionally, it lists what percentage of those components were complete/correct versus what percentage were incomplete/incorrect. For example, the first row of the evidence column shows that 95.71% of student responses coded in the first iteration included an evidence component. It also shows that 35.82% of the evidence components that were included were complete/correct and 67.42% were incomplete/incorrect.

Table 2. Response coding tabulations for framework components

Iteration	Evidence		Reasoning	
	95.71		34.29	
First	+	-	+	-
	35.82	67.42	31.25	68.75
	92.54		44.03	
Second	+	-	+	-
	62.10	37.91	40.68	59.32
	92.16		41.18	
Third	+	-	+	-
	72.34	27.66	23.81	76.19
	97.44		66.03	
Fourth	+	-	+	-
	75.66	24.34	39.81	60.19

Although most students included an evidence component throughout the iterations, Table 2 shows that the quality of student evidence improved throughout the study. In the first iteration, only 35.82% of evidence components were complete/correct. This percentage continued to grow with each iteration, and by the fourth iteration, 75.66% of evidence components were complete/correct. Table 2 also shows that the frequency of a reasoning component increased throughout the iterations. In the first iteration, only 34.29% of student responses included a reasoning component—this would be about 18 students. By the end of the study, 66.03% of student responses included a reasoning component—this would be about 35 students. Additionally, the number of students consistently using complete/correct reasoning grew from about 6 students in the first iteration to about 14 in the fourth iteration.

For reference, the sample response from the first iteration in Table 3 was coded with an incomplete/incorrect evidence component and no inclusion of a reasoning component. Here, the "claim" that "3/4 is more" is correct. The "evidence" is "the unit (4) is smaller than 5, but the quantities are the same." This was coded as "incomplete/incorrect" because the units are fourths and fifths, not 4 and 5, and fourths are greater than fifths, not lesser. There is no reasoning component explaining why the size of the unit helps determine which number is greater or how to know which unit is greater.

Table 3. Incomplete/incorrect evidence or complete/correct evidence and reasoning

Incomplete/Incorrect	Complete/Correct
Task	
Which is more: 3 fourths or 3 fifths? Why?	Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell ate 7 eighths of the other pizza. Who ate more pizza? How do you know?
Response	
¾ is more because the unit (4) is smaller than 5, but the quantities are the same, making ¾ a larger quantity.	Claim: Terrell ate more pizza. Evidence: 1/8 is smaller than 1/6 so Carlos has more pizza left which means Terrel ate more.  Reasoning: One pizza that is split into 8 units has smaller units than a pizza that is split into 6 units. Since each pizza only has 1 unit leftover, the pizza with the smallest unit would have the least left.

Alternatively, the sample response from the fourth iteration in Table 3 was coded with a complete/correct evidence component and a complete/correct reasoning component. Here, the evidence component (labeled by the student) was coded as complete/correct because the student drew a diagram showing that Carlos and Terrel both had one piece left, then stated which of those pieces would be greater based on unit size. That is, they showed enough work and provided enough evidence to lead to the correct claim. Additionally, the response included a reasoning component (labeled by the student) that explains why 1 eighth is lesser than 1 sixth and why the leftover piece mattered.

Student Interaction with the Framework

Based off experiences in a previous study that the researcher conducted, *Using Mathematical Arguments as a Tool to Develop Conceptual Understanding in a Developmental Mathematics Class*, the researcher knew that students needed clear opportunities that necessitated writing conceptual mathematical explanations as well as a framework that supported them in this process. Intentional actions by the teacher created a need within students for the chosen framework and caused them to internalize it.

Creating a Need. During the first iteration up to Test 1, not a lot of class time was given for students to reflect on the quality of their own responses, and the teacher had not yet trained students to evaluate the quality of their responses against the argumentation framework of claim, evidence, reasoning. Argumentation Workshop I did not take place until after Test 1, and before this, students were often frustrated when being questioned in class. Many students held the belief that they already knew quick tricks, such as “keep change flip,” and did not need to go back and explain concepts. This belief had to be disrupted through questions that required a conceptual understanding and/or explanation. To be effective, such questions had to create cognitive dissonance by exposing the limitations of students' tricks and reasoning strategies. Questions that require conceptual explanations exposed limitations by doing at least one of three things for students: (1) provide an experience in which their trick does not work, (2) ask them to find a pattern that takes more exploration than they are typically inclined, or (3) the procedures they need to use are so heavily embedded in context that students need to understand the concepts behind their tricks. Examples of questions doing each of these things are provided in Table 4.

Instead of reflection or evaluation, the major teaching focus during the first iteration but before Test 1 was getting students comfortable sharing mathematical ideas within their groups and familiar with the types of problems outlined above. On Test 1, students were expected to respond to such questions and responses were scored based off the argumentation framework. By having students create written responses, providing feedback on their responses, and providing scores to characterize the quality of their response based on a framework they had not yet seen, students were confronted with the fact that they were not meeting the standards necessary to achieve higher scores. Students desired higher scores, but they did not know what the higher standards for responses included. This drove them to ask, "What should our responses look like?" That is, they saw a need for the framework. The teacher created this need, but the students were the ones that asked for it.

Upon asking for it, students were participated in Workshop I, in which they were exposed to the collaborative argumentation framework. Figure 6 offers a description of the focus and activities of Collaborative Argumentation Workshop I. This workshop defined the components of claim, evidence, and reasoning in both non-mathematical and mathematical contexts. Students had the opportunity to practice crafting arguments and comparing them to the teacher's expectations that they include each component. They also practiced collaborating to make their arguments better. Initially, the researcher noted in observations that students participated in the argumentation workshop, but then reverted to their old response patterns. However, the teacher kept consistently high and unchanging expectations that students satisfy the framework. This led students to realize that reluctance would not change what was required of them to achieve desired scores.

Table 4. Examples of CEQ disruptions

Disruption	Question	Explanation
1. *Provide an experience in which their trick does not work*	Explain why $(-3)^2 * (-3)^3$ has a different answer than $((-3)^2)^3$.	Simply “adding the exponents” or “multiplying the exponents” does not explain why these expressions simplify differently. Students must know where those tricks came from.
2. *Ask them to find a pattern that takes more exploration than they are typically inclined*	What do you think might happen to the area of a rectangle if you increase the perimeter? Explain your reasoning.	If students just come up with one example, say a 3x3 square increasing to a 4x4 square, they will draw an incorrect conclusion. They need to test multiple examples systematically to reach the correct conclusion.
3. *The procedures they need to use are so heavily embedded in context that students need to understand the concepts behind their tricks*	You have 4 pizzas to portion out to a group. You want each person to receive 2 thirds of a pizza. How many people can be fed? Explain your reasoning.	If students do not know the concepts behind division and fractions, they will not recognize that this question prompts them to do $4 \div \frac{2}{3}$. Since they do not set up this expression, they do not use the “keep change flip” trick.

By the end of the first iteration, classroom observations showed an increase in student buy in. Such buy in was shown through students using argumentation language during group work and asking the teacher questions such as, “How would I put my reasoning into words for this response?” or “Is this enough evidence to support my claim?” By asking students questions that required a conceptual explanation, scoring responses before introducing the framework, and keeping expectations consistent, the teacher created a need for the collaborative argumentation framework. After the first iteration, the need for the argumentation framework was motivated by less than desired scores on test one responses. However, for the framework to be truly effective in improving student reasoning skills, students needed to view the process as a tool to improve their understanding rather than just a tool to satisfy teacher expectations. The students needed to internalize the framework.

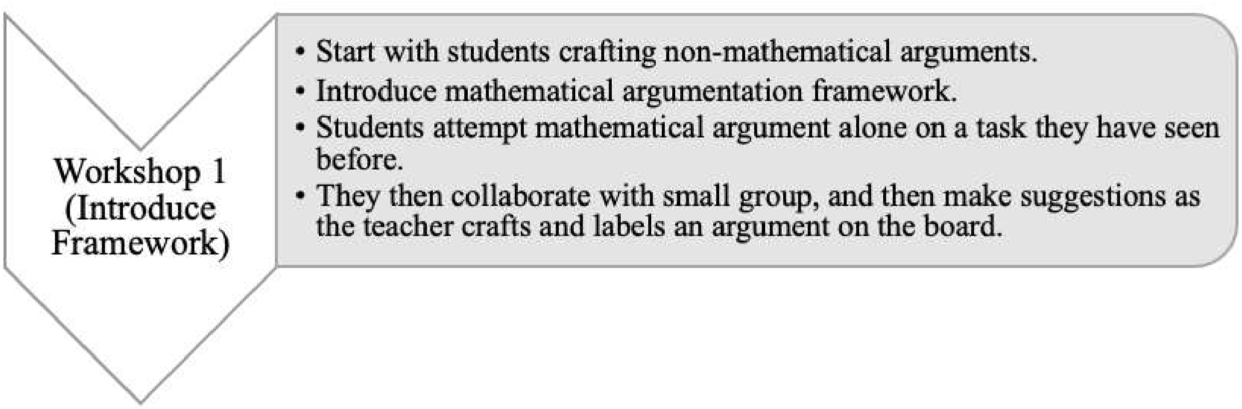


Figure 6. Collaborative argumentation workshop i focus

Internalization. While students recognized the need for an argumentation framework, they were still seeking specific examples of what the teacher was looking for. When shown these examples, many students expressed a lack of confidence in their abilities to generate responses of similar quality. When the researcher inquired about this lack of confidence, students indicated that previous teachers had never had high mathematical expectations for them because they already struggled to meet the “normal” mathematical expectations. That is, many of them had never even been asked to try to produce mathematical arguments. Given this fact and the fact that less than desired test one scores seemed to be their primary motivation, students were more concerned with teacher approval of arguments rather than growing in their understanding of mathematical concepts through the argumentation process. The researcher noted this through observations of student questions centering around “Is this what you are looking for?” or “Is this what you would want us to say on the test?” The researcher identified two things that needed to happen for students to internalize the framework: (1) students needed to feel confident in their ability to identify a solid mathematical argument, and (2) student focus needed to shift from reaching desired scores to reaching a desired understanding.

To address 1—students need to feel confident in their ability to identify a solid mathematical argument—the teacher consistently referred to the pieces of mathematical arguments when they were modeling responses and thinking throughout the second iteration. Additionally, Workshop II included opportunities for students to code an exemplar as well as craft and code one of their own. Figure 7 offers a description of the focus and activities of Collaborative Argumentation Workshop II. Figure 8 shows an example of one of the coded exemplars used in the workshop. Students were first given this response without the coding, and then compared their own coding to that of the teacher.

After this, students crafted and coded a response to the question “Are the expressions $(x^2)^3$ and $x^2 \cdot x^3$ equivalent? Why or why not?” This activity along with other opportunities for self-reflection in class were designed to push students to pay attention to the quality of their responses against the argumentation framework. After this workshop, students were observed referencing the framework when developing responses, which served as evidence that they were beginning to internalize the framework.

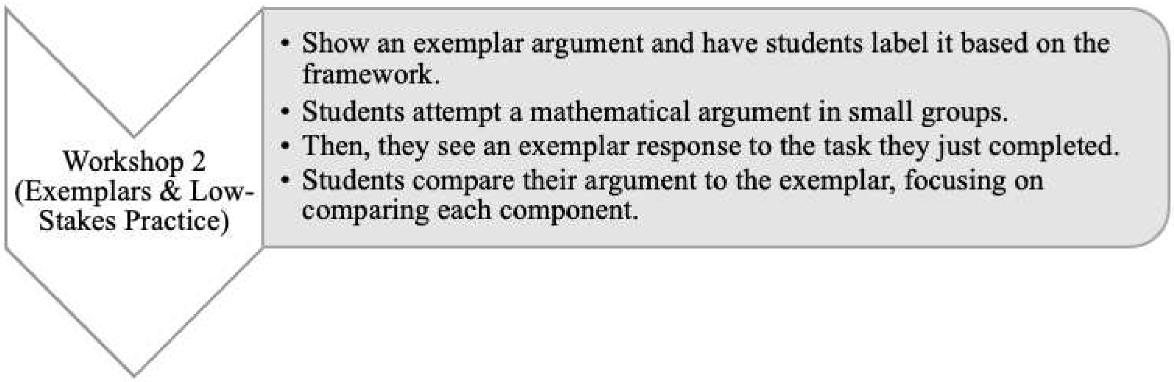


Figure 7. Collaborative argumentation workshop ii focus

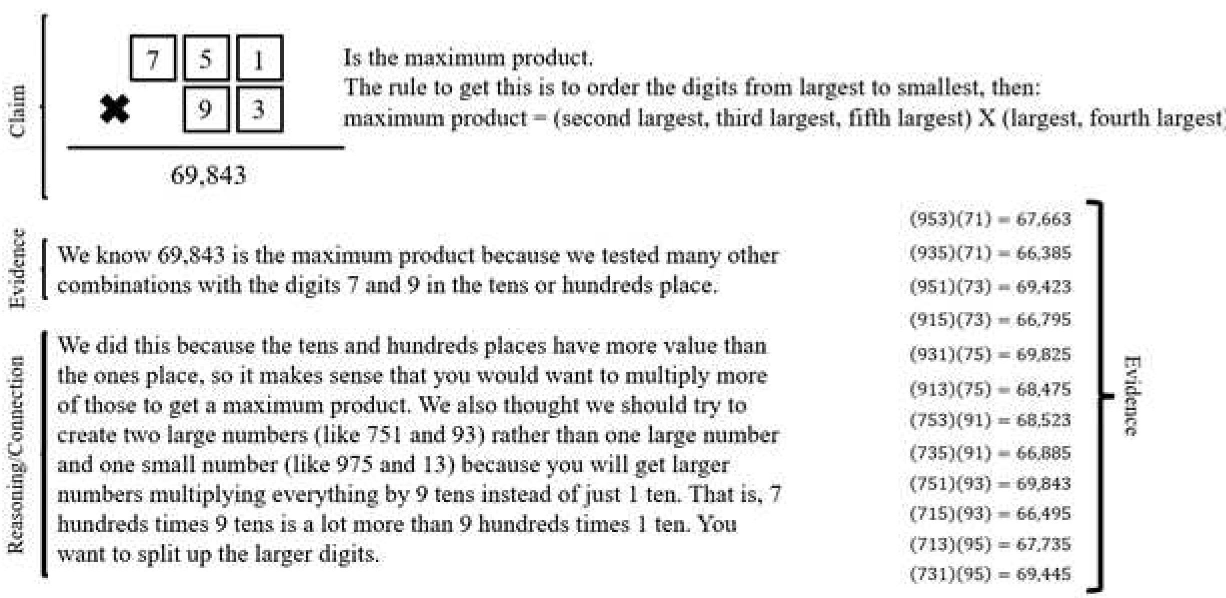


Figure 8. Coded response exemplar

Then, in the third iteration, Workshop III was aimed at pushing students to feel confident in their own evaluations of arguments rather than seek approval from the teacher after each response. Figure 9 offers a description of the focus and activities of Collaborative Argumentation Workshop III.

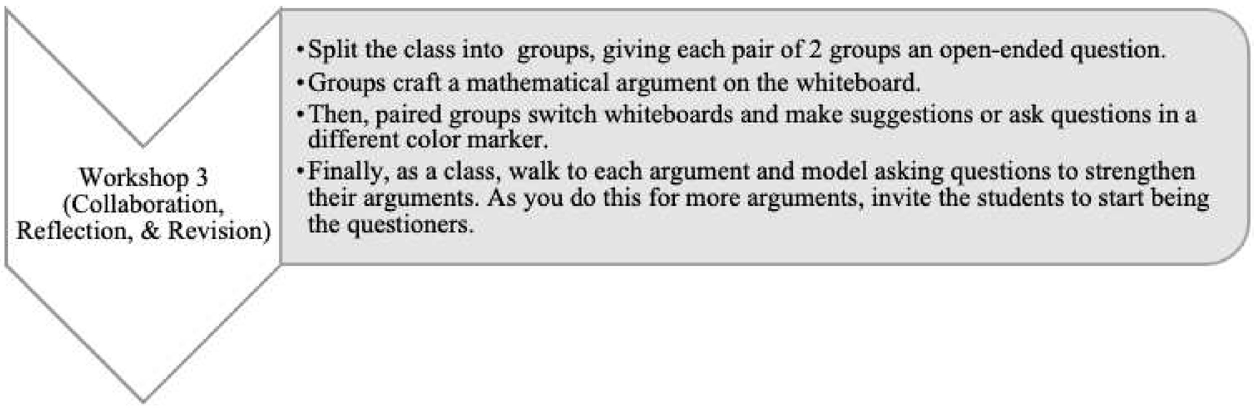


Figure 9. Collaborative argumentation workshop iii focus

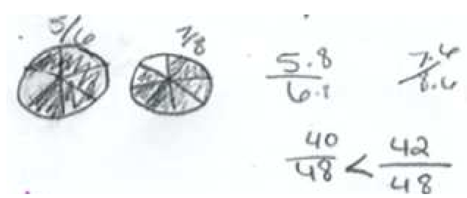
Specifically, Workshop III consisted of an activity in which groups of students developed and presented arguments, and then received real time feedback from the researcher and peers. This made it clear to students that even though their thinking and reasoning seem clear and complete to them, other readers may be left with questions if they do not clearly connect each piece of their argument and fully elaborate on their reasoning. The teacher continued to implement small activities like this during class where students were prompted to share their arguments and others were encouraged to ask questions. In observations after Workshop III, the researcher noted students telling the teacher that they finally felt like they knew what was expected of them and could see the difference between their previous responses and the exemplars.

To address 2—student focus needed to shift from reaching desired scores to reaching a desired understanding—and further solidify student internalization, the researcher wanted to present questions that emphasized reasoning rather than correct answers. To accomplish this, many of the questions in the third and fourth iterations either provided a claim for students and asked them to develop the rest of the argument, or they provided a potential claim and asked students to come up with an argument that supports or refutes it. For example, the CEQ from the fourth iteration, "A student claims that $(x - 3)^2$ is an equal exchange for $x^2 - 9$. Do you agree or disagree? Why?" provides a claim and requires the student to agree or disagree.

By the end of the fourth iteration, the researcher's observations reflected that many students were enthusiastically participating in class discussions and activities, and the comradery among student groups was very high. For example, the researcher noted having to use attention getters to call back the whole class after small group discussions because the groups were so involved in the task and argumentation process. Students were willing to make mistakes in front of their peers, question others' thinking respectfully, and ask others for help when they needed it. The researcher witnessed groups planning their own study sessions, swapping responses to give feedback, and comparing their own work to keys. For example, while it once was common for students to ask the teacher to read their response and let them know if it "made sense" or "was right," the researcher now observed students asking this of their group members. Additionally, some students were coding their own responses even when they were not prompted. The example in Table 5 shows a side-by-side comparison of a student response to a similar question from the first and fourth iterations.

This example shows the student coding their own response without prompting in the fourth iteration response, as well as using reasoning strategies (comparing quantity and unit sizes, looking at what is leftover) and conceptual language (equal exchange, unit) discussed in class. These observations showed that students were using the framework as a helpful tool to write solid mathematical arguments and understand concepts. That is, students had internalized the collaborative argumentation framework.

Table 5. Comparison of first and fourth iteration CEQ responses

First Iteration	Fourth Iteration
Task	
Which is more: 3 fourths or 3 fifths? Why? Use words and pictures.	Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell at 7 eighths of the other pizza. Who ate more pizza? How do you know?
Response	
3 fourths has more than 3 fifths because the fourths is greater than fifths.	Claim: Terrell ate more pizza. Evidence & Reasoning: When it comes to who ate more, we know that it was Terrell because they started with pizzas of the same size which were then cut into 6 and 8 pieces respectively. The 8 pieces were smaller than the 6 pieces so the portions that were leftover from the pizzas were different sizes. The 1 from Carlos' pizza being larger. Also, if you take 5/6 and 7/8 then do an equal exchange to get them to a common unit, it will show 5/6 = 40/48 and 7/8 = 42/48. 42/48 is greater than 40/48 so Terrell ate more pizza. 

Areas of Student Growth

Through coding responses across iterations, the researcher identified three major areas of growth in student reasoning. The first was a shift from procedural understanding and reasoning to conceptual understanding and reasoning (Procedural to Conceptual Understandings). The second was an increase in the rigor with which students evaluated their conclusions (No Longer

Jumping to Conclusions). The third was a shift from creating disjoint and incomplete arguments to creating cohesive and complete arguments (Disjoint to Cohesive Arguments). These findings are supported by the following examples of student work.

Procedural to Conceptual Understandings. Observed from responses coded in the first iteration, students showed that they could write out steps and interpret mathematical procedures done by others, but they struggled to generalize this work and develop reasoning for it in relation to the context. They consistently answered "why" with "because that's the way it is based on what I did" instead of "because this is what's going on conceptually, so I did this work that demonstrates it." For example, Table 6 shows an example of a student response to a task from the first iteration that asks them to interpret different reasoning strategies. The student speaks procedurally about Macy and Ezra's work with phrases like "Macey took two of the 4 pancake recipes and added them" and "they took the number of tbsp in one batch and multiplied it by the number of batches they were making."

A more conceptually focused response would reference the ratio relationship between batches of four pancakes and the amount of each ingredient. That is, "Why do each of the procedures work?" or "How do you know Macey and Ezra arrived at the correct amounts?" Ultimately, the student's procedurally focused explanation proves to the teacher that they know how to interpret a problem. It does not prove to the teacher that the student could apply one of these strategies to a different contextual problem. However, by the fourth iteration, students were demonstrating much more conceptual understanding in their mathematical arguments. This is demonstrated in Table 7 by the fourth iteration student response in which the student converts algebraic symbols to conceptual understandings by explaining what someone would be trying to figure out if they were to put $3 \div 0.01$ into a calculator. They also demonstrate a conceptual understanding of the numbers “3” and “0.01” by discussing wholes, hundredths, and how many hundredths are in a whole.

Additionally, the fourth iteration student response to a question about function rules in Table 8 shows a student demonstrating the ability to consider “what if?” Specifically, they address “What if a product is negative?” This type of reasoning shows a depth in understanding and conceptual understanding of multiplication, function rules, and graphs.

Table 6. CEQ response with procedural explanation

Task
Macey and Ezra want to make 10 pancakes using the recipe for a batch of 4 pancakes which requires 6 tbsp of flour and ½ a cup of milk. They each figured out the amount of these ingredients differently, as shown below. Macey: $4 + 4 + 2 = 10$ pancakes $6 + 6 + 3 = 15$ tbsp flour $\frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4}$ cups milk Ezra: $10 \div 4 = 2.5$ batches $6(2.5) = 15$ tbsp flour $\frac{1}{2}(2.5) = 1.25$ cups milk (See Appendix B for image) Explain each of their reasoning, describing not only what they do, but why it makes sense.
Response
Macey took two of the 4 pancake recipes and added them and then took another 2 pancakes to get 10. Which 2 is half of the 4. Then she did the same thing but with the amount of flour you need. She took the 6 and added another 6 and then another half which is 3 to get 15 tbsp. She then did the same thing with the milk. Ezra takes the amount of pancakes they want, 10, and divides it by the amount the recipe makes, 4, to see how many batches they are making. Then they took the number of tbsp in one batch and multiplied it by the number of batches they were making. Then the same with the milk. They both got the same answer, Macey just did addition, Ezra did multiplication.

Table 7. CEQ response with conceptual explanation

Task
You input the expression $3 \div 0.01$ into your calculator. The output is 300. Explain why this output makes mathematical sense.
Response
The problem is asking how many 1 hundredths fit into 3 wholes. Another way to find the answer is to make an equal exchange. 3 wholes is 300 hundredths if you make an equal exchange of it. Then you take 300 hundredths ÷ 1 hundredth = 300.

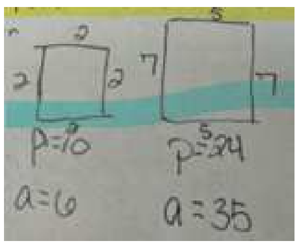
Table 8. CEQ response considering "What if?"

Task
Explain how you know by "looking through" the function rule $y = 3x^2$ that there will be no negative output values. How will this impact the shape of the graph of the function rule?
Response
For a product to be negative, we have to multiply a negative times a positive. 3 is positive, so x^2 would need to be negative. This means the graph will have symmetric points. For example, (2, 12) and (-2, 12).

No Longer Jumping to Conclusions. The content in the second iteration (exponents, maximum area, writing and solving equations) required students to recognize structure and patterns, and then draw conclusions about mathematical concepts and rules from those structures and patterns. Through coding responses, the researcher noted a tendency of jumping to conclusions. That is, many student responses included only one example that supported their claim without testing any other examples. They would use this one example as evidence that their claim was true in general. Not only did this exploration strategy involve faulty reasoning, but it also led many students to the wrong conclusions. For example, the second iteration response in Table 9 shows an instance in which a group of students jumped to a conclusion and developed only one example that supported this.

Table 9. CEQ response that jumps to conclusions

<table>
<tr><th colspan="2">Task</th></tr>
<tr><td colspan="2">What do you think will happen to the area of a rectangle if you increase the perimeter? Explain the reasoning for your choice. Give examples.

a. The area will also increase b. The area will decrease c. It depends</td></tr>
<tr><th colspan="2">Response</th></tr>
<tr><td>
</td><td>A. If the perimeter increases, the area will also increase as shown.</td></tr>
</table>

Here, the students chose “a. the area will also increase” by developing an example where this is true. However, if they would have considered other possible dimensions of perimeters closer in length, they would have discovered a counterexample to their claim. Students still needed to develop a capacity to reason about their conclusions, systematically explore patterns, and know when they had enough evidence.

The researcher’s plan was to develop this capacity through a focus on the reasoning component of the collaborative argumentation framework. For, if the students in the example above had been pushed to provide reasoning for why they chose those particular rectangles and dimensions, they may have been confronted with a dilemma. Providing a reason for pattern exploration forces the thinker to question “Is this always true? What if I change these conditions?” Thus, students continued to participate in tasks that required a conceptual understanding, but now with an emphasis on those that also required pattern exploration. Additionally, Workshop III focused on peer and teacher feedback and questioning, in which students identified holes in their reasoning or times that they jumped to conclusions. By the fourth iteration and final exam, the students were demonstrating the capacity to systematically explore patterns and recognized a need for testing multiple examples. For example, the student responses in Table 10 are from the fourth iteration before Workshop IV and from the final exam after Workshop IV.

Table 10. CEQ responses pre and post workshop iv

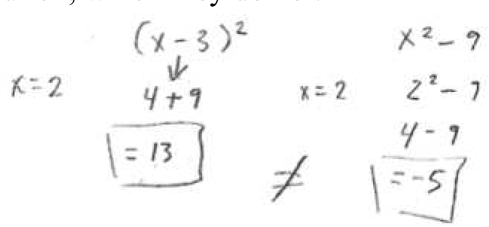
Task	
A student claims that $(x-3)^2$ is an equal exchange for x^2-9. Do you agree or disagree? Provide mathematical evidence to support your claim.	
Pre-Workshop IV Response	Post-Workshop IV Response
No, $(x-3)^2$ is not an equal exchange for x^2-9. If $x=2$ and the two were equal exchanges, they would result in the same solution, which they do not.  	I do not agree. When testing the same inputs in each expression, they give different outputs. 

Figure 10 offers a description of the focus and activities of Collaborative Argumentation Workshop IV. During Workshop IV, the class discussed responses like the pre-workshop one in Table 10 that only test one example. Students were posed with the question, “What if the input we chose to test was 3?" Then, their response approach shifted to include multiple inputs on the final.

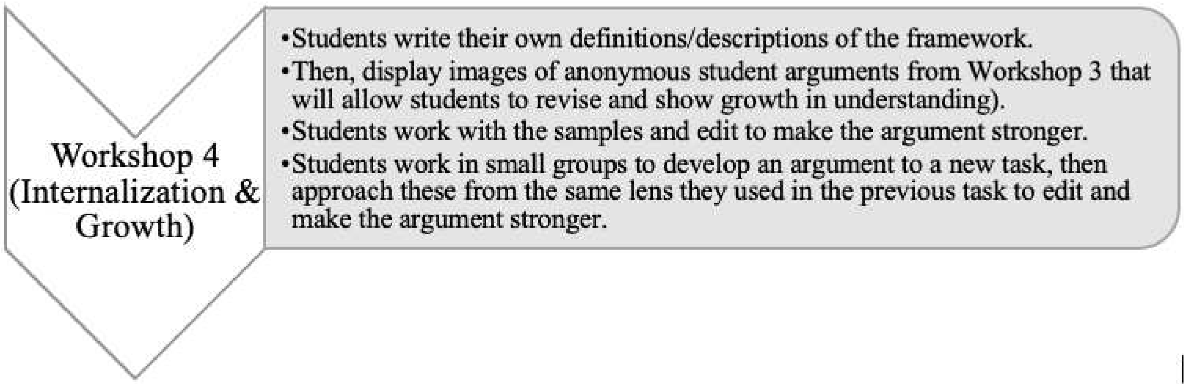


Figure 10. Collaborative argumentation workshop iv focus

Disjoint to Cohesive Arguments. Another target area identified by the researcher during the coding of second iteration responses was disjoint argument components. Observations reflected more attempts at including all three argument components (claim, evidence, reasoning/connection), but these components were often disjoint. That is, the components were either disconnected, meaning students did not use their reasoning to connect the claim and evidence, or contradictory, meaning the evidence or reasoning refuted their claim. Many student responses left the researcher asking, "how do you know?" Table 11 shows an example of a disconnected response and an example of a contradictory response from the second iteration. The task provides a claim for the students and asks them to provide evidence and reasoning.

In the disconnected response, the student provides an incomplete reasoning component, noting that 2^{-2} and 2^{2} are reciprocals, and 4 and ¼ are reciprocals but not explaining how they know or why this is significant. They also provide a list of calculations as evidence, but it is unclear how this evidence relates to the idea of reciprocals. Providing a response like this demonstrates that the student remembers strategies from class but does not have enough understanding of them to apply them in any situation. In the contradictory response, the student's reasoning component states that the exponents indicate getting "4 times bigger" or "¼ smaller." However, their evidence shows that the exponents indicate multiplying by 2 or by ½. Once again, this shows a student regurgitating calculations or language without a complete understanding.

The collaborative feedback element of Workshop III raised conversations about making arguments cohesive and showed students that there was more than one way to craft a solid mathematical argument. When confronted with the task of having to verbally explain their thinking to a group of peers, they were forced to only use evidence and reasoning that they fully understood. These experiences made it clear to students that even though their thinking and reasoning may seem clear and complete to them, other readers may be left with questions if they do not clearly connect each piece of their argument and fully elaborate on their reasoning. The teacher continued to implement small activities like this during class where students were prompted to share their arguments and others were encouraged to ask questions.

By the fourth iteration, observations of student responses indicated that students were beginning to take ownership of reasoning and craft cohesive arguments. Specifically, students recognized that the elements could be intertwined and developed responses in which components had to be labeled with circles and arrows, indicating cohesive arguments. There was also more variation in approaches to questions, indicating an ownership and understanding of reasoning rather than a regurgitation. For example, Table 12 shows a cohesive response from the fourth iteration that includes all three argument components. Each component specifically references the others—the evidence reiterates that Terrel ate more and the reasoning discusses why the evidence refers to unit size and leftovers.

Table 11. CEQ responses with disjoint components

Task	
If 2^2 is 4, why does it make mathematical sense that 2^{-2} is ¼?	
Disconnected Response	Contradictory Response
2^{-2} is the reciprocal of 2^2 and the reciprocal of 4 is ¼. $2^2 = 4$ $2^1 = 2$ $2^0 = 1$ $2^{-1} = \frac{1}{2}$ $2^{-2} = \frac{1}{4}$	2^2 and 2^{-2} are reciprocals, meaning they are opposites. As the exponents go up by 1, they get 4 times bigger, and as we go down by 1, they get ¼ smaller. $2^2 = 2 * 2 = 4$ $2^{-2} = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$

Table 12. CEQ response with cohesive components

<table>
<tr><th>Task</th></tr>
<tr><td>Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell at 7 eighths of the other pizza. Who ate more pizza? Provide a solid mathematical argument for your answer.</td></tr>
<tr><th>Response</th></tr>
<tr><td>

Claim: Terrell ate more pizza.
Evidence: 1/8 is smaller than 1/6 so Carlos has more pizza left which means Terrel ate more.
Reasoning: One pizza that is split into 8 units has smaller units than a pizza that is split into 6 units. Since each pizza only has 1 unit leftover, the pizza with the smallest unit would have the least left.
</td></tr>
</table>

Area for Improvement

Through coding responses across iterations, the researcher also identified an area for improvement or future focus that was not fully addressed through the collaborative argumentation framework and workshops. This was characterized as guiding students from specific to general reasoning strategies.

In observations from the third iteration, the research recorded students expressing some exhaustion with approaching each new task or question with such rigor. Attendance and assignment completion began to decline during this iteration as well. In fact, many students were only getting through one or two problems on lab assignments in class. Additionally, students started to send emails to the teacher asking how long the final would be and if they would be expected to produce mathematical arguments for every question. The researcher postulated that part of this exhaustion may have stemmed from students' lack of generalization. That is, students seemed to approach each task as something completely new and different rather than generalizing strategies to be transferred across problems. For example, pattern recognition and exploration was a strategy referenced in lessons on exponents, maximizing area, multiplication, linear and exponential relationships, and function rule development, but students consistently referred to these concepts as separate entities that each required a new set of skills. Students centered their arguments around details very specific to the problem instead of making more general connections and using reasoning that could be transferred to other problems. Table 13 shows an example of specific reasoning and Table 14 shows an example of general reasoning.

Table 13 reflects specific reasoning because the student relies on computing $80 * 3 * 3 * 3 * 3$ and the context of growing followers to connect representations for scenario 1, and similarly computing $80 + 3 + 3 + 3 + 3$ and using the context of filling a tank to connect representations with scenario 2. General reasoning would instead rely on recognizing scenario 1 is exponential (repeated multiplication) and scenario 2 is linear (repeated addition). This generalized reasoning could be applied to any representation and context.

Table 13. CEQ response with specific reasoning

Task

Match each of the lettered representations to one of the scenarios. Provide an explanation for each match.

Scenario 1: A person has 80 followers on social media. The number of followers triples each year. How many followers will she have after 4 years?

Scenario 2: A tank contains 80 gallons of water and is getting filled at a rate of 3 gallons per minute. How many gallons of water will be in the tank after 4 minutes?

A. $80 * 3 * 3 * 3 * 3$

C. $80 + 3 + 3 + 3 + 3$

E.

x	0	1	2	3	4
y	80	83	86	89	92

B.

x	0	1	2	3	4
y	80	240	720	2160	6480

D. $80 + 4 * 3$

F. $80 * 81$

Specific Reasoning Response

Scenario 1: A, B, F
A shows the initial number of followers, but then it shows the followers increasing 3 times. Since $3 * 3 * 3 * 3 = 81$, A & F are equivalent. B shows the number of followers being tripled each year.

Scenario 2: C, D, E
E shows the amount of water increasing by 3 gallons per minute. C shows the original volume, then increases by 3, 4 times. Since $3 + 3 + 3 + 3 = 4 * 3$, C & D are equivalent.

While the specific reasoning example reflects what most students were doing, the generalized reasoning example in Table 14 was pulled from student work as an exemplar of an achievable goal to work toward. This response uses both specific and general reasoning by referencing the specific ways in which the outputs change, multiplying by 0.75 or adding 11.25

(specific), and the general connection these patterns have with exponential and linear relationships (general). Knowing key differences between linear and exponential relationships and a habit of looking for repeated addition or repeated multiplication can be transferred to other problems and contexts as an entry point to get them started.

Table 14. CEQ response with general reasoning

Task

Which table shows an exponential relationship? Which table shows a linear relationship? Provide evidence to support your claim.

Table 1: The table represents the relationship between the number of years since purchase of a vehicle and its value (in $).

Years since purchase	Value
1	20,000
2	15,000
3	11,250
4	8,437.50

Table 2: The table represents the relationship between time worked (in hours) and the amount of money earned (in $).

Time	Amount Earned
4	45.00
5	56.25
6	67.50
7	78.75

General Reasoning Response

Table 1 shows an exponential relationship because the values are increasing by 0.75 each time by multiplication and because it is multiplication, it makes it exponential. If we were to keep going it would increase by multiplying 0.75.
(calculations shown for how they came up with 0.75 in table)

Table 2 shows a linear relationship because you add 11.25 to each amount earned and that gets you your answer and it's consistent. Because the function is adding, it's linear. If you were to keep adding 11.25 to amount earned, you could figure out any amount of time.

Most student work did not include this type of thinking. Instead, students were observed saying "we haven't had a question like this before" when presented with a new representation or context surrounding a familiar concept. While students were introduced to the idea of specific versus general reasoning and the teacher made a point to show different looking problems that utilized the same concepts, combatting the specific reasoning line of thinking was a need identified too late in the study by the researcher to fully address and needs more research to be developed.

Summary

To answer the research questions “What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?” and “In what ways does the use of such a framework advance student mathematical reasoning?”, the researcher carried out a series of intentional actions, then analyzed and coded student responses, and recorded observations. These moves allowed the researcher to identify details about the frequency of reasoning components in student responses, student interaction with the framework, areas of student growth, areas that still need addressed. The intentional actions taken by the researcher include:

1. Disrupt students’ previous understandings through questions that require conceptual explanations.
2. Score responses based on the collaborative argumentation framework before students have seen it.
3. Implement the four Collaborative Argumentation Workshops throughout the semester that provide (a) coded exemplars and teacher modeling, (b) opportunities to collaborate about reasoning, (c) practice coding and reflecting on arguments, and (d) practice revising and presenting arguments

Then, coding student responses based on the components of claim, evidence, and reasoning and further categorizing these by completeness/correctness showed that only 35.82% of evidence components were complete/correct in the first iteration, and 75.66% of evidence components were complete/correct by the fourth iteration. Additionally, only 34.29% of student responses included a reasoning component in the first iteration, and 66.03% of student responses included a reasoning component. Both the quality of the evidence component and the frequency of reasoning component increased throughout the study.

Additionally, the analyzing and coding process and recorded observations illuminated themes about the nature of these increases. In this study, students expressed a need for the collaborative argumentation framework, specifically requesting a tool for support. They engaged with the framework to a point of internalization, eventually coding their responses without prompting. While most students began the study with strictly procedural understandings, their responses in the fourth iteration provided evidence of improved conceptual understandings. Students began to systematically explore patterns instead of jumping to conclusions, providing multiple examples to support their claims in fourth iteration responses. Lastly, by the fourth iteration, students demonstrated a capacity to revise previously disjoint arguments with disconnected components or contradictory components to make them cohesive.

An area still needing improvement identified through observations was students tending to use specific reasoning rather than general reasoning, approaching each new task without considering utilizing strategies from previous tasks.

CHAPTER V: DISCUSSION

The purpose of this iterative analysis study was to explore how algebra students' reasoning changes over the course of a semester when using a collaborative argumentation framework with specific instructional guidelines motivated by student need. Presented in this section will be: (a) summary of results, (b) relating results to research, (c) recommendations for implementation, (d) recommendations for future research, and (e) conclusion.

Summary of Results

This mixed method, iterative analysis study aimed to answer the research questions:

1. What does using a collaborative argumentation framework in an algebra classroom reveal about student mathematical reasoning?
2. In what ways does the use of such a framework advance student mathematical reasoning?

With the goals of:

a. use suggested framework to make observations about student reasoning
b. identify specific actions to address those observations
c. document the progression of growth in student reasoning over the course of the study

The researcher coded student responses to questions that required a conceptual understanding (CEQs) across four iterations and kept a journal of observations about student engagement, specific teaching moves, areas of student growth within the collaborative argumentation framework, and areas that needed improvement.

Coding responses against the collaborative argumentation framework with components claim, evidence, and reasoning showed an increase in the frequency of a reasoning component and an increase in the quality of evidence components. Inclusion of a reasoning component grew from 34.29% of responses to 66.03% of responses by the end of the study. The percentage of evidence components coded as complete/correct grew from 35.82% to 75.66% by the end of the study.

Observations showed students expressing a need for a collaborative argumentation framework after being exposed to CEQs and receiving scores and feedback on responses prior to introducing the framework. Additionally, after four iterations, students began to internalize the framework, showing this by using it without being prompted and utilizing the language during group work.

Observations also showed student growth in the areas of advancing from procedural to conceptual understandings, no longer jumping to conclusions, and revising disjoint arguments to make them cohesive. At the end of the study, students still needed support to use generalized reasoning strategies rather than approach each CEQ as separate from all others with specific reasoning strategies.

Relating Results to Research

According to the National Council of Teachers in Mathematics (NCTM), using mathematical argumentation in the classroom is an evidence-based practice essential to developing conceptual understanding and mathematical proficiency (Rumsey & Langrall, 2016). Research has suggested that critical thinking skills benefit from argumentation practices, and such processes lead students to create multiple connections between mathematical concepts

(Wagner et al., 2014). Furthermore, there is a positive correlation between argumentation and knowledge construction (Hershkowitz et al., 2001). Based on existing frameworks for using argumentation in the classroom (Toulmin, 2003; Ervin-Kassab et al., 2020; Zwiers, 2019), the researcher developed a framework with the components claim, evidence, and reasoning, and used it to explore student reasoning over the course of a semester.

According to Yoon, Kensington-Miller, Sneddon, and Bartholomew (2011), undergraduate mathematics students consider themselves passive participants in their courses and are reluctant to engage in mathematical activity during a lecture. In concurrance with this existing research, observations from this study indicated that students began the semester reluctant to participate in argumentation practices and confused about what was "wrong" with their old ways of thinking about mathematics.

Additionally, existing research (Nachowitz, 2019; Bieda & Lepak, 2014; Rowland, 2008) revealed that the nature of students' mathematical understandings was primarily procedural. Observations from this study about student responses and coding tabulations in the first iteration also revealed that students tend to answer a mathematical "why" with procedural evidence alone. This meant that, consistent with the findings of Stein, Grover, and Henningsen (1996), there was a disconnect between the teacher's intention with a CEQ and the students' engagement with that CEQ. At the beginning of the semester, the teacher was intending that students engage with mathematical reasoning and communicate conceptual understandings, but typical student responses contained only a claim with evidence in the form of calculations or vague references to tricks and rules. Although using collaborative argumentation in the classroom centers mathematics learning around reasoning (Yackel, 2001), students expressed confusion about what was expected of them and a need for additional support.

Turning to recommendations from other studies and ideas from previous experiences, the researcher adopted a series of instructional guidelines and intential teaching moves to motivate and support students. Part of student motivation to engage with the framework stemmed from the consistent use of CEQs. According to the recommendations of previous research (NCTM, 2014; Smith et al., 2018; Bingölbali & Bingölbali, 2021; Rahayuningsih et al., 2021; Kwon et al., 2006; Aziza, 2021; Fatah et al., 2016; Sholihah et al., 2020; Hancock, 1995), these questions either provided an experience in which previously learned "tricks" do not work, asked students to find a pattern that takes more exploration than they were typically inclined, or had necessary procedures so heavily embedded in context that students needed to understand the concepts behind their tricks.

The other part of student motivation to engage with collaborative argumentation stemmed from their desire to meet teacher expectations and earn positive feedback and scores. High school mathematics teacher Conner (2013) previously noted that clearly outlined and understood expectations were a key to successful implementation of collaborative argumentation in the mathematics classroom. In this study, the researcher exposed students to the types of open-ended questions and engaged them in creating arguments before introducing the framework and expectations. By doing so, the students felt a need for more clarity and found value in the framework once it was introduced. That is, they had previous experiences to apply it to and motivation in the form of wanting to improve scores.

Introducing the argumentation framework and expectations at a time when students had previous experiences to apply it to and a sense of relevance fits with existing research and suggestions about developing shared understandings to engage students in argumentation. Shared understanding is developed through small group answer construction and representative tools

(Michalchik et al., 2008), gestures (Alibali et al., 2013), shared experiences (Cobb et al., 1992), and whole class reflection/comparison of student responses as well as intentional positive reinforcement of desired response characteristics (Yackel & Cobb, 1996). Each of these things were done during argumentation workshops.

Aside from motivation, observations about student responses and interactions with CEQs showed that students needed support moving toward conceptual understandings, recognizing and systematically exploring patterns, and communicating arguments in a cohesive manner.

To support students moving from procedural to conceptual understandings, the researcher considered an argument from Yoon, et al. (2011) that stated "the success of small group interactions during large-scale lectures depends on students and lecturers establishing supportive social norms, and adjusting their lecture goals from 'covering the content' to 'developing mathematical understanding'" (p. 1107). Supportive social norms were developed through time spent at the beginning of the semester establishing student groups and facilitating discussions to get to know one another. Additionally, by the recommendations of Sadler and Good (2006), no ideas were shut down by the teacher. Instead, collaboration and questioning were used to help students recognize areas of growth. With students acting as their own evaluators, there was a focus on growth and improvement rather than an anxiety about avoiding mistakes. To focus students on conceptual understandings instead of covering procedural content, the researcher developed CEQs that provided students with an answer or procedure and asked them to respond explaining the concept and reasoning. For example, the CEQ "Explain why $(-3)^2 * (-3)^3$ has a different answer than $((-3)^2)^3$." gives students the claim that the solutions are different and focuses their attention the concept of exponentiation rather than on doing exponentiation.

To support students recognizing and systematically exploring patterns, the researcher used suggestions from Hattie and Timperley (2007) by implementing opportunities for real-time self-reflection and teacher feedback during which the teacher modeled reasoning strategies and argument development and revision. These opportunities were whole class discussions analyzing samples of student work. Specifically, the researcher pulled samples of student work in which students had jumped to a conclusion without fully exploring a pattern. That is, with systematic exploration, a counterexample could have been found that refuted their claim or the reader could be satisfied that the claim was true in all cases. The teacher presented such samples and asked targeted questions to poke holes in the existing pattern exploration, then modeled a more thorough reasoning approach.

To support a transition from disjoint to cohesive arguments, the researcher used suggestions from Nordin and Boistrup (2018) to pay attention to all forms of communication from the students and assist them in redirecting their ideas into one cohesive and presentable argument. Because of the nature of the CEQs students were engaging in, the teacher knew they were using some reasoning strategies. However, students struggled to communicate these on paper or whiteboards. When they did communicate their reasoning, it was often entirely separate from the rest of their argument. It became imperative that the teacher listen to small group discussions, look at scratch work, and ask students to talk through their thought process so she could aid students in incorporating a reasoning component into their written arguments and relate all components to one another.

Previous research had shown that argumentation could be used as an evidence-based practice in the mathematics classroom to develop reasoning skills (Thompson et al., 2012) and deepen conceptual understandings (Rumsey & Langrall, 2016). In this study, the frequency of reasoning inclusion in written arguments increased from the first to the fourth iteration, and fourth iteration student responses reflected conceptual understandings, demonstrated systematic pattern exploration, and communicated thinking in a cohesive manner. These findings support similar findings from a study analyzing the interactions between students in an inquiry/argument culture, in which students built and checked for consensus and took over the traditional teacher's role of validating mathematical ideas (Woods et al., 2006). Furthermore, a study regarding science education showed that collaborative argumentation helped students develop accurate conceptual understandings over time (Li et al., 2021), as seen through the growth of student mathematical arguments during this study.

Additionally in this study, students internalized the collaborative argumentation framework. Their questions shifted from asking the teacher "Is this right?" and "Is this what you want?" to asking their peers "Does my reasoning make sense here?" and "Why did you use this evidence instead of this evidence?" They also used the language of the framework to organize arguments on a regular basis and without prompting. Specifically, they labeled arguments on the final and homework assignments without being asked and used the framework language when collaborating in class. This indicated that they viewed the collaborative argumentation framework as a helpful tool to reach understanding of concepts and meet teacher expectations. In a different study, other mathematics teachers found that with consistent implementation of their argumentation framework, students began to use it as a tool for approaching problems without being prompted (Ervin-Kassab et al., 2020).

By the third and fourth iterations, the researcher identified a need to move students toward a more generalized and transferrable reasoning approach because, as suggested by Ellis (2007), the use of iterative action/reflection styles and generalizing with deductive reasoning improves mathematical understanding. Furthermore, getting students in the habit of a more generalized reasoning approach would give them the tools to confidently approach different kinds of problems with different contexts because they are trained to look for underlying concepts and patterns as opposed to procedures and specific details. This type of reasoning and connection making has been shown to develop in students through engaging consistently in argumentation (Wagner et al., 2014). However, even once the frequency of reasoning components in student responses increased, much of it was context and problem specific (inductive) rather than generalized and conceptual (deductive). This focus was identified as an area for further study.

Recommendations for Implementation

The results indicate that using a collaborative argumentation framework in a mathematics classroom can improve the frequency of reasoning, the quality of evidence, conceptual understanding, pattern exploration, and communication of thinking. The researcher attributes this growth to the chosen framework, Collaborative Argumentation Workshops, and instructional guidelines.

The chosen framework was developed by the researcher as a synthesis of existing frameworks from Toulmin (2003), Ervin-Kassab et al. (2020), and Zwiers (2019). The framework identifies three major components of a solid mathematical argument and recommends that arguments be developed or discussed with peers as this provides opportunity for reflection and improvement. Figure 11 shows the three components of the framework—claim, evidence, reasoning—and provides a description of each. Reasoning is on the bottom of the pyramid because without it, the argument falls apart.

In order to introduce and practice using the framework, the researcher developed four Collaborative Argumentation Workshops to use during class every four weeks. The key effective criterion for each workshop is outlined in Figure 12. Additionally, the workshop slides are provided in the appendices. At the beginning of each iteration, the researcher used observations from previous iterations and existing research to add to a list of instructional guidelines for implementation shown in Table 15.

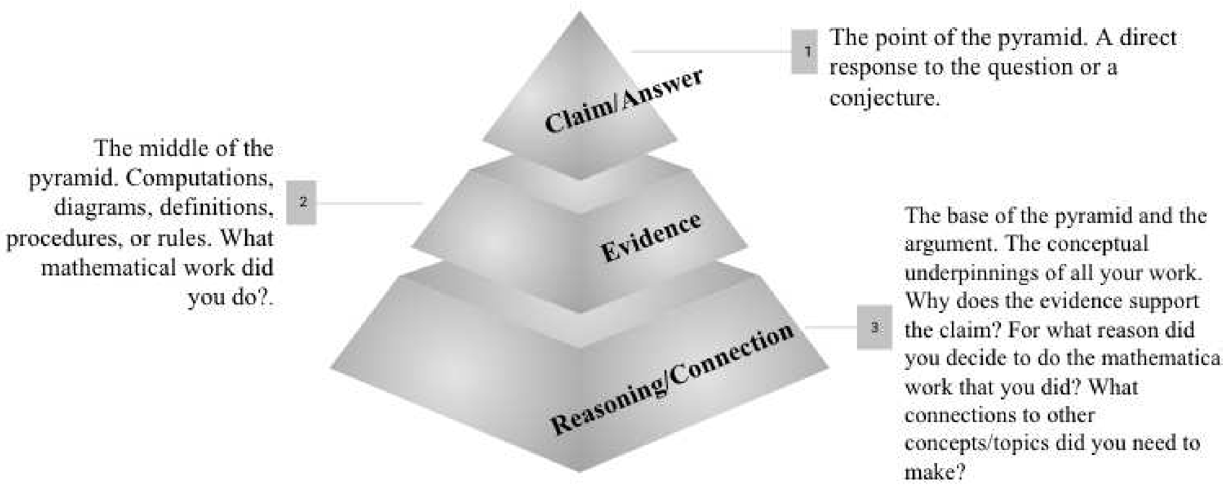


Figure 11. Collaborative argumentation framework components

Based on the observed positive student response, improvement in frequency of reasoning and quality of evidence, growth in conceptual understanding and pattern recognition, and students demonstrating an increased ability to develop solid mathematical arguments, this framework, Collaborative Argumentation Workshops, and instructional guidelines are recommended by the researcher for implementation in high school and undergraduate mathematics classrooms.

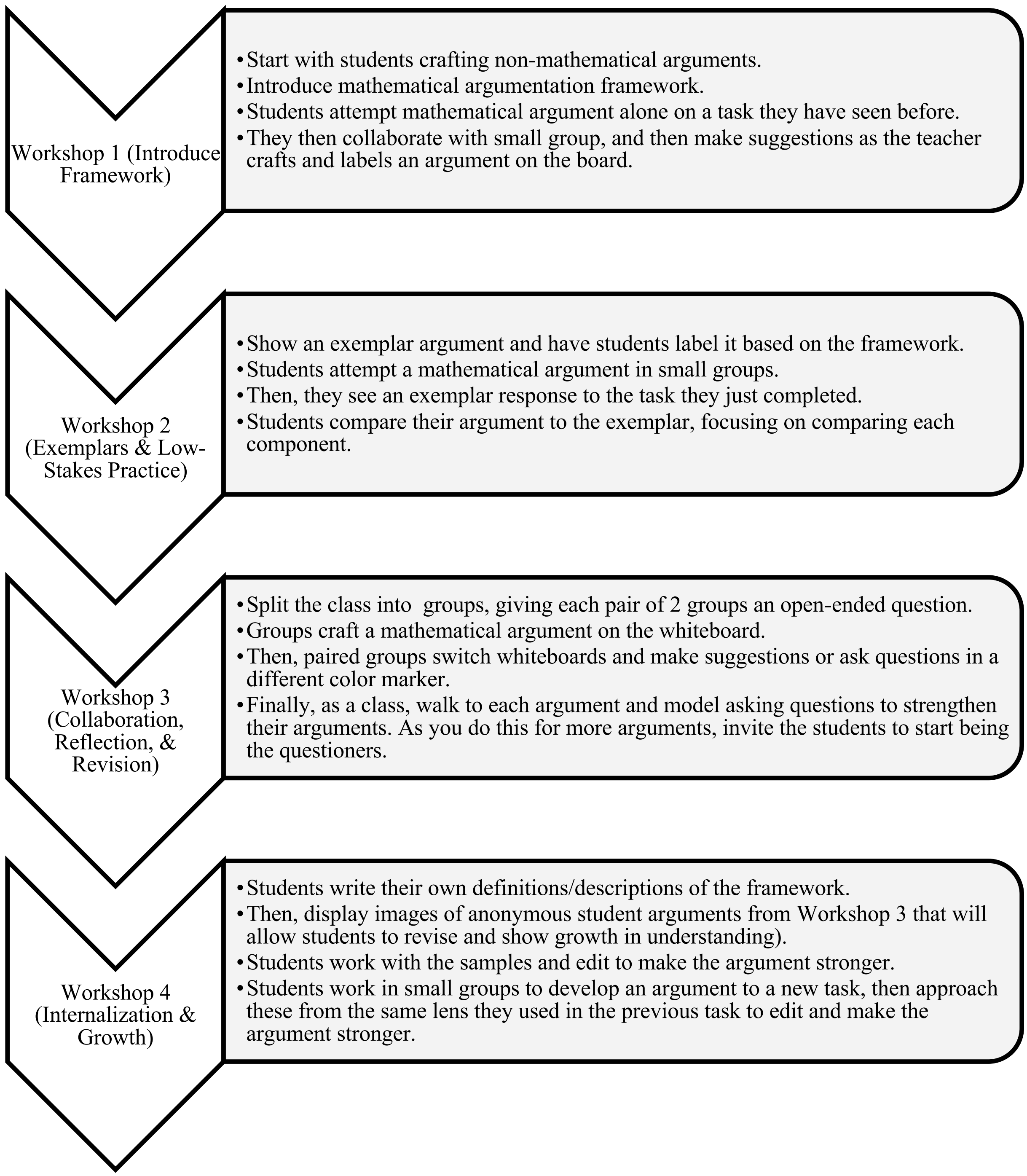


Figure 12. Collaborative argumentation workshop foci

Table 15. Instructional guidelines for implementation

Guideline	Implementation
1. *Questions requiring conceptual explanations (CEQs) to disrupt students' current understandings*	- provide an experience in which their trick does not work - ask them to find a pattern that takes more exploration than they are typically inclined - the procedures they need to use are so heavily embedded in context that students need to understand the concepts behind their tricks
2. *Collaboration to expose students to communicating mathematical thinking*	- put students in groups of 3 and spend time developing community - have groups work at whiteboards and frequently display thinking publicly - facilitate whole class discussions between groups
3. *Use what motivates students to create a need for the collaborative argumentation framework*	- engage students in CEQs before introducing framework - score test or assignment according to framework before students have seen it (do not actually enter this score as a grade)
4. *No ideas are shut down by the teacher. Focus on the difference in current and desired understandings.*	- provide exemplars to compare with sample student responses and let students be the judge of what is acceptable - allow students to develop arguments without help, then question their thinking to push understanding
5. *Students view exemplar arguments and practice coding them against the framework. Practice whole class reflection and revision of arguments.*	- consistently use the language of the framework when modeling responses - ask students to identify framework components during instruction and activities
6. *Focus on difference between concepts and procedures*	- anytime a procedure or trick can be used, do it, then ask the students why it works - eliminate the desire to use procedures by asking CEQs
7. *Focus on components as cohesive rather than disjoint*	- combine evidence component from one student response with the reasoning component of a different student response to the same CEQ, then discuss the flaws in this argument and how to relate components to one another

Recommendations for Future Research

Observations from the study also highlighted areas for future improvement. After analysis and reflection about the results, the researcher identified three areas needing additional research and focus that would strengthen the recommendations for implementation. These areas are supporting students through using general over specific reasoning strategies, ways to realistically implement more reflection and feedback time, and developing more frequent workshops.

The researcher noted a tendency in students to approach each CEQ as separate from all others and provide reasoning specific to and reliant on the context of the problem. This caused exhaustion with each new task and a lack of ability to transfer conceptual knowledge to CEQs in formats students had not seen before. For example, students tended to identify and relate different representations of linear and exponential relationships based on specific numbers or contextual labels. While this strategy worked, it was less transferrable to different CEQs than if they had focused on the differences between linear and exponential relationships in general. Student reluctance to reason generally was identified as an issue through observations late in the study. In future studies, this focus should be at the forefront with researchers testing different supports to address it and working to develop targeted activities and instruction.

Furthermore, while the researcher did develop a workshop focused on reflection and feedback, they also identified a need for more. Obstacles preventing the inclusion of more reflection and feedback on student responses during the course of this study included a lack of class time for such activities, two-week long units, and attendance. Reflection activities eat up class time revisiting old topics and responses. While valuable, there is also a pressure to keep moving forward and introduce new content. Additionally, with each unit lasting only two weeks, there was little turnaround time for the teacher to collect assignments or tests, give thorough feedback, and return them while such responses were still relevant or before students had an opportunity to show improvement. Couple this short turnaround time with frequent student absences and it becomes unlikely that students will have time to reflect on feedback before they have taken another test. Because of this predicament, the researcher recommends further study using the recommendations for implementation but restructuring the course to allow time for more effective feedback and reflection. Also, researchers could develop activities that may help the teacher give feedback in a timely manner.

Related to the structure of the class, the researcher also recommends the development of more Collaborative Argumentation Workshops. They observed a definite increase in student use of the framework after each workshop. Developing more workshops would allow the teacher to target one area of improvement at a time. Shorter, more frequent workshops could help students internalize the framework earlier on in the study, allowing their reasoning to improve more.

Conclusion

One of the most valuable applications of mathematics in everyday life is the transferrable reasoning skills developed through engaging with mathematical tasks. However, many students go their whole elementary and secondary mathematics education without developing these skills. That is, students are showing up to college with little conceptual knowledge or ability to recognize and systematically explore patterns. Additionally, they struggle to effectively communicate their thinking in a clear and cohesive manner. Teachers have noted struggles with motivating students to engage in argumentative mathematical tasks because of such challenges, and students fall into a pattern of thinking they are bad at mathematics but can get by if they memorize procedures and meet low expectations. This study revealed that students do indeed have the less than desirable reasoning tendencies described above, but it also showed that these tendencies can be advanced through instruction using a collaborative argumentation framework, a series of workshops, and instructional guidelines for implementation.

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APPENDICES

Appendix A: Human Subjects IRB Approval Notice

IRB #: IRB-FY2022-530
Title: Collaborative Argumentation as a Means to Improve Reasoning and Connection Making Skills in Mathematics
Creation Date: 4-7-2022
End Date:
Status: Approved
Principal Investigator: Patrick Sullivan
Review Board: MSU
Sponsor:

Study History

Submission Type Initial	Review Type Expedited	Decision Approved

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Appendix B: Collaborative Argumentation Workshop I

Answer and explain: Is Water Wet?

On your own: write your answer and argument/explanation for your claim

With your group: as a team with the same claim, compare the strength of your different arguments/explanations and develop the best argument that you can

With an opposing group: present your arguments and determine which team has the strongest argument and why (they don't have to have changed your mind, just who presented the strongest argument)

3

What do you think makes a solid argument?

- Confidence
- Evidence, and explanation
- Research
- Logical reasoning
- Ability to see other points of view
- Define big words if we use them
- Acknowledge other side
- Confidence, passion
- Explaining why you are right and the other side is wrong
- Explaining each piece of evidence, elaborating
- Evidence
- Reasoning
- Best evidence first

4

An example: Water is not wet.

The definition of wet is "covered or saturated with water or another liquid." You can't saturate water with more water and if you add water to another liquid, it doesn't make it wetter. "Wet" is an adjective like "sweaty". Sweat is not sweaty, sweat makes someone or something else sweaty. Likewise, water makes someone or something else wet.

Claim
Evidence
Reasoning/Connection

5

A solid mathematical argument/explanation consists of:

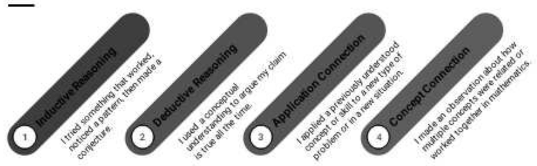
1 The point of the pyramid. A direct response to the question or a conjecture.

2 The middle of the pyramid. Computations, diagrams, definitions, procedures, or rules. What mathematical work did you do?.

3 The base of the pyramid and the argument. The conceptual underpinnings of all your work. Why does the evidence support the claim? For what reason did you decide to do the mathematical work that you did? What connections to other concepts/topics did you need to make?

6

Reasoning/Connection



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Cameron is driving 70mph for 35 miles. Craig is driving 60mph for 25 miles. Who will finish their trip first? How do you know?

Craig will finish his trip first.

We had to compare their rates of 70mph (which is a ratio 70 miles : 1 hour OR 70 miles : 60 minutes) and 60mph (60 miles : 1 hr OR 60 miles : 60 minutes). To do this, we made two ratio tables.

miles	70	35
minute	60	30

miles	60	1	25
minute	60	1	25

We used Cameron's relationship of 70 miles in 60 minutes and multiplied both amounts by ½ to maintain that multiplicative relationship. It will take him 30 minutes to finish his trip. For Craig, since we couldn't easily multiply to get from 60 to 25, we found the unit ratio by multiplying both miles and minutes by 1/60. The unit ratio would tell us how long it took Craig to travel 1 mile. Then, we could multiply that amount by 25 to find out how long it will take him to travel 25 miles. This works because the relationship between miles and time should stay the same. Each mile should take the same amount of time. It will take Craig 25 minutes to finish his trip. 25 minutes is less than 30 minutes.

Claim: "Craig will finish his trip first." This directly answers the question.

Evidence: Ratio tables and "It will take Cameron 30 minutes to finish. It will take Craig 25 minutes. 25 minutes is less than 30 minutes." This gives mathematical work that supports the claim.

Reasoning/Connection: Everything else. This explains how we used ratios and multiplicative relationships to find values in the ratio tables, why we chose specific values, and how our work related to the question.

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Now, with your groups, answer the question and write a solid mathematical explanation.

While exploring in the forest, you find a cave full of treasures. You are about to start collecting your reward when a wizard appears. They tell you that they are the guardian of this cave and all its diamonds, rubies, and emeralds. They explain that <u>2 diamonds are worth as much as 3 rubies, and 5 rubies are worth as much as 9 emeralds.</u> If you can <u>make a pile of diamonds and a pile of emeralds that have exactly the same worth</u>, then you can take the entire contents of the cave. <u>**What will be the quantities of your piles?**</u>

<u>**Explain your answer.**</u>

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The Collaboration

- Compare your argument/explanation with other groups.
- Use your **Reasoning/Connections** to evaluate each others' **Evidence**.
- Do you need to change your **Claim**? Do you need to find more **Evidence**? Do you need to provide more clarity in your **Reasoning/Connections**?

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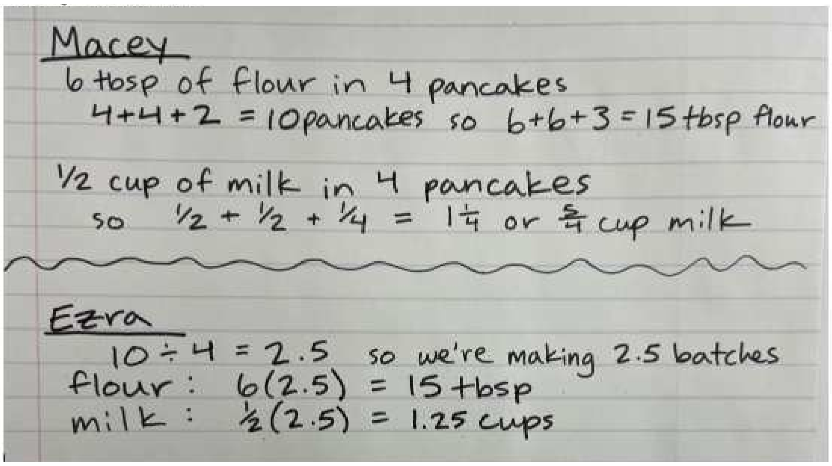
Appendix C: First Iteration CEQs

1. Which is more: 3 fourths or 3 fifths? Why? Use words and pictures.

2. Write an example of a multiple digit addition problem that would lead you to "carry the one." Explain why we do this and what is really going on.

3. Here are the indredients to make 4 pancakes: 6 tablespoons of flour, ½ cup of milk, ½ cup of water, 1 pinch of salt, 1 egg.
   Macey and Ezra want to make 10 pancakes. Below is how each of them figured out how much flour and milk they would need.


Macey
6 tbsp of flour in 4 pancakes
4+4+2 = 10 pancakes so 6+6+3 = 15 tbsp flour
1/2 cup of milk in 4 pancakes
so 1/2 + 1/2 + 1/4 = 1 1/4 or 5/4 cup milk
Ezra
10 ÷ 4 = 2.5 so we're making 2.5 batches
flour: 6(2.5) = 15 tbsp
milk: 1/2(2.5) = 1.25 cups

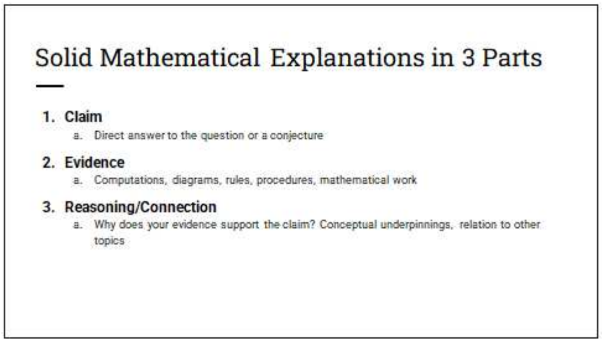
   Explain each of their reasoning, describing not only what they do, but why it makes sense. How are they each thinking about the problem? What are the similarities and differences in their approaches?

4. Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell ate 7 eighths of the other pizza. Who has more pizza left to eat the next day? Provide an explanation that would convince anyone of your answer.

5. Shown in the table below is a relationship between temperature in degrees Fahrenheit and temperature in degrees Celsius. Is this type of relationship a ratio relationship? Explain.

Fahrenheit	41	50	68	104
Celsius	5	10	20	40

Appendix D: Collaborative Argumentation Workshop II



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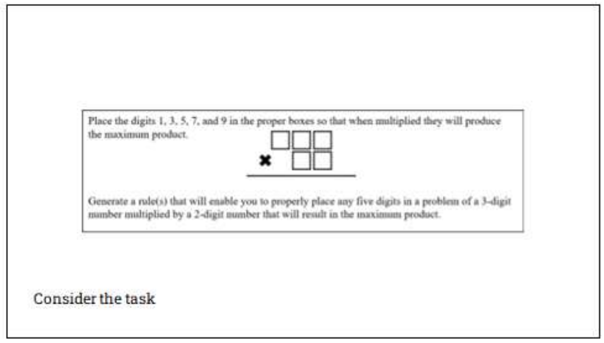

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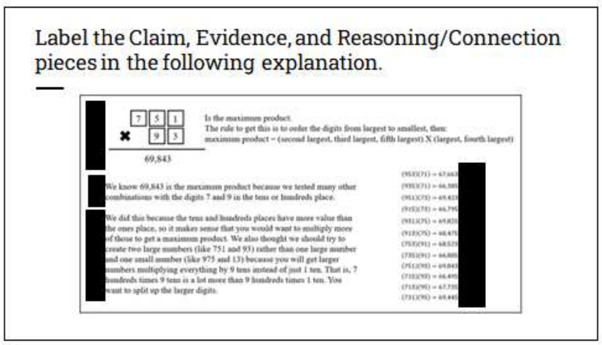

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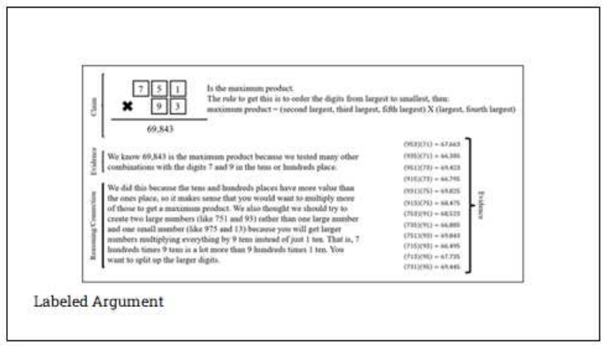

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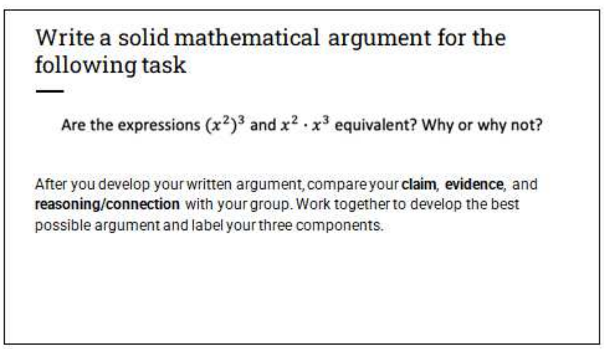

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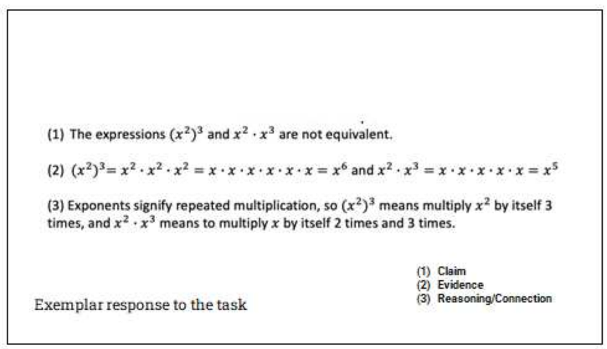

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Appendix E: Second Iteration CEQs

1. What do you think might happen to the area of a rectangle if you increase the perimeter?
    a. The area will also increase.
    b. The area will decrease.
    c. It depends.

   Explain the reasoning for your choice. Give examples.

2. Fill in the equivalent expressions table below. Answer "yes" or "no" to say whether the expression is equivalent to x^6. Then, back up your decision by either expanding/simplifying the expression or plugging in a number.

Expression	Equivalent to x^6? (yes/no)	Back it up
$x^3 + x^3$		
$(x^2)^3$		
$\frac{1}{2}x^6 + \frac{1}{2}x^6$		
$\frac{x^{12}}{x^2}$		
$x^{-3} \times x^9$		

3. Explain mathematically why $(-3)^2 * (-3)^3$ has a different answer than $((-3)^2)^3$.

4. Think about the language you associate with simplifying expressions involving exponents such as $3^5 * 3^2$ and $(3^5)^2$. You learned a "trick," but not the "why." Exponentiation is considered repeated multiplication. Use this idea to explain why $3^5 * 3^2 = 3^7$ and $(3^5)^2 = 3^{10}$.

5. If 2^2 is 4, why does it make mathematical sense that 2^{-2} is ¼?

6. Grace ate 3 fourths of a whole Subway sandwich and Molly ate 2 thirds of a whole Subway sandwich. Explain how you could use the concept of equal exchanges to determine who ate more.

Appendix F: Collaborative Argumentation Workshop III

Directions

Complete the appropriate task with your group. Write your argument and work on the whiteboard as if you were answering it on a test. Be sure to include a Claim, Evidence, and Reasoning/Connection.

We will be evaluating your arguments as a class when you are done.

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Groups 1 & 2

Are $\boldsymbol{x^2 + 9}$ and $\boldsymbol{(x + 3)^2}$ equivalent expressions? Convince me of your claim.

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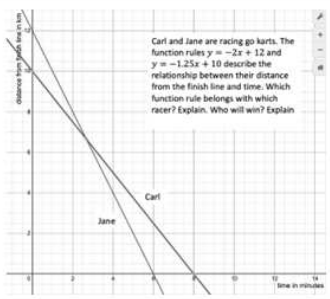
Groups 3 & 4



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Appendix G: Third Iteration CEQs

1. The actions on the inputs of three different function rules are shown below.
   Function Rule 1: "multiply by 3, square it"
   Function Rule 2: "square it, multiply by 3"
   Function Rule 3: "square it, multiply by 9"
   a. Complete the input/output table for each function rule.

	-2	-1	0	1	2
Function Rule 1					
Function Rule 2					
Function Rule 3					

   b. Use algebraic symbols to write the three function rules.
   c. Two of these function rules are equal exchanges of each other. Explain why this makes mathematical sense.

2. Match each of the lettered representations with one of the following situations. Provide an explanation for each match.
   Situation 1: A person has 80 followers on social media. The number of followers triples each year. How many followers will she have after 4 years?
   Situation 2: A tank contains 80 gallons of water and is getting filled as a rate of 3 gallons per minute. How many gallons of water will be in the tank after 4 minutes?

A. $80 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

B.

x	0	1	2	3	4
y	80	240	720	2,160	6,480

C. $80 + 3 + 3 + 3 + 3$

D. $80 + 4 \cdot 3$

E.

x	0	1	2	3	4
y	80	83	86	89	92

F. $80 \cdot 81$

3. Which of the tables below represents a linear relationship? Which table represents an exponential relationship? Provide evidence to support your claim.

Table 1: The table represents the relationship between the number of years since purchase of a vehicle and its value (in $).

Years since purchase	Value
1	20,000
2	15,000
3	11,250
4	8,437.50

Table 2: The table represents the relationship between time worked (in hours) and the amount of money earned (in $).

Time	Amount Earned
4	45.00
5	56.25
6	67.50
7	78.75

Appendix H: Collaborative Argumentation Workshop IV

How do you define Claim, Evidence, and Reasoning/Connection?

In my mind, the claim is...

In my mind, the evidence is...

In my mind, the reasoning/connection is...

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Label the Claim, Evidence, and Reasoning/Connection in the following argument.

Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell at 7 eighths of the other pizza. Who ate more pizza? Provide a solid mathematical argument for your answer.

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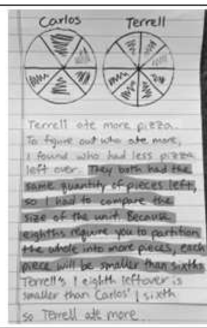

Claim: yellow
Evidence: blue
Reasoning/Connection: pink

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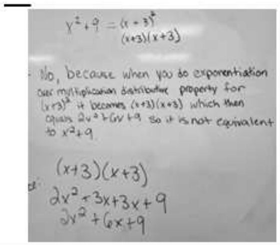
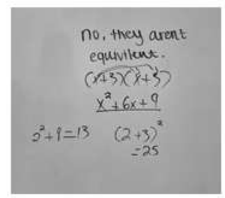
Look for any existing Claim, Evidence, or Reasoning/Connection. Then, edit the argument to make it even stronger.





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Determine what type of relationship is shown in each representation (linear, quadratic, or exponential) and provide a solid mathematical explanation for each claim.

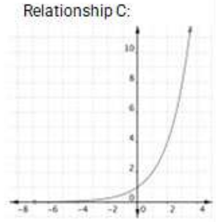
Relationship A:

Input	Output
-1	2
0	0
1	2
2	8
3	18

Relationship B:

Amber is completing a last minute cupcake order for her baking business. She already has 6 cupcakes leftover from a previous order. She can make 12 cupcakes every hour. What is the relationship between time she works (t) and cupcakes she has for the order (c)?

Relationship C:



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Appendix I: Fourth Iteration CEQs

1. Explain how you know by "looking through" the function rule $y = 3x^2$ that there will be no output values that will be negative. How does this impact the shape of the graph of the function rule?

2. You input the expression $3 \div 0.01$ into your calculator. The output is 300. Using ideas from this class, explain why this output makes mathematical sense.

3. A student claims that $(x - 3)^2$ is an equal exchange for $x^2 - 9$. Do you agree or disagree? Provide mathematical evidence to support your claim.

4. Also see "My Best Mathematical Arguments" problems 1 and 3.

Appendix J: My Best Mathematical Arguments

The following 5 questions are ones you have seen before on previous tests. Throughout the semester, we have worked on developing our conceptual understandings of mathematics and using those understandings to write solid mathematical arguments including a *Claim*, *Evidence*, and *Reasoning*.

Your Task: Choose 3 of the following questions and respond with your best mathematical argument. Clearly label your *Claim*, any *Evidence* you used, and the *Reasoning* that links your evidence to your claim.

1. Two pizzas are the same size. Carlos ate 5 sixths of one of the pizzas and Terrell at 7 eighths of the other pizza. Who ate more pizza? Provide a solid mathematical argument for your answer.

2. The table shows a relationship between temperature in degrees Fahrenheit and temperature in degrees Celsius. Is this relationship a ratio relationship? Provide a solid mathematical argument for your answer.

Fahrenheit	41	50	68	104
Celsius	5	10	20	40

3. If 2^2 is 4, what is 2^{-2} ? Provide a solid mathematical argument for your answer.

4. A student claims that the algebraic expression $(x + 1)^2$ is an equal exchange for $x^2 + 1$. Do you agree or disagree? Provide a solid mathematical argument for your answer.

5. Two different relationships are shown below. Which one represents an exponential relationship, and which one represents a linear relationship? Provide a solid mathematical argument for your answer.

Table 1: relationship between the number of years since purchase of a vehicle and its value in dollars

Years since purchase	Value
1	20,000
2	15,000
3	11,250
4	8,437.50

Table 2: relationship between time worked in hours and the amount of money earned in dollars

Time	Amount Earned
4	45.00
5	56.25
6	67.50
7	78.75