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On Covering Groups With Proper Subgroups

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ON COVERING GROUPS WITH PROPER SUBGROUPS

A Master's Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

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ABSTRACT

In this paper, we explore groups that can be expressed as a union of proper subgroups. Using “covering number” to denote the minimal number of proper subgroups required to cover a group, we explore the nature of groups with covering numbers 3 and 4, while also finding covering numbers for p-groups, dihedral, and generalized dihedral groups.

KEYWORDS: group, dihedral, covering, generalized dihedral, subgroup

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1. INTRODUCTION

A few key terms will be addressed. We will assume the reader's familiarity with the basics of groups using general notation, terminology, and some results from Dummit and Foote [3].

DEFINITION 1.1: A group covering is a set of proper subgroups of a given group whose union is the group itself. The group is said to be covered by these subgroups.

THEOREM 1.2: No cyclic group has a covering.

Proof. By it's definition, a cyclic group is generated by a single element. Assume to the contrary that there is a covering for some cyclic group, then one of these subgroups must contain a generating element. Therefore, that subgroup is the group itself and thus not a proper subgroup. This contradicts our definition for covering. □

The converse for this is also true, giving a more powerful conclusion.

THEOREM 1.3: A group has a covering if and only if it is not cyclic.

Proof. Because of our previous theorem, all that needs to be shown is that every non-cyclic group has a covering. Suppose G is a non-cyclic group. Therefore, no single element generates it. Collect the subgroups generated by each element. While wildly inefficient, we have created a set of proper subgroups such that every element of G is contained in at least one of them, hence a covering. □

DEFINITION 1.4: The covering number of a group, denoted $\sigma(G)$, is the minimal number of subgroups required to cover the group. If a group cannot be covered, then $\sigma(G) = \infty$.

A covering using the fewest number of proper subgroups may be referred to as a minimal covering. Our goal going forward is to observe the nature of minimal coverings and covering numbers. Since the subgroups must be proper, it should be clear that no group has covering number 1. We continue by observing the possibility of other covering numbers.

THEOREM 1.5: No group has a covering number of 2.

Proof. Let G be any group such that its covering number is 2. Therefore, there are two proper subgroups, which we will call A and B such that $G = A \cup B$. Pick elements $a \in A$ such that $a \notin B$ and likewise $b \in B$ such that $b \notin A$.

If this cannot be done then either all elements of A are contained in B or vice versa. This means that one of the groups is not necessary for the covering, contradicting that 2 subgroups are needed.

If $ab \in A$, then since $a^{-1} \in A$, we have $b \in A$. This is a clear contradiction. This similarly holds if ab is contained in B. So, G cannot be covered by only two proper subgroups. □

2. PRELIMINARIES

The first thing to note is that the focus of this thesis will be on finite groups, and so it will be assumed of all groups going forward. To start, we will need a few tools introduced by Cohn [2].

LEMMA 2.1: If $N \trianglelefteq G$, then $\sigma(G) \leq \sigma(G/N)$.

Proof. Begin with a minimal covering of G/N, labeled $\{H_1/N, H_2/N, ..., H_n/N\}$ for some $n = \sigma(G/N)$ where each H_i is a subgroup of G containing N. In other words, $G/N = \bigcup_i^n (H_i/N)$. We then have that $G/N = (\bigcup_{i=1}^n H_i)/N$ and so $G = \bigcup_i^n H_i$. Hence $\sigma(G)$ is at most n. □

COROLLARY 2.2: Given a surjective homomorphism $\phi : G \to H$, $\sigma(G) \leq \sigma(H)$.

COROLLARY 2.3: $\sigma(H \times K) \leq \min(\sigma(H), \sigma(K))$.

These corollaries are useful tools for finding an upper bound for the covering number of groups. Finding a lower bound on the covering number tends to be more of a case-by-case situation unique to each type of group. The case where the first upper bound is equal to the covering number prompts the following definition.

DEFINITION 2.4: A group G is called primitive if there exists no normal subgroup N for which $\sigma(G) = \sigma(G/N)$.

When looking for specific covering numbers, we can focus on primitive groups.

LEMMA 2.5: A minimal covering for any given group G can always be expressed as a union of maximal subgroups.

Proof. Since we assume that G is finite, any proper subgroup is contained in a maximal subgroup. Therefore, any non-maximal proper subgroup used in a covering can be substituted with a maximal subgroup that contains it. The number of subgroups has not increased, and so this is still a valid covering. □

LEMMA 2.6: If $\gcd(|H|, |K|) = 1$ then $\sigma(H \times K) = \min(\sigma(H), \sigma(K))$.

Proof. Let G be the group $G = H \times K$. If G is cyclic, then H and K are cyclic and we are done. We therefore will assume that G is non-cyclic and that $\sigma(G) = n$ for some n. Because $\gcd(|H|, |K|) = 1$, any subgroup of G will be of the form $X \times Y$, where $X \leq H$ and $Y \leq K$. By Lemma 2.5 we can cover G with maximal subgroups, which satisfy either $X = H$ or $Y = K$. This gives us a covering for G, using n maximal subgroups:

$$G = \left(\bigcup_{r=1}^{p} H \times Y_r \right) \cup \left(\bigcup_{s=1}^{q} X_s \times K \right) = G_1 \cup G_2$$

for some non-negative integers p, q such that $p + q = n$. We aim to show that either $G_1 = G$ or $G_2 = G$.

If $p \neq 0$, then G_1 is non-trivial, implying there exists some element $(h_1, k_1) \in G$ that is not an element of G_2. More generally, $(h_1, k) \notin G_2$ for any $k \in K$. Therefore, $(h_1, k) \in G_1$. So, if (h_1, k) is contained in G_1 for any $k \in K$ and $H \leq G_1$, then $(h, k) \in G_1$ for any $h \in H$, $k \in K$. Hence $G_1 = G$ and so $q = 0$ and $p = n$.

This then gives us that $G = G_1 = H \times (\bigcup_{r=1}^{n} Y_r)$. This means that $K = (\bigcup_{r=1}^{n} Y_r)$ and thus $\sigma(K) \leq n$. The same argument with $q \neq 0$ yields $\sigma(H) \leq n$. Corollary 2.3 then gives us equality, the desired result. $\square$

3. COVERING NUMBER THREE

The route for this section follows [1] as it cleverly uses very little outside results from group theory.

We start with a group G such that $\sigma(G) = 3$. So, $G = A \cup B \cup C$, where A, B, and C are proper subgroups of G. Our goal is to find out what kind of group G can be.

Let K denote the intersection of the three proper subgroups. Being the intersection of groups, K is also a subgroup of G. As well, let $\tilde{A} = A - (B \cup C)$, $\tilde{B} = B - (A \cup C)$, and $\tilde{C} = C - (A \cup B)$. See figure 1.

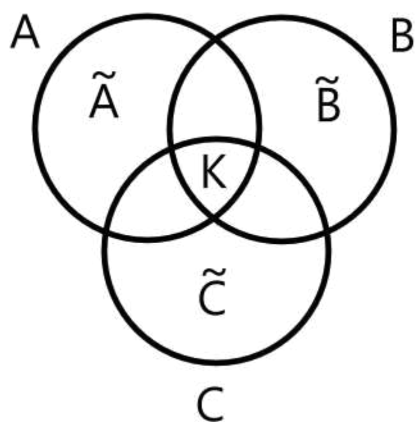

Figure 1. Venn Diagram of Covering Groups A, B, and C, with labeled regions.

An important note is that these four regions are all non-empty. For the tilde groups, we only need to observe that if one of them is empty, say $\tilde{A}$, then $G = B \cup C$, which contradicts our assumption of needing three. Therefore, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are all non-empty. We have $e \in K$, so K is also non-empty.

With those four regions labeled, we are left with only three more: $(A \cap B) - C$, $(A \cap C) - B$, and $(B \cap C) - A$. These sets can be thought of as elements belonging to exactly two of the subgroups.

THEOREM 3.1: The set $(B \cap C) - A$ is empty.

Proof. Let X be the set $(B \cap C) - A$. Assume that X is non empty, choosing $h \in X$. Then, pick an element a from the set $\tilde{A}$. We have $ah \in G$, and so ah is contained in A, B, or C.

If $ah \in A$, then we know $a^{-1} \in A$, and so $h = a^{-1}ah \in A$, a contradiction. If $ah \in B$, then since $h^{-1} \in B$, $a = ahh^{-1} \in B$, but $a \in \tilde{A}$, once again a contradiction. Likewise, $ah \in C$ implies that $a \in C$, a contradiction. Therefore, H is empty. □

This same rationale shows that the other two sets, $A \cap C - B$ and $A \cap B - C$ are empty as well.

LEMMA 3.2: Suppose $\tilde{a} \in \tilde{A}$, $\tilde{b} \in \tilde{B}$, and $\tilde{c} \in \tilde{C}$. Then, $\tilde{a}\tilde{b} \in \tilde{C}$, $\tilde{a}\tilde{c} \in \tilde{B}$, and $\tilde{b}\tilde{c} \in \tilde{A}$.

Proof. We will show $\tilde{a}\tilde{b} \in \tilde{C}$, since the other two relations will follow by analogous arguments. Given $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$.

First, $\tilde{a}\tilde{b} \notin A$, since this would imply $\tilde{b} \in A$, a contradiction. Likewise, $\tilde{a}\tilde{b} \in B$ implies $\tilde{a} \in B$, another contradiction. Therefore, $\tilde{a}\tilde{b} \in C$ and $\tilde{a}\tilde{b} \notin A \cup B$. So, $\tilde{a}\tilde{b} \in \tilde{C}$, and hence we are done. □

LEMMA 3.3: The sets $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ are each closed under inverses.

Proof. As before, we will work with $\tilde{A}$, but note that arguments hold for $\tilde{B}$ and $\tilde{C}$.

Pick $\tilde{a} \in \tilde{A}$. Since $\tilde{a} \in A$, $\tilde{a}^{-1} \in A$. This gives us two cases for its location: either $\tilde{a}^{-1} \in \tilde{A}$, or $\tilde{a}^{-1} \in K$.

Assume $\tilde{a}^{-1} \in K$. Then, since $\tilde{a}\tilde{a}^{-1} = e \in K$, as is $\tilde{a}^{-1}$, this then implies that $\tilde{a} \in K$. Since this is a contradiction, the only option is $\tilde{a}^{-1} \in \tilde{A}$. □

This result lets us extend Lemma 3.2 a little further, giving the converse.

LEMMA 3.4: An element belongs to a given tilde set if and only if it is the product of elements from the other two.

Proof. We already have one direction from Lemma 3.2, so all that is left to show is that every tilde element is expressed as a product from the other two tilde sets.

Pick any $\tilde{a} \in \tilde{A}$. Given any $\tilde{c} \in \tilde{C}$, we have that $\tilde{c}^{-1} \in C$, and so $\tilde{a}\tilde{c}^{-1} = \tilde{b}$ for some $\tilde{b} \in \tilde{B}$. Therefore, $\tilde{a} = \tilde{b}\tilde{c}$.

A similar proof works for $\tilde{B}$ and $\tilde{C}$. □

LEMMA 3.5: If $\tilde{a_1}$ and $\tilde{a_2}$ are elements in $\tilde{A}$, then $\tilde{a_1}\tilde{a_2} \in K$.

Proof. We start with $\tilde{a_1}$ and $\tilde{a_2}$ as elements in $\tilde{A}$. Since these are elements also contained in A, we know $\tilde{a_1}\tilde{a_2}$ is as well. By our previous lemma, $\tilde{a_2} = \tilde{b}\tilde{c}$ for some elements $\tilde{b} \in \tilde{B}$ and $\tilde{c} \in \tilde{C}$. Then $\tilde{a_1}\tilde{a_2} = \tilde{a_1}\tilde{b}\tilde{c}$. Notice then that $\tilde{a_1}\tilde{b} \in C$, and also $\tilde{c} \in C$, hence $\tilde{a_1}\tilde{a_2} \in C$. Likewise, we could define $\tilde{a_2}$ as the product of some $\tilde{b_2}$ and $\tilde{c_2}$. Now, $\tilde{a_1}\tilde{c_2}$ is an element of B and thus the product $\tilde{a_1}\tilde{a_2}$ is contained in B.

This means $\tilde{a_1}\tilde{a_2}$ is contained in each of A, B, and C. Hence, it is contained in K. □

A similar argument holds for the other two tilde sets.

LEMMA 3.6: $K \trianglelefteq G$.

Proof. Normality holds in A if $aka^{-1} \in K$ for any $a \in A$ and $k \in K$. If $a \in K$, then we are done. So, pick an element $\tilde{a} \in \tilde{A}$ and any $k \in K$. Now, $\tilde{a}k$ is contained in A. More specifically, $\tilde{a}k \in \tilde{A}$ or $\tilde{a}k \in K$. If $\tilde{a}k \in K$, then $\tilde{a} \in K$, and clearly this is false. Since $\tilde{a}k \in \tilde{A}$ and $\tilde{a}^{-1} \in \tilde{A}$, Lemma 3.5 gives us $\tilde{a}k\tilde{a}^{-1} \in K$.

A similar process can be done to show normality in B and C. Hence K is normal in G. □

LEMMA 3.7: If $\tilde{a} \in \tilde{A}$, then $\tilde{a}K = \tilde{A}$.

Start with elements $\tilde{a} \in \tilde{A}$ and $k \in K$. Then, the product is contained in either $\tilde{A}$ or K. If $\tilde{a}k \in K$, then $\tilde{a} \in K$, a contradiction, so $\tilde{a}k \in \tilde{A}$. Therefore, $\tilde{a}K \subseteq \tilde{A}$.

For the other direction, let $\tilde{a}$ be an element of $\tilde{A}$. Then $\tilde{a}_1 = \tilde{a}\tilde{a}^{-1}\tilde{a}_1 \in \tilde{a}K$ since $\tilde{a}^{-1}\tilde{a} \in K$ using lemmas 3.3 and 3.5.

THEOREM 3.8: A group has covering number 3 if and only if $G/K \cong C_2 \times C_2$ for some normal subgroup K.

We already have proved the forward direction. The previous lemmas show that $G/K = \{K, \tilde{a}K, \tilde{b}K, \tilde{c}K\}$. The identity element is K, every element has an order of 2, and the product of 2 non-identity elements results in the remaining non-identity element. This is precisely $C_2 \times C_2$, the Klein 4-group.

For our other direction, we start by assuming that there exists $K \trianglelefteq G$ such that $G/K \cong C_2 \times C_2$. We can express $C_2 \times C_2$ as $\{1, x, y, xy\}$. This gives all non-trivial proper subgroups of $\langle x\rangle$, $\langle y\rangle$, and $\langle xy\rangle$. These cover $C_2 \times C_2$ minimally, so $\sigma(C_2 \times C_2) = 3$. From this, we have that $\sigma(G) \leq \sigma(G/K) = \sigma(C_2 \times C_2) = 3$. Since no group has covering number less than 3, $\sigma(G) = 3$.

4. COVERING p-GROUPS AND ABELIAN GROUPS

Our investigation will follow the techniques used by Cohn [2]. If G is our group in question and is covered by n many proper subgroups so that $G = \bigcup_{i=1}^{n} H_i$, then order these subgroups so that

$$|H_1| \geq |H_2| \geq ... \geq |H_n|$$

This ordering will be assumed for our proofs going forward. It should also be useful to recognize that this ordering is reversed in terms of index:

$$[G : H_1] \leq [G : H_2] \leq ... \leq [G : H_n]$$

THEOREM 4.1: If $G = \bigcup_{i=1}^{n} H_i$, where each H_i is a proper subgroup of G, then $|G| \leq \sum_{i=2}^{n} |H_i|$.

Proof. First, the number of elements contained in any H_i that are not contained in H_1 is $|H_i| - |H_1 \cap H_i|$. Factoring makes this $|H_i|\left(1 - \frac{|H_1|}{|H_1 H_i|}\right)$, since $|H_1 \cap H_i| = \frac{|H_1||H_i|}{|H_1 H_i|}$. As well, $|G| \geq |H_1 H_i|$ leads to $|H_i| - |H_1 \cap H_i| \leq |H_i|\left(1 - \frac{|H_1|}{|G|}\right)$.

We then have the inequality $|G| \leq |H_1| + \left(1 - \frac{|H_1|}{|G|}\right) \sum_{i=2}^{n} |H_i|$. Now, $1 - \frac{|H_1|}{|G|} = \frac{|G| - |H_1|}{|G|}$. Our inequality can then be expressed as $|G| - |H_1| \leq \left(\frac{|G| - |H_1|}{|G|}\right) \sum_{i=2}^{n} |H_i|$. Multiplying both sides by $\frac{|G|}{|G| - |H_1|}$ then gives the desired inequality. □

LEMMA 4.2: If $\sigma(G) = n$, then $[G : H_2] \leq n - 1$.

Proof. First, we have that $\frac{|G|}{|H_2|} \leq \frac{|G|}{|H_3|}$, otherwise written as $\frac{|H_3|}{|H_2|} \leq 1$. This likewise holds so that $\frac{|H_i|}{|H_2|} \leq 1$ so long as $i \geq 2$. So, we take our result from Theorem 4.1

and divide out by $|H_2|$, giving

$$\frac{|G|}{|H_2|} \leq \sum_{i=2}^{n} \frac{|H_i|}{|H_2|} \leq \sum_{i=2}^{n} 1 = n - 1$$

which completes the proof. □

COROLLARY 4.3: If $[G : H_2] = n$, then $\sigma(G) \geq n + 1$.

These tools give an alternative route for finding the groups with a covering number of 3.

Suppose that $\sigma(G) = 3$, and so we have that $G = H_1 \cup H_2 \cup H_3$, where each H_i is a proper subgroup. By Lemma 4.2, $[G : H_2] \leq 2$. Since $H_2 \neq G$, $[G : H_2] = 2$, thus implying that $[G : H_1] = 2$. So, G has two subgroups of index 2.

If G has any group with two subgroups of index 2, labeled A and B, then these subgroups are both normal in G. We then have that the subgroup $AB \leq G$ while $A < AB$. Therefore, $[G : AB][AB : A] = [G : A] = 2$, where $[AB : A] > 1$. This implies that $[G : AB] = 1$ and hence $G = AB$.

Therefore, there is a surjective homomorphism from G to $G/A \times G/B$ with kernel $A \cap B$. If we let K be the kernel of this action, then $G/K \cong G/A \times G/B \cong C_2 \times C_2$, implying that $\sigma(G) \leq 3$. The smallest possible covering number is 3, and so $\sigma(G) = 3$.

LEMMA 4.4: If p is a prime integer, then the group $C_p \times C_p$ has the covering number $p + 1$, and it is primitive.

Proof. Every proper subgroup of $C_p \times C_p$ has index p. Corollary 4.3 gives the lower bound $\sigma(C_p \times C_p) \geq p + 1$. So, if there exists a set of $p + 1$ proper subgroups that cover $C_p \times C_p$, then we have found its covering number.

The group $C_p \times C_p$ is generated by two elements, which we will call a and b. Let $H_1 = \langle ab \rangle, H_2 = \langle ab^2 \rangle, ..., H_{p-1} = \langle ab^{p-1} \rangle$, as well as letting $H_p = \langle a \rangle$ and $H_{p+1} = \langle b \rangle$. To show that the union of these covers $C_p \times C_p$, pick any element from the group, which can be written in the form of $a^x b^y$, where x and y are non-negative integers. If $x = 0$, then this element is contained in H_{p+1}, likewise if $y = 0$ it is contained in H_p. Otherwise, let $i = yx^{-1} (\text{mod } p)$. We have that $a^x b^y = (ab^i)^x$ and is thus contained in H_i.

Lastly, showing that it is primitive comes from the fact that every quotient group of $C_p \times C_p$ is cyclic, and so no quotient group shares its covering number. □

THEOREM 4.5: If G is a non-cyclic p-group, then $\sigma(G) = p + 1$, with it being primitive if and only if $G = C_p \times C_p$.

Proof. Since every proper subgroup of a p-group will have an index of p, Corollary 4.3 gives us that $\sigma(G) \geq p + 1$.

For the other direction, we have two case: When G is abelian, and when G is not abelian. If G is abelian, then it is the direct product of other p-groups. So,

$$G = H_1 \times H_2 \times ... \times H_k$$

for some k, where each H_i is also an abelian p-group. Since the direct product of $H_2 \times ... \times H_k$ is itself an abelian p-group, G can be expressed as the direct product of exactly two p-groups. These two p-groups both have a quotient group that is isomorphic to C_p, hence G has a quotient group isomorphic to $C_p \times C_p$. So, $\sigma(G) \leq \sigma(C_p \times C_p) = p + 1$.

The non-abelian case will utilize mathematical induction. To start, $|G| = p^k$, where k is some positive integer. Note that by assuming G is non-cyclic, we have that $k \geq 2$. For the case $k = 2$, G is simply $C_p \times C_p$, which has the covering number $p + 1$. For the inductive step, assume that the covering number is $p + 1$ given $k \leq n$ for some n. For the case of $k = n + 1$, we find that since the center of any p-group is non-trivial, $G/Z(G)$ would be a smaller non-cyclic p-group. So, $\sigma(G) \leq \sigma(G/Z(G)) = p + 1$.

The last part of this theorem is that the only primitive case is $C_p \times C_p$. What we have already shown is that every non-cyclic p-group satisfies the condition that there exists a normal subgroup N such that $G/N \cong C_p \times C_p$ and that $\sigma(G/N) = \sigma(C_p \times C_p)$. So, there is nothing more needed. $\square$

A valuable result following from this is the covering number for any non-cyclic finite abelian group.

THEOREM 4.6: Given G is a non-cyclic finite abelian group, $\sigma(G) = p + 1$, where p is the smallest prime number such that G has two p-groups in its cyclic decomposition.

Proof. Decomposing G into the direct product of cyclic groups gives us $G = C_{p_1^{\alpha_1}} \times C_{p_2^{\alpha_2}} \times ... \times C_{p_n^{\alpha_n}}$ for some non-negative integer n where each p_i is some prime. Since G is not cyclic, there must exist some j, k such that $p_j = p_k$. Reorder these cyclic groups so that the groups with p_j power order are first. We then have that G is the direct product of $C_{p_j^{\alpha_1}} \times C_{p_j^{\alpha_2}} \times ... \times C_{p_j^{\alpha_m}}$ for some m with the remaining direct product of cyclic p-groups, which we'll label H. These two groups have relatively prime order, and so we can utilize lemma 2.5 to obtain $\sigma(G) = min(\sigma(C_{p_j^{\alpha_1}} \times C_{p_j^{\alpha_2}} \times ... \times C_{p_j^{\alpha_m}}), \sigma(H))$.

The group $C_{p_j^{\alpha_1}} \times C_{p_j^{\alpha_2}} \times ... \times C_{p_j^{\alpha_m}}$, being a non cyclic p-group, has covering number $p_j + 1$. If H is non-cyclic, then we can repeat the process, finding more p-groups until we are left with a direct product of non-cyclic p-groups and eventually a cyclic group. The result is then that $\sigma(G)$ is the minimum covering number over all of these non-cyclic p-groups. This concludes our proof. $\square$

5. COVERING NUMBER FOUR

For this problem, we first need what is called the normal core of a subgroup. Let G be a group with subgroup A. Let G act on the set of left cosets of A, denoted G/A, by left multiplication. We define this operation, $\sigma_g : G/A \to G/A$ as for each $g \in G$, $\sigma_g(xA) = gxA$ for any coset xA in G/A. We can see that this is one-to-one and onto as $\sigma_g(x_1A) = \sigma_g(x_2A)$ means that $gx_1A = gx_2A$, which left multiplication by g^{-1} gives us that $x_1A = x_2A$, covering that this is one-to-one. For onto, given any xA that we wish for output, $\sigma_g(g^{-1}xA) = gg^{-1}xA = xA$.

This means that this action permutes the elements of G/A.

DEFINITION 5.1: The core of A, is defined to be the kernel of this action. We denote it by core(K).

Note that since this is the kernel of a homomorphism from G to Sym(G/A), the core of A is normal in G. From this we have core$(A) = \{g \in G \mid gxA = xA, \forall x \in G\}$. This distinction that $gxA = xA$ can then become $x^{-1}gxA = A$. This is the same as $x^{-1}gx \in A$. We then can simplify this last piece to get our other way of constructing the normal core: core$(A) = \{g \in G \mid g \in xAx^{-1}\ \forall x \in G\} = \bigcap\limits_{x \in G} xAx^{-1}$.

LEMMA 5.2: Given a group G, core(A) is the largest subgroup of A that is normal in G.

Proof. Suppose N is a normal subgroup of G and also a normal subgroup of A. Then, for every $x \in G$, $N = xNx^{-1} \le xAx^{-1}$. So, $N \le \bigcap\limits_{x \in G} xAx^{-1} = core(A)$. □

THEOREM 5.3: Given a group G such that $\sigma(G) \neq 3$, $\sigma(G) = 4$ if and only if there exists some $K \lhd G$ such that G/K is isomorphic to either $C_3 \times C_3$ or S_3. The groups $C_3 \times C_3$ and S_3 are the only primitive 4-coverable groups.

Proof. Suppose that $\sigma(G) = 4$, so that $G = H_1 \cup H_2 \cup H_3 \cup H_4$. Lemma 4.2 then gives $[G : H_2] \le 3$. Now, if $[G : H_2] = 2$, then $[G : H_1] = 2$ as well. As previously shown, this implies that $\sigma(G) = 3$, so we conclude that $[G : H_2] = 3$. Following with Theorem 4.1, $|G| \leq |H_2| + |H_3| + |H_4|$. Dividing out by $|H_2|$, we have

$$3 = \frac{|G|}{|H_2|} \leq 1 + \frac{|H_3|}{|H_2|} + \frac{|H_4|}{|H_2|} \leq 3$$

as these groups are ordered in non-increasing order. Therefore, we have equality and so H_2, H_3, and H_4 all have an index of 3.

Going forward, we just need to work with knowing that G has 2 subgroups of index 3. So, let G be such a group and $\sigma(G) \neq 3$. Then we have the following cases: Either both A and B are normal, or at least one of them is not.

Case 1: A and B are both normal. Then their intersection, $A \cap B$, as well as AB must be normal as well. So, $3 = [G : A] = [G : AB][AB : A]$, which implies that $[AB : A] = 1$ or 3. Since B is assumed to be distinct from A, $[AB : A]$ must be equal to 3, and so $[G : AB] = 1$. Thus, $G = AB$. Using an exercise result from Dummit and Foote [1],

$$G/(A \cap B) \cong G/A \times G/B \cong C_3 \times C_3$$

Since $C_3 \times C_3$ is a p-group, we know $\sigma(G) \leq \sigma(C_3 \times C_3) = 4$. The assumption $\sigma(G) \neq 3$ implies equality here.

Case 2: Without loss of generality, let $A \ntrianglelefteq G$. Let G act on the set G/A of left cosets of A, with K being the kernel of this action, $\phi : G \to \mathrm{Sym}(G/A)$ so that K is the normal core of A. Therefore, $G/K \cong \mathrm{im}(\phi) \leq \mathrm{Sym}(G/A) \cong S_3$. In other words, G/K is isomporphic to a subgroup of S_3.

The left cosets of A are A, xA, and yA, for some $x, y \in G$. By prior work, $K = A \cap xAx^{-1} \cap yAy^{-1}$. Note that this intersection is of 3 distinct subgroups of G, all of which K is normal in. Therefore, A/K, xAx^{-1}/K, and yAy^{-1}/K are all three distinct subgroups of G/K, all of which have the same order. Because of this, G/K cannot be cyclic. Another way to reach this conclusion is that these subgroups are in fact proper subgroups of G/K and cover it, implying that G/K cannot be cyclic. Since all proper subgroups of S_3 are cyclic, G/K must not be a proper subgroup and hence, $G/K \cong S_3$.

With our forward direction covered, the reverse is a direct application of Lemma 2.1 with the assumption that $\sigma(G) \neq 3$. $\square$

6. THE COVERING NUMBER OF A DIHEDRAL GROUP

To clarify notation, we use D_{2n} to denote the dihedral group of order $2n$. More specifically, we express this as $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$. We begin by exploring the proper subgroups of any such group.

LEMMA 6.1: No proper subgroup of D_{2n} contains both the elements r and sr^i for any i.

Proof. By way of contradiction, assume that D_{2n} has some proper subgroup, H, containing the elements r and sr^i. By group closure, $r^{n-i} \in H$. Therefore, $sr^i r^{n-i} = s \in H$, implying that $H = G$. So, H cannot be proper and we are done. □

Since no proper subgroup can contain both r and sr^i, it naturally follows that in a valid covering for D_{2n}, at least one of the subgroups used must be $\langle r \rangle$.

LEMMA 6.2: Given prime integer p and any distinct non-negative integers i, j less than p, no proper subgroup of D_{2p} contains both sr^i, sr^j.

Proof. By way of contradiction, let H be a subgroup of D_{2p} such that $sr^i \in H$ and $sr^j \in H$ for some distinct positive i, j less than p. Without loss of generality, let $j > i$.

It follows that, $sr^i sr^j = S^2 r^{j-i} = r^{j-i} \in H$. Call this element r^a, where $a = j - i$. Since $j > i$, a is a positive integer less than p. Therefore, there exists an integer b such that $ab \equiv 1 \pmod{p}$. So, $(r^a)^b = r \in H$.

By Lemma 6.1, H cannot be a proper subgroup as it contains r as well as sr^i. Therefore, a contradiction is met, and our proof is complete. □

COROLLARY 6.3: If H is a proper subgroup of D_{2p} and $sr^i \in H$ for some integer i, then $H = \langle sr^i \rangle = \{1, sr^i\}$.

THEOREM 6.4: Given prime integer p, $\sigma(D_{2p}) = p + 1$.

Proof. As mentioned prior, at least one subgroup used in a covering must be $H_1 = \langle r \rangle$. By Corollary 6.3, there must also be p many subgroups of the form $\langle sr^i \rangle$ for each $0 \leq i < p$. Any element of D_{2n} is contained in these $p + 1$ total subgroups, as all powers of r are contained in H_1, and any sr^i has its own subgroup. Hence, this covers D_{2n} and we are done. □

The natural direction is to extend this towards all dihedral groups. We can do this by broadening Lemma 6.2.

LEMMA 6.5: Given non-negative integers i, j, n, where $i < j < p$, with p being the smallest prime integer dividing n, then the only subgroup of D_{2n} containing sr^i and sr^j is the group D_{2n}.

Proof. We start with such integers, i, j, n, p, noting that p is the smallest prime divisor of n and $i < j < p$.

The first thing to note is that $0 < j - i < p$. Since p is the smallest prime factor of n, $j - i$ is relatively prime to n. Therefore, $j - i$ a unit in $\mathbb{Z}_n$ and so there exists an integer a such that $a(j - i) \equiv 1$ (mod n).

Let H be any subgroup of D_{2n} that contains the elements sr^i and sr^j. By group closure, $sr^i sr^j = s^2 r^{j-1} = r^{j-i} \in H$. We then also have that $(r^{j-i})^a = r \in H$, and therefore all powers of r are all contained in H.

Therefore, $Sr^j r^{-j} = S \in H$. We then have that H contains generators for D_{2n}, and so $H = G$. □

THEOREM 6.6: Given positive integer n with p being the smallest prime factor of n, $\sigma(D_{2n}) = p + 1$.

Proof. Just as before, we start with this notion that given a set of proper subgroups that cover D_{2n}, we have an initial one being $H_1 = \langle r \rangle$. Knowing that p is the smallest prime divisor of n, each unique sr^i for $0 \leq i < p$ will be contained in separate subgroups. This gives us an additional p subgroups, meaning we have a lower bound of $\sigma(D_{2n}) \geq p + 1$

For an upper bound, the subgroup generated by r^p is normal in D_{2n}. Taking a quotient with it yields $D_{2n}/\langle r^p \rangle \cong D_{2p}$, which has the covering number $p + 1$. By Lemma 2.1, $\sigma(D_{2n}) \leq \sigma(D_{2p}) = p + 1$ and hence $\sigma(D_{2n}) = p + 1$. □

7. COVERING NUMBER OF A GENERALIZED DIHEDRAL GROUP

DEFINITION 7.1: A generalized dihedral group is the semi-direct product of an abelian group H with an element x that inverts elements through conjugation. We denote it as $\mathrm{Dih}(H) = \langle H, x \mid x^2 = 1,\, xhx^{-1} = h^{-1} \text{ for every } h \in H\rangle$.

The standard dihedral groups are the case where H is a cyclic group generated by r, with x being replaced by s. Since the cyclic case is already proven, we will be assuming H is non-cyclic going forward.

THEOREM 7.2: Every subgroup of $G = \mathrm{Dih}(H)$ is either a subgroup of H or a generalized dihedral group.

Proof. Let $K \leq G$. If $K \leq H$, then we are done, so assume otherwise. Therefore, there exists $hx \in K$ for some $h \in H$. Let $A = H \cap K$. We show that $K = A \rtimes \langle hx\rangle$ by showing (i) $A \trianglelefteq K$, (ii) $A \cap \{hx\} = 1$, and (iii) $K = A\langle hx\rangle$.

For (i), $H \trianglelefteq G$, $H \cap K \trianglelefteq K$ and hence $A \trianglelefteq K$.

To show (ii), $hxhx = hh^{-1} = 1$ and thus $\langle hx\rangle = \{1, hx\}$. Since $hx \notin H$, $A \cap \langle hx\rangle = \{1\}$.

For (iii), since $A \leq K$ and $hx \in K$, it follows that $A\langle hx\rangle \leq K$. For our other direction, let $k \in K$. If $k \in H$, then $k \in A$ and hence $A\langle hx\rangle$. If $k \notin H$, then $k = h_1x$ for some $h_1 \in H$. Consider $k(hx)^{-1} \in K$. Then $k(hx)^{-1} = h_1xxh^{-1} = h_1h^{-1} \in H$ and thus $k(hx)^{-1} \in H \cap K = A$. So, $k \in A\langle hx\rangle$. Therefore, $K \leq A\langle hx\rangle$ and thus $K = A\langle hx\rangle$.

Note that for any $k \in K$, $(hx)k(hx)^{-1} = hxkxh^{-1} = hk^{-1}h^{-1} = k^{-1}$. It then follows that $K = \mathrm{Dih}(A)$. □

LEMMA 7.3: If $G = \mathrm{Dih}(H)$, then each maximal subgroup used to cover G contains a non-trivial subgroup of H.

Proof. By way of contradiction, let M be a maximal subgroup of G that contains no non-trivial subgroup of H. Therefore, the only non-identity elements in M are of the form ax where $a \in H$. If M contains distinct elements ax and bx, then $axbx = ab^{-1} \in H$ is contained in M, a contradiction.

So, M only has two elements: some ax and the identity. However, M is thus not maximal, as $M < \langle a, x \rangle < G$. This completes our proof. □

LEMMA 7.4: Each maximal subgroup of $G = \text{Dih}(H)$ that is not H contains at least one element of the form hx for some $h \in H$.

Proof. This follows directly from Theorem 7.2. Subgroups of G are either subgroups of H or generalized dihedral groups utilizing some element hx. The only subgroup of H that is maximal in G is H itself. □

LEMMA 7.5: The maximal subgroups of $G = \text{Dih}(H)$ are H and those of the form $A \cup hxA$ where A is maximal in H.

Proof. Start with a maximal subgroup of G, $M \neq H$. By Lemmas 7.2 and 7.3, $M = A \cup Bx$, where A is a proper subgroup of H and B is a subset of H. It follows that B consists of cosets of A. Assume further that B contains two distinct cosets of A, b_1A and b_2A, for some $b_1, b_2 \in H$. Since $b_1x \in M$ and $b_2x \in M$, $b_1xb_2x = b_1b_2^{-1} \in M$ and hence $b_1b_2^{-1} \in A$. Therefore, $b_2A = b_2(b_1b_2^{-1})A = b_1b_2b_2^{-1}A = b_1A$. Therefore, B is exactly a coset of A.

To show A is maximal in H, assume otherwise. Then pick A' such that $A < A' < M$. It follows that $M = A \cup hxA < A' \cup hxA' < G$, for any $hx \in A$. Therefore, M is not maximal, and we have our desired result. □

THEOREM 7.6: $\sigma(\text{Dih}(H)) = p + 1$, where p is the smallest prime dividing the order of H.

Proof. Suppose $\mathcal{C}$ is a minimal covering of G.

Case 1: $H \notin \mathcal{C}$. Then every element of $\mathcal{C}$ is of the form $A_i \cup h_i x A_i$ for some $A_i < H$ and some $h_i \in H$. Thus $G = \bigcup_{i=1}^{n} A_i \cup h_i x A_i$. Hence, $\bigcup_{i=1}^{n} A_i = H$ and $\bigcup_{i=1}^{n} h_i x A_i = G - H = Hx$. If we take each $h_i = 1$, then we still have a valid covering. It follows then that we simply choose A_i to be a minimal covering for H, giving us $\sigma(G) \leq \sigma(H)$.

Case 2: $H \in \mathcal{C}$. Then G is covered by H and subgroups of the form $A_i \cup h_i x A_i$. This reduces to minimizing n where $G - H = Hx = \bigcup_{i=1}^{n} h_i x A_i$. Let $k = \max_{1 \leq i \leq n}\{|A_i|\}$. We then have:

$$|H| = |Hx| = |\bigcup_{i=1}^{n} h_i x A_i| \leq \sum_{i=1}^{n} |h_i x A_i| \leq nk$$

This gives us $|H|/k \leq n$.

The group H can be expressed as $H = C_{p_1^{\alpha_1}} \times C_{p_2^{\alpha_2}} \times ... \times C_{p_n^{\alpha_n}}$ such that $p_1 \leq p_2 \leq ... \leq p_n$. It follows that $A = C_{p_1^{\alpha_1 - 1}} \times C_{p_2^{\alpha_2}} \times ... \times C_{p_n^{\alpha_n}}$ is a maximal subgroup in H with index p_1 and that p_1 is the smallest non-trivial divisor of H. Therefore, $Hx = \bigcup_{i=1}^{p_1} h_i x A$ where each $b_i A$ is a distinct coset of A. Hence $n \leq p_1$.

Now, $|H|/k \leq n \leq p_1$, where $|H|/k$ is a non-trivial divisor of $|H|$ and p_1 is the smallest non-trivial divisor of H. Therefore, $n = p_1$. This implies a covering of G using $p + 1$ subgroups where p is the smallest prime divisor of $|H|$.

Since the two cases cover all possible coverings of G, the smallest result between them must be $\sigma(G)$. It then follows that $p + 1 \leq \sigma(H)$, and we are done. $\square$

8. CONCLUSION

Finite non-cyclic groups can be expressed as the set-wise union of proper subgroups, called a covering. We have given the term covering number to denote the minimal number of proper subgroups required to cover a group.

The above work has shown that no groups exist with covering numbers 1 or 2, while finding fundamental groups for 3 and 4 for which all other groups with covering numbers 3 and 4 have a homomorphism mapping onto them. These fundamental groups we have called primitive groups. The only primitive group for covering number 3 is $C_2 \times C_2$, while the covering number 4 has two primitive groups: $C_3 \times C_3$ and S_3.

As well, covering numbers for non-cyclic p-groups have been found to be $p+1$, with the primitive p-groups being $C_p \times C_p$. This lead to the covering number for non-cyclic finite abelian groups, being $p+1$, where p is the smallest p-group for which the group has 2 in its cyclic decomposition. The covering number for any dihedral group D_{2n} is $p+1$, where p is the smallest prime factor of n. Similarly, the generalized dihedral groups generated by an abelian group H we found to have the covering number $p+1$, where p is the smallest prime factor of $|H|$.

Going forward, a further pursuit of research may be in attempting to generalize our approach on the dihedral and/or generalized dihedral groups, looking instead on the potential interactions between covering numbers and semi-direct products.

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