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Geometric Dissections

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GEOMETRIC DISSECTIONS

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

In the study of geometry, the notion of dissection and its mechanics are occasionally overlooked. We consider and trace the history and theorems surrounding geometric dissections in both recreational and academic mathematics. We explore the important advancements in this particular topic from antiquity through the nineteenth and early twentieth centuries. We conclude with an exploration of the Banach-Tarski paradox.

KEYWORDS: geometry, geometric dissections, Banach-Tarski paradox, Dehn invariant, Wallace-Bolyai-Gerwien theorem

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DEDICATION

This work is dedicated to those I lost

Donald L. Turner

April 30, 1936 – February 13, 2017

Charles G. Martin, Jr.

January 31, 1954 – April 12, 2019

And to those I found along the way

Haleigh Q. Martin

Married June 6, 2017

Sterling E. Martin

November 18, 2019

Owen H. Martin

April 2, 2021

Elijah R. Martin

May 14, 2023

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1. INTRODUCTION

Through its long history, geometry has been instrumental in several areas of significance, among which are architecture, construction, and navigation. The compass and straightedge constructions and proofs found thousands of years ago still form the basis of geometry education today. While such constructions and proofs are often a frustration to modern students, geometry puzzles have been and remain a more entertaining introduction to geometric concepts. One common class of geometry puzzle is the dissection puzzle where the aim is to take one geometric figure, decompose it into smaller figures, and arrange them to obtain another geometric figure. Students attempting to solve a dissection puzzle, such as a tangram puzzle, will learn and utilize the isometries of Euclidean geometry, even if the students cannot express them as such at first.

As shall be shown, dissection puzzles are an ancient creation that persist along with compass and straightedge proofs. First, we discuss the oldest known dissection puzzle, the stomachion. After that will come a history and exploration of the tangram. The last of the dissection puzzles examined is a curiosity discovered by British puzzle master, Henry Dudeney.

Despite the ancient nature of geometric dissection puzzles, the proof of the concept underlying dissection puzzles is relatively modern. That concept, the Wallace-Bolyai-Gerwien theorem, was proved in the early nineteenth century. As we shall see, the progression from two dimensions to three dimensions culminated in the work of Max Dehn and the polyhedral measure that bears his name: the Dehn invariant.

Following the historical progression, we will see what happens when regular geometric figures are replaced by sets and points. Particularly, several paradoxes arise in this examination. The culmination of these paradoxes, and this work, is the infamous Banach-Tarski paradox.

A recurrent theme throughout is that of paradox. We shall see two forms of paradox: illusory and actual. An illusory paradox, as the name implies, is a mere illusion. Such a paradox is really some visual deception that, though convincing, is merely a trick. An actual paradox, by contrast, yields a result that seems contradictory or impossible, and yet is demonstrably true.

2. STOMACHION

The oldest known dissection puzzle dates back over 2000 years to ancient Greece and was first discussed by Archimedes. Known as the stomachion, the puzzle has precious little historical attestation: the two main sources are the Roman historian Ausonius and a badly damaged manuscript known as the Archimedes palimpsest. As noted in [3], The two sources disagree on several details of the puzzle.

Ausonius refers to the puzzle as being a dissection of a square, the pieces of which could be arranged into a variety of shapes such as a warrior or an elephant. Noted in [6], Ausonius also refers to the puzzle by a different name: the ostomachion. Roughly translating as "bone fight", some scholars insist this is the puzzle's actual name.

The Archimedes palimpsest, by contrast, refers to the puzzle in a diagram as a dissection of a double-square rectangle. One of the objectives of play, according to the palimpsest, was to fit the pieces back into the box in which they came. Whether the box was also a rectangle or a square is unclear: both are referred to and both are possible with the pieces of the double-square dissection.

For this work, we will refer to the puzzle as the stomachion. Although [3] leans toward the puzzle being a double-square rectangle, the square version of the dissection will be considered as more modern work has been done with it. However it was played and whether it was originally a rectangle or a square, the compass and straightedge construction is identical for each, differing only by the relative dimensions of the resulting pieces. Let $ABCD$ be a square. As in Figure 1, The stomachion pieces can be constructed thus:

1. Draw the segment $\overline{AC}$.
2. Find the midpoints of $\overline{AB}$ and $\overline{CD}$. Label them E and F respectively.
3. Draw the segments $\overline{CE}$ and $\overline{DE}$.
4. Draw the segment $\overline{EF}$.

5. Find the midpoint of $\overline{CE}$. Label it G.

6. Draw the segment $\overline{FG}$.

7. Along the ray $\overrightarrow{DG}$, draw the segment from G through $\overline{BC}$. Call the point of intersection H.

8. Label the intersection of $\overline{AC}$ and $\overline{DE}$ as I.

9. Find the midpoint of $\overline{AI}$. Label it J.

10. Draw the segment $\overline{DJ}$.

11. Find the midpoint of $\overline{DF}$. Label it K.

12. Find the line perpendicular to $\overline{DF}$ passing through K and draw the segment from K to the intersection of $\overline{DE}$. Label it L.

13. Along the ray $\overrightarrow{AK}$, draw the segment from the intersection of $\overline{DE}$ to K. Label the intersection M.

The resulting dissection forms the pieces of the stomachion. If the original aim of play was to fit the pieces back together as a square, a puzzler would have a variety of ways to accomplish this: as discovered by Bill Cutler per [6], there are 536 distinct ways to rearrange the pieces into the original square, counting arrangements equivalent by reflection and rotation as being the same.

3. TANGRAM

Perhaps the most well-known dissection puzzle, the tangram has fascinated puzzlers the world over for the last two centuries. According to [3], one puzzle was even among Napoleon's collection of amusements. The tangram was and remains a popular introduction to geometry for students. Before it was used primarily for educating children, the tangram was famously popular in Europe and China for the early part of the nineteenth century. Though it was introduced in America around the same time, the puzzle was not nearly as popular in the United States until the 1860s.

The Chinese mathematicians of antiquity were as familiar with geometric dissections as their Greek counterparts. Particularly, the Pythagorean theorem was demonstrated in the third century by Liu Hui as a dissection and rearrangement of two squares into a single, larger square. Hui's original dissection has, unfortunately, been lost to time. Many guesses have been made as to the exact dissection used, but none are confirmed. As found in [5], Figures 2 and 3 are two proposed interpretations of Liu Hui's original instructions.

Such dissections found expression beyond mathematical or puzzling curiosity in China. Discussed at length in [3], Butterfly Wing Tables were set of tables of various shapes that could be arranged as a wide assortment of shapes. While other designs predate the Butterfly Wing Tables, this particular design is of interest due to its similarity with the tangram. Invented by Ko Shan early in the seventeenth century, the tables were a collection of tables sold as a set. Each set had a total of thirteen tables in a variety of shapes and sizes. The tables included a book authored by Shan himself detailing possible arrangements of the tables. The book detailed configurations ranging from basic geometric shapes such as squares and triangles to larger, more ornate patterns such as a cave or a pavilion. Of particular interest is the possible relation between these tables and the tangram. Note in Figures 5a and 5b to see the similarity of the tangram with half of the Butterfly Wing

Tables.

The tangram is fairly modern: the puzzle was first created during the reign of Chia-ch'ing, placing the puzzle's creation close to 1800. The objective of play is the same as Ausonius's description of the stomachion: arrange the pieces to obtain a given problem silhouette with all of the pieces used and no pieces overlapping. These dissections and arrangements could and still can be found in a variety of books and range from simple geometric shapes to far more creative silhouettes, such as a crane or a man. We exhibit a few of the people puzzle silhouettes in Figure 4.

The construction is simpler than that of the stomachion as seen in Figure 6. Begin with a square $ABCD$. The tangram can be constructed through the following steps:

1. Draw the segment $\overline{AC}$
2. Find the midpoint of $\overline{AD}$, label it E
3. Find the midpoint of $\overline{CD}$, label it F
4. Draw the segment $\overline{EF}$
5. Find the midpoint of $\overline{AC}$, label it G
6. Find the midpoint of $\overline{CG}$, label it H
7. Find the midpoint of $\overline{EF}$, label it I
8. Draw the segment $\overline{IB}$. Note that G will be colinear.
9. Find the midpoint of $\overline{AG}$, label it J.
10. Draw the segment $\overline{FJ}$
11. Draw the segment $\overline{IH}$

A particularly interesting subset of tangram silhouettes are the "paradoxical pairs." For these silhouettes, one of the pair appears to be identical to the other but with some addition (or subtraction). The seeming paradox arises from the fact that the two figures can be made with the same seven pieces, despite one appearing to have a larger area than the other. This apparent paradox is, as noted in [3], merely an illusion: though the figures seem to be such that one is a proper subset of the other, the figures actually retain the same area.

Figure 7 is an example of a paradoxic pair. Even seeing the arrangement of the pieces, it appears as if the two squares are the same size despite the one having an hourglass cutout. The illusion is that the second figure is not a square at all, but rather a near-square rectangle. By taking the area of the tangram square to be 1 and the resulting dimensions of the constituent pieces, it becomes clear that the dimensions of the second figure are $1 \times 3\sqrt{2}/4$. Similarly, Figure 8 appears to show a missing triangle. However, examining the central isosceles triangle of the complete figure shows that it has legs of length 1 and base $\sqrt{2}$. By contrast, the incomplete triangle has legs length $3\sqrt{2}/4$ and base length $3/2$.

4. HABERDASHER'S PUZZLE

Physical puzzle sets, such as the tangram and the stomachion, have enjoyed much popularity but they are far from the only source of puzzles. Puzzle books have been popular through the years as well. One such by Britain's "first and greatest puzzle master" Henry E. Dudeney is the source of our next geometric curiosity. In [1], Dudeney introduces a plethora of characters, among whom is the haberdasher. In the setup for his eponymous puzzle depicted in Figure 9, the haberdasher produced a piece of cloth in the shape of a perfect equilateral triangle and said, "Show me, then, if ye can, in what manner this piece of cloth may be cut into four several pieces that may be put together to make a perfect square." The story went on to relate that the haberdasher admitted to having no answer after several attempts found a dissection with five pieces but not four. Dudeney, however, had not only found a solution, but one with a rather unique property. As he describes in [1], "... I have found that the feat may really be performed in so few as four pieces, and without turning over any piece when placing them together. The method of doing this is subtle." Below are the steps to the dissection of the haberdasher's puzzle.

As shown in Figure 10, given an equilateral triangle $\triangle ABC$ The steps to Dudeney's dissection are as follows:

1. Find the midpoint of $\overline{AB}$, call it D.
2. Find the midpoint of $\overline{BC}$, call it E.
3. Take the arc from $\overline{EB}$ and the ray $\overrightarrow{AE}$, label their intersection F.
4. Find the midpoint of $\overline{AF}$, label it G.
5. Along the ray $\overrightarrow{EB}$, find the intersection with the arc from $\overline{GE}$, label the intersection H.
6. Take the arc from $\overline{EH}$ and label its intersection with $\overline{AC}$ as J.
7. Copy the length of $\overline{BE}$ and find point K on $\overline{AC}$ such that $\ell(\overline{BE}) = \ell(\overline{JK})$.

8. Draw the perpendicular to $\overline{JE}$ through D, label the point on $\overline{JE}$ as L.

9. Draw the perpendicular to $\overline{JE}$ through K, label the point on $\overline{JE}$ as M.

Not only is this dissection a rather clever one, but it also has a fascinating property as described by Dudeney. In the solution to the puzzle, Dudeney notes of the dissection that, "... the four pieces form a sort of chain, and that when they are closed up in one direction they form the triangle, and when closed in the other direction they form the square." That is to say, the dissection is *hinged* (see Figure 11). By hinged, what is meant is that the subregions of the dissection can remain attached to certain neighboring pieces by one vertex per neighbor during the process of translation and rotation.

5. WALLACE-BOLYAI-GERWIEN THEOREM

The preceding puzzles are all rearrangement puzzles, relying on the isometries of translation, rotation, and reflection for their particular dissections. For each puzzle, it follows necessarily that the solution arrangements all share the same area as the original, undissected polygon: the stomachion's square or double-square rectangle, the tangram's square, and the haberdasher's triangle. Each dissection requires several, often subtle, steps. From this, a question arises: is it possible to take any two polygons of the same area and dissect the first of them in such a way that the resulting pieces can be rearranged to cover the second? Could such a dissection be arbitrarily described? The answer to both questions turns out be yes.

As is often the case in mathematics, the discovery of these answers and the resulting theorem and proof was made independently by a few different mathematicians around the same time: William Wallace in 1807, Farkas Bolyai in 1833, and Paul Gerwien in 1835. Their collective discovery is the Wallace-Bolyai-Gerwien (WBG) theorem which states that two polygons are congruent by dissection if and only if they have the same area.

DEFINITION 5.1 (Scissors Dissection): For an arbitrary polygon P, a finite union of smaller polygons ($P = P_1 \cup ... \cup P_n$) such that the smaller polygons have pairwise disjoint interiors is called a *scissors dissection*.

DEFINITION 5.2 (Scissors Congruence): Two polygons P, Q are *scissors congruent* if there are dissections ($P = P_1 \cup ... \cup P_n$, $Q = Q_1 \cup ... \cup Q_n$) such that $P_k \cong Q_k$. The term is also interchangeable with *congruent by dissection*.

The process to prove the WBG theorem is straightforward: since the isometries in two-dimensions are transitive, it suffices to show that any simple polygon is congruent by dissection with an intermediate polygon. The two common choices for this intermediate are a square of equivalent area or a rectangle with side lengths 1 and A where A is the area of the original polygon. We opt for the latter in this work.

Definition 5.3 (Simple polygon): A polygon P with no two non-consecutive intersecting edges is called a *simple polygon*. There is a well-defined bounded interior and unbounded exterior for a simple polygon, where the interior is surrounded by edges. When referring to P, the convention is to include the interior of P.

Definition 5.4 (Convex vertex): A vertex is said to be a *convex vertex* if the measure of the internal angle formed by it and its two adjacent vertices is contained in the open set (0, 180).

Definition 5.5 (Reflex vertex): A vertex is said to be a *reflex vertex* if the measure of the internal angle formed by it and its two adjacent vertices is contained in the open set (180, 360).

Lemma 5.1: Every simple polygon has at least one convex vertex.

Proof. Let P be an arbitrary simple polygon. Choose a right-most vertex of P. Suppose this vertex is reflex. This immediately contradicts our choice of a right-most vertex: were the vertex reflex, it could not be right-most. Thus the vertex is convex. □

The preceding proof is a simple way to demonstrate an apparently obvious truth. As we shall see, not all such "obvious truths" are as simple to demonstrate.

Definition 5.6 (Triangulation): For a given simple polygon P, a *triangulation* of P is a scissors dissection of P where every polygonal subregion is a triangle. If such a dissection exists for P, it is said that P can be triangulated.

Lemma 5.2: Any simple polygon can be triangulated.

Proof. Let n be the number of vertices (and sides) of a given simple polygon P. For $n = 3$, the given polygon is a triangle and the statement holds trivially. Proceeding inductively, for a given polygon P with $n > 3$ vertices, suppose that every polygon with $k < n$ vertices can be triangulated. Label the vertices of P $v_1, v_2, ..., v_n$ such that v_j is adjacent to v_{j+1} and v_n is adjacent to v_1.

Choose a convex vertex v_i and consider its neighbors, v_{i-1} and v_{i+1}. Draw the segment $\overline{v_{i-1}v_{i+1}}$. One of two cases will be true.

1. The open segment $\overline{v_{i-1}v_{i+1}}$ will pass entirely through the interior of P.
2. The open segment $\overline{v_{i-1}v_{i+1}}$ will intersect the boundary of P in at least one point.

For case 1, the open segment $\overline{v_{i-1}v_{i+1}}$ divides P into two regions: $\triangle v_i v_{i-1} v_{i+1}$ and $P' = P$ - $\triangle v_i v_{i-1} v_{i+1}$. By construction, P' has $n-1$ vertices and, by the induction hypothesis, can also be triangulated. Hence then entire polygon P can be triangulated by adding $\triangle v_i v_{i-1} v_{i+1}$ to the triangulation of P'.

For case 2, $\triangle v_i v_{i-1} v_{i+1}$ contains at least one vertex of P in its interior. Let L be the collection of lines parallel to $\overline{v_{i-1}v_{i+1}}$ passing through $\triangle v_i v_{i-1} v_{i+1}$. Starting from v and moving towards $\overline{v_{i-1}v_{i+1}}$, let l be the first line in L that intersects a vertex of P. Call this vertex v_j. Draw the segment $\overline{v_i v_j}$. Since $\overline{v_i v_j}$ passes entirely through the interior or P, it separates P into two polygonal regions, both with fewer than n vertices and the result follows by induction. □

Knowing that any polygon can be triangulated greatly simplifies the proof as it reduces the scope of polygons to one of the easiest to work with. While any such triangulations would immediately be a rearrangement for a number of other polygons, as demonstrated by both the tangram and the stomachion, it is not sufficient for the statement of the WBG theorem. For that, more must be known about the individual triangles.

LEMMA 5.3: Any triangle is scissors congruent with a rectangle of the same area

Proof. See Figure 12 for an image of the following dissection. Let $\triangle ABC$ be an arbitrary triangle. Without loss of generality, assume angles $\angle ABC$ and $\angle ACB$ are acute. Let D and E be the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Segment $\overline{DE}$ is parallel to $\overline{BC}$. To prove this, consider the following:

1. $\ell(\overline{AD}) = \frac{1}{2}\ell(\overline{AB})$

2. $\ell(\overline{AE}) = \frac{1}{2}\ell(\overline{AC})$

3. $\triangle ABC \sim \triangle ADE$, by SAS similarity

4. $\measuredangle ABC = \measuredangle ADE$, since the triangles are similar

5. $\overline{AB}$ is a transversal cutting $\overline{BC}$ and $\overline{DE}$ with $\angle ABC$ and $\angle ADE$ corresponding angles

6. Since the corresponding angles are of equal measure, $\overline{BC} \parallel \overline{DE}$

Let point F on $\overline{DE}$ be such that $\overline{AF} \perp \overline{DE}$. Extend $\overline{DE}$ from D to point G, where G is such that $\overline{BG} \perp \overline{DE}$ and $\overline{BG} \perp \overline{BC}$. Similarly, extend $\overline{DE}$ from E to point H such that $\overline{CH} \perp \overline{DE}$ and $\overline{CH} \perp \overline{BC}$. Now we claim that $\triangle BGD$ is congruent to $\triangle DAF$ and $\triangle CHE$ is congruent to $\triangle AFE$. First, we show that $\triangle BCD \cong \triangle AFD$.

1. $\angle DBG \cong \angle DAF$ by alternate interior angles

2. $\angle BDG \cong \angle ADF$ by alternate interior angles

3. $\overline{BD} \cong \overline{DA}$ since D is the midpoint of $\overline{AB}$

4. $\triangle BCD \cong \triangle AFD$ by ASA

Next, we show that $\triangle CHE \cong \triangle AFE$.

1. $\angle CEH \cong \angle AEF$ by alternate interior angles

2. $\angle ECH \cong \angle EAF$ by alternate interior angles

3. $\overline{CE} \cong \overline{EA}$ since E is the midpoint of $\overline{AC}$

4. $\triangle CHE \cong \triangle AFE$ by ASA

Since $BGHC$ forms a rectangle with regions BCD, CHE, and $BDEC$ congruent to the regions AFD, AFE, and $BDEC$ of $\triangle ABC$, the rectangle $BGHC$ is congruent by dissection with $\triangle ABC$. Since the triangle was arbitrarily chosen, any triangle can be dissected and rearranged into a rectangle of the same area. □

It is at this point where the two previously mentioned approaches diverge. The steps involved for the dissection into a square are similar ending with an appeal to the Pythagorean theorem. We proceed with dissecting into a $1 \times A$ rectangle.

LEMMA 5.4: A rectangle of side lengths a and b can be dissected into a rectangle of side lengths c and d if $ab = cd$.

Proof. Let $ABCD$ and $DEFG$ be rectangles arranged as in Figure 13 or Figure 14. In either case, let $\ell(\overline{AB}) = b$, $\ell(\overline{EF}) = d$, $\ell(\overline{AD}) = a$, and $\ell(\overline{ED}) = c$. Draw the line segment $\overline{AG}$. This segment with either pass entirely through the interior of both rectangles or part of the segment will be outside both rectangles. We first consider the case where the segment passes entirely through the interior.

Case 1: Line segment stays in the interior of at least one rectangle from endpoint to endpoint.

As in Figure 13, $\overline{AG}$ passes through the edge of rectangle $ABCD$ at point H and through the edge of rectangle $DEFG$ at point I. We seek to describe the coordinates of H and I. By considering the points as being on a Cartesian grid with point D being the origin, the equation describing the line segment $\overline{AG}$ is

$$\frac{x}{d} + \frac{y}{a} = 1$$

For point H, $y = c$ since point H is on the edge of $DEFG$. Hence,

$$\begin{aligned} \frac{x}{d} + \frac{c}{a} &= 1 \\ x &= d - \frac{cd}{a} \\ x &= d - \frac{ab}{a} \\ x &= d - b \end{aligned}$$

Thus, point H is at $(d - b, c)$. Similarly, point I is on the edge of $ABCD$ and thus $x = b$

for point I. So,

$$\begin{aligned}\frac{b}{d}+\frac{y}{a}&=1\\ y&=a-\frac{ab}{d}\\ y&=a-\frac{cd}{d}\\ y&=a-c\end{aligned}$$

Thus, point I is at $(b, a-c)$.

Having established where $\overline{AG}$ intersects $ABCD$ and $DEFG$, we need to show that the resulting regions are congruent. We first show that $\triangle AEH \cong \triangle ICG$. Considering the diagram in Figure 13, note that the following relations hold.

$$\ell(\overline{AE}) = a-c = \ell(\overline{IC})$$
$$\angle AEH = 90^\circ = \angle ICG$$
$$\ell(\overline{EH}) = d-b = \ell(\overline{CG})$$

Thus, by SAS, $\triangle AEH \cong \triangle ICG$. The other pair of triangles, $\triangle ABI$ and $\triangle HFG$, are similarly shown to be congruent.

$$\ell(\overline{AB}) = d-(d-b) = b = \ell(\overline{HF})$$
$$\angle ABI = 90^\circ = \angle HFG$$
$$\ell(\overline{BI}) = a-(a-c) = c = \ell(\overline{FG})$$

Thus by SAS, $\triangle ABI \cong \triangle HFG$. The remaining region, $DEHIC$, is clearly congruent to itself. Thus, for the first dissection case, the rectangles $ABCD$ and $DEFG$ are scissors congruent.

Case 2: Part of the line segment is external to both rectangles.

While this case can be handled directly, it is simpler to reduce it to Case 1 through intermediate steps. To best describe this process of intermediate steps, it is useful to know when this situation arises. The result is the following statement:

For two rectangles with side lengths a, b, *and* c, d, *if* $\max\{a, b\} < \frac{1}{2}\max\{c, d\}$, *then the dissecting segment used in Figures 13 and 14 will be partially external to both rectangles.*

Proof: Without loss of generality, let $b = \max\{a, b\}$ and $d = \max\{c, d\}$. Align the rectangles as in Figure 14. The dissecting segment $\overline{AG}$ will exit rectangle $ABCD$ at point H, which, as before, exists as point $(b, a - c)$. For $\overline{AG}$ to be external to both $ABCD$ and $DEFG$, the inequality $a - c > c$ must be true. And so,

$$\begin{aligned} a - c = a - \frac{ab}{d} > a - \frac{a\frac{1}{2}d}{d} = \frac{1}{2}a \\ \implies \frac{1}{2}a > c \end{aligned}$$

And the desired inequality follows. Thus, if $\max\{a, b\} < \frac{1}{2}\max\{c, d\}$, $\overline{AG}$ is partially external to both $ABCD$ and $DEFG$.

Knowing when case 2 occurs, we reduce it to case 1 by dissecting $ABCD$ into a rectangle with side lengths $a/2$ and $2b$. While it may be necessary to repeat this process more than once, each step will halve the height and double the width. Since Euclidean geometry is Archimedean, there exists some number $k \in \mathbb{N}$ such that $2^k b > d/2$. And so, for case 2, rectangle $ABCD$ will need to be halved k times before the result rectangle can be dissected into $DEFG$ via case 1. □

Theorem 5.5 (Wallace-Bolyai-Gerwein): Two polygons are scissors congruent if and only if they have the same area.

Proof. For any two polygons P, Q of equal area A, both can be dissected first into separate collections of triangles by Lemma 5.2, then the collections of triangles into collections of rectangles by Lemma 5.3. Next, by Lemma 5.4 the collections of rectangles can be transformed into collections of $1 \times A_k$ rectangles where A_k is the area of the k-th rectangle in the given collection. Finally, these collections can each be transformed into singular $1 \times A$ rectangles by stitching rectangles of each collection together on their edges. The resulting $1 \times A$ rectangles are clearly scissors congruent with each other, and P and Q are each scissors congruent with the $1 \times A$ rectangle. And thus, by the transitivity of congruence, $P \cong Q$ and both directions of the proof are satisfied. □

The resulting dissection is useful only in demonstrating the possibility of such a dissection and relation between any two polygons of equal area. Attempting to actually perform the dissection would result in pieces too small to really use and far more pieces than necessary. Indeed, simpler dissections for a pair of specific polygons is usually the point of a dissection puzzle. Rather, the beauty of the WBG dissection lies in its arbitrary nature: all that needs to be known is that the polygons are simple and have the same area.

It follows readily that scissors congruence is an equivalence relation for the polygons of Euclidean two-space with the area of the polygon being its equivalence class. As proven above in the WBG theorem, all one needs to know to determine whether or not two polygons are scissors congruent is the area of the two polygons being compared. The natural question about whether or not volume is likewise the sole determinant of equivalence class for Euclidean three-space is the topic of the next section.

6. DEHN INVARIANT

After the proof of the Wallace-Bolyai-Gerwein theorem, the question of whether an equivalent result holds in three dimensions followed naturally. This question about scissors congruence in three Euclidean dimensions became the third of Hilbert's unanswered questions for the International Congress of Mathematics. Prior to the convening of the Congress, Max Dehn, a protégé of Hilbert's, successfully demonstrated the conclusion was negative. That is, equal volume alone is not sufficient for two polyhedra to be scissors congruent.

We will see how Dehn reached his conclusion by tracing similar argumentation and proofs as found in [2]. First, we expand a few of our earlier definitions into their three-dimensional equivalents. We will also define a critical term for understanding Dehn's work: the dihedral angle.

Definition 6.1 (Scissors Dissection (Polyhedra)): For an arbitrary polyhedron P, a finite union of smaller polyhedra ($P = P_1 \cup ... \cup P_n$) such that the smaller polyhedra have pairwise disjoint interiors is called a *scissors dissection of a polyhedron.*

Definition 6.2 (Scissors Congruence (Polyhedra)): Two polyhedra P, Q are *scissors congruent* if there are dissections ($P = P_1 \cup ... \cup P_n$, $Q = Q_1 \cup ... \cup Q_n$) such that $P_k \cong Q_k$.

Definition 6.3 (Dihedral Angle): The angle between two faces of a polyhedron from a shared edge is called the *dihedral angle.*

Suspecting that equal volume was insufficient for two polyhedra to be scissors congruent, Dehn set out to find another invariant property of scissors congruent polyhedra. What he found was far from intuitive: a tensor group joining edge lengths with their respective dihedral angles. As shall be shown, a given polyhedron has a value from this group that does not change under scissors dissection. As is custom, this invariant bears the name of its discoverer: the Dehn invariant. We first define the underlying group, which we call the Dehn tensor group.

Definition 6.4 (Dehn tensor group): Let G be $\mathbb{R} \bigotimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$. This group contains the set of all expressions of the form

$$(a_1 \otimes \phi_1) + (a_2 \otimes \phi_2) + ... + (a_n \otimes \phi_n)$$

where $a_i \in \mathbb{R}$ and $\phi_i \in \mathbb{R}/2\pi\mathbb{Z}$ with the elements of G subject to the following relations:

1. $a_1 \otimes b + a_2 \otimes b = (a_1 + a_2) \otimes b$
2. $a \otimes b_1 + a \otimes b_2 = a \otimes (b_1 + b_2)$
3. $a * z \otimes b = a \otimes b * z$, where $z \in \mathbb{Z}$
4. $a \otimes b = a \otimes (b + z\pi)$, where $z \in \mathbb{Z}$

Properties 1 and 2 of the group $\mathbb{R} \bigotimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z}$ each parallels a property of three-dimensional scissors dissections. Property 1 indicates that if a cut splits an edge into two pieces, the individual pieces together yield the same value as the entire edge. Similarly, Property 2 indicates that if a cut splits a dihedral angle into two sub-angles, the two sub-angles put together are equivalent to the original dihedral angle. Property 3 is a consequence of the definition of the tensor group, but it does have a geometric application: for any integer multiple of identical polyhedral pieces, the overall value does not change whether the edge lengths or dihedral angles are joined together. Property 4 is a consequence of both Property 2 and the nature of modulo group $\mathbb{R}/2\pi\mathbb{Z}$ that means, geometrically speaking, the addition of any integer multiple of π does not change the value measured.

By definition, the operation of addition in G is both associative and commutative. By establishing the existence of an identity of G and additive inverses, we will see that G is an abelian group.

Theorem 6.1 (Dehn tensor group is abelian): *Proof.* Note that for any $a \in \mathbb{R}$,

$$\begin{aligned} a \otimes 0 &= a \otimes 0 + 0 \otimes 0 = a \otimes 0 + (a - a) \otimes 0 \\ &= a \otimes 0 + a \otimes 0 + -a \otimes 0 = a \otimes 0 + -a \otimes 0 \\ &= 0 \otimes 0 \end{aligned}$$

by Properties 1 and 2. By similar argumentation, it can be shown that $0 \otimes \phi = 0 \otimes 0$. Thus, for any $a \in \mathbb{R}$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$,

$$0 \otimes 0 + a \otimes \phi = a \otimes 0 + a \otimes \phi = a \otimes (0 + \phi) = a \otimes \phi$$

establishing that $0 \otimes 0$ is the additive identity of G. The existence of additive inverses also follows readily: for any $a \otimes \phi$, both $-a \otimes \phi$ and $a \otimes -\phi$ yield $0 \otimes 0$. Therefore, since an identity element and additive inverses exist, G is an abelian group. □

Knowing the identity of the Dehn tensor group, there is a useful fact about what values are not equivalent to $0 \otimes 0$.

FACT 6.1: If $x \notin \mathbb{Q}$, $y \otimes x\pi \neq 0$.

With the underlying space defined, we may now define the Dehn invariant. There are also a few useful theorems concerning the Dehn invariant in application. We prove these as well.

Definition 6.5 (Dehn invariant): We define the *Dehn invariant of* P (denoted as $\delta(P)$) by taking each edge length of P, l_i, and the corresponding dihedral angle, ϕ_i, and consider their sum: $\delta(P) = \sum_{i=1}^{m} l_i \otimes \phi_i$ where m is the number of edges of P.

Theorem 6.2: If P, P' are polyhedra such that $P \cong P'$, then $\delta(P) = \delta(P')$.

Proof. This immediately follows from the fact that any congruent polyhedra necessarily have congruent edges and dihedral angles and thus equal Dehn invariants. □

Theorem 6.3: If P_1, P_2 are polyhedra with nonoverlapping interiors, then $\delta(P_1 \cup P_2) = \delta(P_1) + \delta(P_2)$.

Proof. For $P_1 \cup P_2$, the edges, dihedral angles, and faces that are unchanged by the union of P_1 and P_2 clearly do not change with respect to their Dehn measures. For those portions that are affected, there are three cases to consider for joining two polyhedra together: 1) an edge of P_1 and P_2 can be fused together to form $P_1 \cup P_2$; 2) two edges of P_1 and P_2, each having a common dihedral angle θ, can be fused at the end to produce a single edge of $P_1 \cup P_2$; or 3) two edges of P_1 and P_2 having the same edge lengths and supplemental dihedral angles can be joined to make a new face of $P_1 \cup P_2$.

Case 1: An edge from each P_1 and P_2 are fused into a single edge of $P_1 \cup P_2$. This new joined edge in $P_1 \cup P_2$ will have a dihedral angle $\theta = \theta_1 + \theta_2$ where θ_1 and θ_2 are the corresponding dihedral angles from P_1 and P_2 respectively. Thus the Dehn invariant of the edge in $P_1 \cup P_2$ is $(a \otimes \theta) = (a \otimes \theta_1 + \theta_2) = (a \otimes \theta_1) + (a \otimes \theta_2)$ and thus the overall contribution to $\delta(P_1 \cup P_2)$ is unchanged.

Case 2: Two edges of P_1 and P_2 have the same dihedral angle θ and are joined at the end to form one longer edge with dihedral angle θ. The edge in $P_1 \cup P_2$ will have an edge length of $a = a_1 + a_2$ where a_1, a_2 are the lengths of the corresponding edges from P_1 and P_2 respectively. For this edge in $P_1 \cup P_2$, the Dehn measure is thus $(a \otimes \theta) = (a_1 + a_2 \otimes \theta) = (a_1 \otimes \theta) + (a_2 \otimes \theta)$ and, as before, the contribution to $\delta(P_1 \cup P_2)$ does not change.

Case 3: Two edges of P_1 and P_2 with the same length and supplemental dihedral angles are joined into a new face of $P_1 \cup P_2$. In this case, there is neither edge nor dihedral angle to work with in $P_1 \cup P_2$. Even so, considering the sum of the separate Dehn measures of P_1 and P_2 yields $(a \otimes \theta_1) + (a \otimes \theta_2) = (a \otimes \pi) = 0$. Thus, even without a corresponding edge or dihedral angle to measure in $P_1 \cup P_2$, the $\delta(P_1 \cup P_2)$ does not change. □

With the Dehn invariant defined, we are equipped to see why volume was insufficient for scissors congruence in three dimensions. Dehn himself worked with the cube and the tetrahedron in his attempts to find an equivalent of the WBG theorem. When he could not find a suitable dissection, he found the tensor space and invariant above. We shall here demonstrate Dehn's result: the cube and tetrahedron have different Dehn invariants, and are thus not scissors congruent.

Theorem 6.4: The Dehn invariant of a cube is 0.

Proof. Without loss of generality, let P be the unit cube. It is obvious that every dihedral angle of P is $\frac{\pi}{2}$. Thus,

$$\begin{aligned}\delta(P) = \sum_{i=1}^{n}(a_i \otimes \theta_i) &= \sum_{i=1}^{12}(1 \otimes \frac{\pi}{2}) = 12(1 \otimes \frac{\pi}{2}) \\ &= 1 \otimes 12\frac{\pi}{2} = 1 \otimes 6\pi = 1 \otimes 0 = 0\end{aligned}$$

as claimed. □

The various values of δ can be thought of as equivalence classes corresponding to entire families of polyhedra that, volume being equal, are scissors congruent. To see that the Dehn invariant of the tetrahedron is not 0, we first prove a useful theorem concerning rational values of the cosine function.

Theorem 6.5: If $\cos\theta = \frac{m}{n}$ where $m,\ n \in \mathbb{Z}$ and $\theta = \frac{p}{q}\pi$ where $\frac{p}{q} \in \mathbb{Q}$, then $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$.

Proof. By DeMoivre's theorem, we have that

$$\begin{aligned}(\cos\theta + i\sin\theta)^q &= \cos(q\theta) + i\sin(q\theta) \\ &= \cos(p\pi) + i\sin(p\pi) \\ &= \pm 1\end{aligned}$$

Similarly,

$$\begin{aligned}(\cos\theta - i\sin\theta)^q &= \cos(q\theta) - i\sin(q\theta) \\ &= \cos(p\pi) - i\sin(p\pi) \\ &= \pm 1\end{aligned}$$

The above indicates that both $(\cos\theta + i\sin\theta)^q$ and $(\cos\theta - i\sin\theta)^q$ are algebraic integers. Since the algebraic integers form a ring, $(\cos\theta + i\sin\theta)^q + (\cos\theta - i\sin\theta)^q = 2\cos\theta$ is an algebraic integer by closure. By hypothesis, $2\cos\theta = \frac{2m}{n}$ and so $\frac{2m}{n}$ is also an algebraic integer. But every rational algebraic integer is an integer, so $\frac{2m}{n} = k \in \mathbb{Z}$ and that $\frac{m}{n} = \frac{k}{2}$. Finally, since $\cos\theta = \frac{k}{2}$, $k = 0, \pm 1, \pm 2$ and the desired outcome follows. □

THEOREM 6.6: The Dehn invariant of a regular tetrahedron is non-zero.

Proof. Without loss of generality, let P be a regular tetrahedron of side lengths 1. To find the dihedral angles of P, we place the vertices of P at four convenient points:

$$(0,0,0), (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}), (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

Next, we take the midpoint of $(0,0,0)$ and $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$: $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, 0)$. Taking this point and the remaining two vertices of P forms an isosceles triangle with two legs of length $\frac{\sqrt{3}}{2}$ and one leg of length 1. The dihedral angle θ is opposite the leg of length 1, and thus by the law of cosines

$$\begin{aligned}1 &= (\frac{\sqrt{3}}{2})^2 + (\frac{\sqrt{3}}{2})^2 - 2(\frac{\sqrt{3}}{2})^2\cos\theta \\ &= \frac{3}{4} + \frac{3}{4} - 2\frac{3}{4}\cos\theta = \frac{3}{2}(1 - \cos\theta), \\ \cos\theta &= \frac{1}{3}\end{aligned}$$

By 6.5, $\theta = \arccos(\frac{1}{3}) \notin \mathbb{Q}$ and thus $\frac{1}{2} \otimes \theta \neq 0$ and $\delta(P) \neq 0$ by Fact 6.1. □

Since the Dehn invariants of the cube and tetrahedron are different, it is thus impossible for them to be scissors congruent. With this result, we have Max Dehn's counterexample and answer to Hilbert's Third Question. Max Dehn showed that it was necessary for two polyhedra to have the same volume and Dehn invariant for them to be scissors congruent. With the narrow focus of Dehn's work being limited to a comparison of the cube and tetrahedron, it remained an open question whether equal volume and Dehn invariant were sufficient in all cases to establish scissors congruence. Jean-Pierre Sydler showed in 1965 that the combination of volume and Dehn invariant was both necessary and sufficient for scissors congruence in three dimensions. We omit the proof of the Dehn-Sydler theorem here as its proof is sufficiently complicated to warrant its own work.

THEOREM 6.7 (Dehn-Sydler): Two polyhedra are scissors congruent if and only if they have the same volume and Dehn invariant.

While Dehn was able to answer one of Hilbert's infamous questions, there was more to come in the early twentieth century for our understanding of geometry. The Third Question and Dehn's work were concerned with straight planar cuts. It left open the question of what would happen if the pieces were not so cleanly defined. As we shall see, that question led to one of the most astonishing results in modern mathematics.

7. BANACH-TARSKI PARADOX

With the discovery of the Dehn invariant, the next progression in dissections was to consider arbitrary point sets of a given region instead of straight-edge cuts. The intuitive expectation is that by employing a more malleable class of sets in three dimensions, volume alone is sufficient to determine dissection equivalence. The surprising truth is that using arbitrary point sets out of a bounded region in $\mathbb{R}^3$ results in a true paradox. In this section, we trace the structure of proofs and arguments laid out in [4]. The first step is to give a precise definition of paradoxical.

Definition 7.1 (G-Paradoxical): Let X be a set and G a group acting on X. Suppose E is a non-empty subset of X. E is said to be *G-paradoxical* if there exist $A, B \subset E$ each having finite pairwise disjoint partitions $A_1, ..., A_n$ and $B_1, ..., B_m$ and $g_1, ..., g_n$ and $h_1, ..., h_m \in G$ such that $E = \bigcup_{i=1}^{n} g_i(A_i)$ and $E = \bigcup_{j=1}^{m} h_j(B_j)$.

Thus far, we have considered the equivalence (or non-equivalence) of particular regions through scissors congruence. The consideration of the boundaries of these regions, as previously mentioned, is ignored in this approach. If we wish to consider regions as sets and not as geometric regions, a different approach is required.

Definition 7.2 (Equidecomposability): Let X be a set and G a group acting on X and $A, B \subseteq X$. A and B are said to be *equidecomposable with respect to G* (or *G-equidecomposable*) if A and B can be finitely partitioned such that

$$A = \bigcup_{i=1}^{n} A_i, B = \bigcup_{i=1}^{n} B_i$$

$$A_i \cap A_j = \emptyset = B_i \cap B_j, i < j \leq n$$

and there are $g_1, ..., g_n \in G$ such that $g_i(A_i) = B_i$. We denote this relation by $A \sim_G B$. For simplicity, the subscript will be dropped if G is the complete isometry group for the underlying space or is otherwise obvious.

THEOREM 7.1: Let X be a set and G a group acting on X. Further, let E, $E' \subseteq X$ be equidecomposable. If E is paradoxical with respect to G, so is E'.

Proof. Let $\{E_i\}_{i=1}^n$, $\{g_i\}_{i=1}^n$ respectively be the partition of E and the collection of elements of G such that $X = \bigcup_{i=1}^n g_i(E_i)$. Let $\{E'_j\}_{j=1}^m$, $\{h_j\}_{j=1}^m$ respectively be the partition of E' and the collection of elements of G such that $E = \bigcup_{j=1}^m h_j(E'_j)$. To show that E' is G-paradoxical, first, for each E'_j and E_i, let $\{E'_{j,i}\}_{j=1}^m$ be the partition of E'_j such that $h_j(E'_{j,i}) \subset E_i$. Then,

$$\begin{aligned} X &= \bigcup_{i=1}^n g_i(E_i) \\ &= \bigcup_{i=1}^n g_i(\bigcup_{j=1}^m h_j(E'_{j,i})) \\ &= \bigcup_{i=1}^n \bigcup_{j=1}^m g_i h_j(E'_{j,i}) \end{aligned}$$

Since G is a group, $g_i h_J \in G$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Further, by construction, $E_{j,i} \cap E_{k,l} = \emptyset$ for any $j \neq k$, $i \neq l$. Thus, E' is G-paradoxical. $\square$

This approach to equivalence drops any reference to polygonal regions and is purely focused on sets. The set-theoretic approach allows for a far broader set of regions to be considered. We will establish more about the equidecomposable relation as it will be used extensively throughout this section. First, we notate a relation between two sets, $\preceq$, by $A \preceq B$ if and only if there exists $B' \subset B$ such that $A \sim B'$.

THEOREM 7.2 (Banach-Schröder-Bernstein Theorem): Let X be a set, G a group acting on X, and A, $B \subseteq X$. If $A \preceq B$ and $B \preceq A$, then $A \sim_G B$. This establishes $\preceq$ as a partial ordering on $\mathcal{P}(X)$ over the $\sim_G$-classes.

Proof. The relation $\sim$ satisfies the following two conditions:

(a) If $A \sim B$, then there exists a bijection $g : A \to B$ such that, for $C \subseteq A$, $C \sim g(C)$

(b) If $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, $A_1 \sim B_1$, and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.

The only other assumption for this proof is that $\sim$ satisfies (a) and (b) as an equivalence relation on $\mathcal{P}$. Let f and g be bijections as guaranteed by (a) defined as $f : A \to B_1$, $g : A_1 \to B$. Let $\{C\}_n$ be a sequence of sets defined by $C_0 = A \setminus A_1$ and $C_{n+1} = g^{-1}f(C_n)$. Further define C such that $C = \bigcup_{n=0}^{\infty} C_n$. We verify that $g(A \setminus C) = B \setminus f(C)$.

$$\begin{aligned} g(A \setminus C) &= g(A) \setminus g(C) = B \setminus g(C) \\ &= B \setminus [g(C_0) \cup gg^{-1}f(C_0) \cup ...] \\ &= B \setminus [\emptyset \cup f(C_0) \cup f(C_1) \cup ...] \\ &= B \setminus f(\bigcup_{n=0}^{\infty} C_n) \\ &= B \setminus f(C) \end{aligned}$$

Thus, the choice of g implies that $A \setminus C \sim B \setminus f(C)$. Our choice of f implies that $C \sim f(C)$ and, by (b), $(A \setminus C) \cup C \sim (B \setminus f(C)) \cup f(C)$ and $A \sim B$ follows. □

Though the applications of scissors congruence and equidecomposability are quite different, the underlying operations are similar. Both argue that if the subregions described in each are equivalent, then the overall regions are equivalent. A natural question arises: is there any relation between scissors congruence and equidcomposability? Recall that scissors congruence ignores particular subsets of the compared regions, namely the boundaries. For there to be any relation between scissors congruence and equidcomposability, these subsets must be accounted for. It turns out that there is such a way to relate the two.

Theorem 7.3: If A is a bounded set in the plane with non-empty interior and T is a set of finitely many bounded line segments disjoint from A, then $A \sim A \cup T$.

Proof. Let $D \subseteq A$ be a disc with radius r. For every $t \in T$, we may assume that t may be subdivided so that each piece is of length less than r since each $t \in T$ is bounded. Choose any rotation of D about its center with infinite order, call it θ. Choose R to be any radius of D excluding its center and let $\overline{R} = R \cup \theta(R) \cup \theta^2(R) \cup ....$ Then, for $t \in T$, $D \cup t \preceq D$ because $\theta(\overline{R})$ is disjoint from R and $D \setminus \overline{R}$. Thus, for any isometry τ mapping t into R, $D \cup t = (D \setminus \overline{R}) \cup \overline{R} \cup t \sim (D \setminus \overline{R}) \cup \theta(\overline{R}) \cup \tau(t) \subseteq D$. Clearly $D \preceq D \cup t$, and so by the Banach-Schröder-Bernstein theorem, $D \sim D \cup t$. Repeating the above process for each element of T, it follows that $D \sim D \cup T$. But then, $(A \setminus D) \cup D \sim (A \setminus D) \cup D \cup T$, and $A \sim A \cup T$ as required. □

The method used in Theorem 7.3 illustrates a particular feature of working with sets and group actions: absorption. Troublesome sets can be handled by simply absorbing them into another set. The notion of absorption to bring otherwise troublesome sets into an equidecomposable relationship will be used extensively throughout this section. First, we apply this approach to establish a relationship between scissors congruence and equidecomposability.

THEOREM 7.4: If two polygons P_1 and P_2 are scissors congruent, they are equidecomposable.

Proof. For P_1 and P_2, let Q_1 and Q_2 be the union of the open sets in the respective polygon given by the hypothesized dissection. It follows readily that $Q_1 \sim Q_2$. What needs to be shown is that $P_1 \sim Q_1$ and $P_2 \sim Q_2$; that is, that the boundaries of the polygonal pieces can be accounted for. This follows readily from Theorem 7.3 by setting $A = Q_i$ and $T = P_i \setminus Q_i$. Thus, $P_1 \sim Q_1 \sim Q_2 \sim P_2$ and $P_1 \sim P_2$ follows. □

We shall see that the converse of Theorem 7.4 does not hold as a consequence of a later theorem. That later result is a consequence of what sets are equidecomposable. Theorem 7.3 gave a glimpse of what counter-intuitive sets are equidecomposable. As curious as it is to be able to absorb a set into another, we shall see that the reverse is also possible: it is possible to remove subsets without affecting the overall set. Our focus turns to exploring a famous result of this sort of set extraction: the Banach-Tarski paradox.

Informally, the set E is said to be paradoxical with respect to G if there are two disjoint subsets of E – A and B – such that A and B can each be made to cover E via rearrangements through G. Further, A and B can be chosen such that $A \cup B = E$ and thus $A \cup B$, $g(A)$, and $h(B)$ are partitions of E. That A and B can be chosen in such a way is a provable consequence of the Banach-Schröder-Bernstein theorem.

One such group which will be instrumental throughout the rest of this work is the free group of rank 2. Recall that a group F is said to be *free* if there exists a set M such that every element of F is a word consisting of letters of M. M is said to be the generating set of F. Letting M be of the form $\{\sigma, \sigma^{-1} : \sigma \in M\}$, the elements of F are said to be equivalent if they differ only by finitely many pairs of letters of the form $\sigma\sigma^{-1}$. A word is said to be reduced if it contains no such adjacent inverse letter pairs. For simplicity, all subsequent references to free groups will be in terms of reduced words only. The identity of F, e, is the empty word. F has concatenation as its group action. For any two words a, b in F, ab will be the unique reduced word after any and all resulting inverse pairs are resolved. The rank of the free group is the number of inverse pairs in the generating set.

Theorem 7.5: A free group F of rank 2 is F-paradoxical where F acts on itself by left multiplication.

Proof. Let σ and τ be the free generators of F. If ρ is any reduced combination of σ, σ^{-1}, τ, and τ^{-1}, then let $W(\rho)$ be the set of all words, after reduction, begin on the left with ρ. Consider then that,

$$F = \{e\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup \bigcup_{n=1}^{\infty} \tau^{-n} \cup \bigcup_{n=1}^{\infty} \tau^{-n}(W(\sigma) \cup W(\sigma^{-1}))$$

Separate F into A and B with

$$A = W(\sigma) \cup W(\sigma^{-1}),$$
$$B = \{e\} \cup W(\tau) \cup \bigcup_{n=1}^{\infty} \tau^{-n} \cup \bigcup_{n=1}^{\infty} \tau^{-n}(W(\sigma) \cup W(\sigma^{-1})).$$

Then it follows readily that

$$F = W(\sigma) \cup \sigma W(\sigma^{-1}),$$
$$F = \{e\} \cup W(\tau) \cup \bigcup_{n=1}^{\infty} \tau^{-n} \cup \tau(\bigcup_{n=1}^{\infty} \tau^{-n}(W(\sigma) \cup W(\sigma^{-1}))$$

and thus F is paradoxical with respect to left-multiplication. □

In addition to groups, semigroups are also of interest. Recall that a semigroup is a set with an associative binary operation and an identity. Definition 7.1 cannot be readily applied to semigroups as they lack inverses. Even so, there are useful results involving semigroups. A free semigroup is similar to a free group: both are the set of all words generated from a generating set through concatenation. The primary difference is that the generating set of the free semigroup consists only of unique letters and not inverse pairs. The measure of rank is also the number of these letters instead of inverse pairs. Should a free semigroup be embedded within a group, it is referred to as a semisubgroup.

THEOREM 7.6: Any group having a free semigroup of rank 2 contains a paradoxical set.

Proof. Let G be a group with a free semisubgroup, call it S. Further, let σ and τ be the free generators of S. Let $A \subset S$ be such that every element of A has σ as its left-most element; similarly for $B \subset S$ with τ. It follows readily that $A = \sigma(S)$ and $B = \tau(S)$. Then it immediately follows that S is G-paradoxical as σ^{-1}, $\tau^{-1} \in G$ and $\sigma^{-1}(A) = S = \tau^{-1}(B)$. □

This result is useful in that it allows for an easier examination of the paradoxicality of a given set. Specifically, it only needs to be shown for a particular set that it contains two independent elements generating a free semigroup for the underlying set to be paradoxical. Another useful result and its corollary will serve as the basis of our understanding of the Banach-Tarski paradox.

Theorem 7.7: If G is paradoxical and acts on X without nontrivial fixed points, then X is G-paradoxical.

Proof. Supposing that G is paradoxical, let $\{A_i\}$, $\{B_j\}$, $\{g_i\}$, and $\{h_j\}$ be such that $\bigcup g_i A_i = G = \bigcup b_j H_j$. By the Axiom of Choice, we may select exactly one element from each G-orbit of X – call this set M. It follows that $\{g(M) : g \in G\}$ is a partition of X. Each family of values in the partition is pairwise-disjoint as G has no nontrivial fixed points in X. The partition thus serves as a way to transform any given subset of G into a subset of X: for $S \in G$, $S^* = \{g(M) : g \in S\} \in X$. When the paradoxical sets in G are transformed in this manner, they result in paradoxical sets in X. Particularly, $\bigcup g_i A_i = G = \bigcup h_j B_j$ implies $\bigcup g_i(A_i^*) = X = \bigcup h_j(B_j^*)$. Disjointedness between the sets of $\{A_i^*\}$ and $\{B_j^*\}$ is preserved through the transformation. Thus $\{A_i^*\}$ and $\{B_j^*\}$ form a paradoxical decomposition of X. □

Corollary 7.7.1: If F is a free group of rank 2 acting on X with no nontrivial fixed points, X is F-paradoxical.

Proof. Follows immediately from Theorems 7.5 and 7.7. □

Knowing that a set is paradoxical if it contains a free group of rank 2, we seek to demonstrate the existence of such a group among the isometry group of Euclidean 3-space. This is the first of the Euclidean isometry groups where such a group is possible; both the one and two dimensional isometry groups are solvable and thus do not admit a free group of rank 2. While there are many ways to find a generating set of independent rotations of $\mathbb{S}^2$ – the unit sphere – we focus on one particular pair (and their inverses) found by K. Satô.

Definition 7.3 (Satô Rotation):

$$\sigma = \frac{1}{7}\begin{bmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{bmatrix} \qquad \tau = \frac{1}{7}\begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \\ -3 & 2 & 6 \end{bmatrix}$$

$$\sigma^{-1} = \frac{1}{7}\begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{bmatrix} \qquad \tau^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ -6 & 3 & 2 \\ 3 & 2 & 6 \end{bmatrix}$$

These rotations are not unique: there are many other independent rotations of $\mathbb{S}^2$ that could be used. It must be shown that these rotations are, in fact, independent. As we work with rotation matrices, recall that $SO_n(\mathbb{R})$ is the special orthogonal group. We add an additional piece of notation for our purposes: $SO_n(\mathbb{Q})$ is the subset of $SO_n(\mathbb{R})$ with rational entries.

Theorem 7.8: The two Satô rotations are independent. Thus, if $n \geq 3$, $SO_n(\mathbb{Q})$ has a free group of rank 2

Proof. It must be demonstrated that there is no reduced, nontrivial word W comprised of $\sigma^{\pm 1}$, $\tau^{\pm 1}$ such that $W = I$. Suppose that W is such a word. By conjugating W by a sufficiently high enough power of σ, we can assume that W has σ as its right-most term. For simplicity, consider the Satô rotations without the scalar $\frac{1}{7}$ and instead consider the values of W modulo 7. This suffices as the norm of each row of the Satô matrices is 7. Define four vector sets:

$$\begin{aligned} V_\sigma &= \{(3,1,2),(5,4,1),(6,2,4)\}, \\ V_{\sigma^{-1}} &= \{(1,5,4),(2,3,1),(4,6,2)\}, \\ V_\tau &= \{(3,2,6),(5,1,3),(6,4,5)\}, \\ V_{\tau^{-1}} &= \{(3,5,1),(5,6,4),(6,3,2)\} \end{aligned}$$

Observe that $W = \sigma^{k_n}\tau^{k_{n-1}}...\tau^{k_2}\sigma^{k_1}\sigma$ with $k_i \in \mathbb{Z}$ and $k_1 \geq 0$ by construction. Starting with the right-most σ term, note that its first column c is $(6, 2, -3) \equiv (6, 2, 4) \bmod 7 \in V_\sigma$. The elements of the V sets behave according to the following four properties:

1. For any $v \in V_\sigma \cup V_\tau \cup V_{\tau^{-1}}$, $\sigma v \in V_\sigma$.
2. For any $v \in V_{\sigma^{-1}} \cup V_\tau \cup V_{\tau^{-1}}$, $\sigma^{-1} v \in V_{\sigma^{-1}}$.
3. For any $v \in V_\tau \cup V_\sigma \cup V_{\sigma^{-1}}$, $\tau v \in V_\tau$.
4. For any $v \in V_{\tau^{-1}} \cup V_\sigma \cup V_{\sigma^{-1}}$, $\tau^{-1} v \in V_{\tau^{-1}}$.

Given that our first column c is in V_σ, we note that, by (1), c remains in V_σ through the remaining k_1 powers of σ that are right-most in W. The next right-most element of W is either a power of τ or τ^{-1}; either way, c lands in $V_\tau \cup V_{\tau^{-1}}$ by (3) and (4). The antepenultimate element of W is either a power σ or σ^{-1}; either way, c lands in $V_\sigma \cup V_{\sigma^{-1}}$ by (1) and (2). This alternation continues throughout the elements of W and so c is never (1, 0, 0), contradicting our choice of W. Since the choice of W is arbitrary, there is no W such that $W = I$. □

Knowing that the Satô rotations are independent, we must show that they act on some set with no nontrivial fixed points in order to invoke 7.7. The set in question is the rational sphere ($\mathbb{S}^2 \cap \mathbb{Q}^3$); the Satô rotations act on the rational sphere in a closed manner, taking rational points only to rational points. Since the Satô rotations do indeed act on the rational sphere without nontrivial fixed points, the rational sphere admits a paradoxical decomposition.

Theorem 7.9: The Satô rotations act on the rational sphere with no nontrivial fixed points.

Proof. Suppose that ω is a nontrivial word comprised of $\sigma^\pm$ and $\tau^\pm$ is a minimal fixed point. Computing the eigenvectors of the Satô matricies yields axes $(2, 1, 0)$ and $(0, 1, 2)$ for σ and τ respectively. These axes intersect the unit sphere at either $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0)$ or $(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

As such, ω cannot be a pure power of a single atom and ω must be of the form $\omega = \sigma^{\pm} \ldots \tau^{\pm}$ or $\omega = \tau^{\pm} \ldots \sigma^{\pm}$. Since a fixed point remains fixed under conjugation, we may assume that $\omega = \tau^{\pm} \ldots \sigma^{\pm}$. It is useful at this point to consider the quaternion representations of $\sigma^{\pm}$ and $\tau^{\pm}$. Note that any quaternion denotes a unique, up to a sign, quaternion of norm 1. As such, we may represent σ and τ uniquely with quaternions. A direct computation yields the quaternions $\frac{1}{\sqrt{14}}(3, \pm(2, 1, 0))$ and $\frac{1}{\sqrt{14}}(3, \pm(0, 1, 2))$ for σ and τ respectively. The quaternion representations of the Satô matricies are far easier to work with for the necessary algebra. To further simplify the computations, we ignore the coefficient of $\sqrt{14}^{-k}$ where k is the length of a given word. For a given word w, let $q(w) = (c_w, (X_w, Y_w, Z_w))$; that is, let q denote the integer portion of a given quaternion. Then for ω, the axis of rotation intersects the unit sphere at the points
$\pm(X_\omega, Y_\omega, Z_\omega)/\sqrt{(X_\omega^2 + Y_\omega^2 + Z_\omega^2)}$, and $X_\omega^2 + Y_\omega^2 + Z_\omega^2$ would have to be a perfect square for ω to be a fixed point as supposed. We will show that $X_\omega^2 + Y_\omega^2 + Z_\omega^2 \in \{3, 5, 6\}$ mod 7 and hence the sum cannot be a perfect square. For an integer-valued quaternion $q(w)$, let $\bar{q}(w)$ denote the same quaternion reduced mod 7. Note that if, for any such $\bar{q}(w)$, we multiply $\bar{q}$ by a mod 7 integer, the quadratic character of $X_w^2 + Y_w^2 + Z_w^2$ is unchanged. This is due to the fact that, for any mod 7 integer m, $X^2 + Y^2 + Z^2$ is a mod 7 square if and only if $m^2(X^2 + Y^2 + Z^2)$ is. Due to this property, we may assume that $c = 1$ when working with $\bar{q}(w)$. This allows us to demonstrate that, for any of the atoms, the quadratic character of $X^2 + Y^2 + Z^2$ mod 7 is the same for the atom and any power of that atom. We demonstrate the fixed nature of the quadratic character by showing that, for any atom ρ, $\bar{q}(\rho)$ is equivalent to a certain fixed quaternion. For the context of this proof, we use $\equiv$ to denote the modulo 7 equivalence relation. For the four atoms and k a positive integer, the following equivalence relations hold:

1. $\bar{q}(\sigma^k) \equiv (1, (3, 5, 0))$
2. $\bar{q}(\sigma^{-k}) \equiv (1, (4, 2, 0))$
3. $\bar{q}(\tau^k) \equiv (1, (0, 5, 3))$

4. $\bar{q}(\tau^{-k}) \equiv (1, (0, 2, 4))$

To demonstrate these equivalences, we first consider the base case where $k = 1$. Note that the equivalence relation in question is, for a quaternion $(c, (x, y, z))$, to multiply by c^{-1} mod 7.

1. $\bar{q}(\sigma) = (3, (2, 1, 0) \equiv (1, (3, 5, 0))$
2. $\bar{q}(\sigma^{-1}) = (3, (-2, -1, 0)) \equiv (1, (4, 2, 0))$
3. $\bar{q}(\tau) = (3, (0, 1, 2)) \equiv (1, (0, 5, 3))$
4. $\bar{q}(\tau^{-1}) = (3, (0, -1, -2)) \equiv (1, (0, 2, 4))$

Thus, the assertion holds for $k = 1$. The next step is to show equivalence holds for $k = 2$ as then we may proceed inductively through any integer power k.

1. $\bar{q}(\sigma^2) = (4, (5, 6, 0) \equiv (1, (3, 5, 0))$
2. $\bar{q}(\sigma^{-2}) = (4, (2, 1, 0)) \equiv (1, (4, 2, 0))$
3. $\bar{q}(\tau^2) = (4, (0, 6, 5)) \equiv (1, (0, 5, 3))$
4. $\bar{q}(\tau^{-2}) = (4, (0, 1, 2)) \equiv (1, (0, 2, 4))$

With these first two cases confirmed, it follows that, no matter the power, the equivalent quaternion does not change. Thus, in evaluating the modulo 7 quadratic character of ω, any power of a given atom may be replaced by the atom itself. This allows us to reduce ω to the form $\omega = \tau^{\pm}\sigma^{\pm} \ldots \tau^{\pm}\sigma^{\pm}$. To demonstrate that the overall quadratic character of ω does no change, let $V = \{(1, (1, 1, 5)), (1, (5, 1, 1)), (1, (4, 3, 4)), (1, (6, 5, 6))\}$. This set is an invariant set for words of the form $\tau^{\pm}\sigma^{\pm}$ and can be easily verified. Thus, moving leftward through ω, $\bar{q}(\omega) \in V$. For each $v \in V$, the sum of squares is equal to 6 mod 7; the actual sum of squares for ω will be 3 or 5 or 6 mod 7 since we are using equivalence classes. All that remains to be done is to show invariance among pairs of the component words. Any such pair of words can be verified to yield a quaternion that exists in V. The

16 calculations are omitted here, but are easily verifiable. Thus, the sum $X_\omega^2 + Y_\omega^2 + Z_\omega^2$ is never a perfect square, contradicting our choice of ω. Since our choice of ω was arbitrary, there are no nontrivial rational fixed points generated by $\sigma^\pm$ and $\tau^\pm$. □

Corollary 7.9.1: The rational sphere is paradoxical

Proof. By 7.7 and 7.8. □

The ultimate aim is to demonstrate that the entire unit sphere is paradoxical. Recall that 7.7 requires that the action act without nontrivial fixed points. Clearly the Satô rotations act on the unit sphere with nontrivial fixed points: the collection of points where the axes of rotation intersect the sphere. This collection can be excluded and the remainder of the sphere is immediately paradoxical.

Theorem 7.10 (Hausdorff Paradox): There is a countable subset D of $\mathbb{S}^2$ such that $\mathbb{S}^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical.

Proof. Let θ, ϕ be a pair of independent rotations of $\mathbb{S}^2$ that are the generators of a free group, F. For each non-identity rotation of θ or ϕ, there are exactly two fixed points: the points where the axis of rotation intersects $\mathbb{S}^2$. Call the collection of all of these points D. Since the free group F is countable, D is also countable. For any point $P \in \mathbb{S}^2 \setminus D$, and any $f \in F$, $f(P) \in \mathbb{S}^2 \setminus D$. This is due to the fact that, if $g \in F$ fixed $f(P)$, P would be a fixed point of $f^{-1}gf$. Thus F acts on $\mathbb{S}^2 \setminus D$ without nontrivial fixed points. Hence, by corollary 7.7.1, $\mathbb{S}^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical. □

There is still a countable set to contend with before we can declare the entire sphere is paradoxical. Fortunately, a method for handling such as set has already been demonstrated in 7.3: the process of absorption. We utilize a similar approach for the sphere.

Theorem 7.11: If D is a countable subset of $\mathbb{S}^2$, then $\mathbb{S}^2$ and $\mathbb{S}^2 \setminus D$ are $SO_3(\mathbb{R})$-equidecomposable.

Proof. We must demonstrate the existence of a rotation ρ such that the resulting rotational sets $D, \rho(D), \rho^2(D), ...$ are all pairwise disjoint. Finding such a rotation is sufficient as it would then follow that, for $D^* = \{\rho^n(D) : n = 0, 1, 2, ...\}$, $\mathbb{S}^2 = D^* \cup (\mathbb{S}^2 \setminus D^*) \sim \rho(D^*) \cup (\mathbb{S}^2 \setminus D^*) = \mathbb{S}^2 \setminus D$. To find such a rotation, let ℓ be a line passing through the origin that misses the countable set D. Then let A be the set of all angles α such that, for some $n \in \mathbb{N}$ and some $P \in D$, $\rho_\alpha(P) \in D$ where ρ_α is the rotation about ℓ by $n\alpha$ radians. By construction, A is countable, so we may choose an angle $\theta \notin A$. Let ρ_θ be the corresponding rotation about ℓ. Thus, $\rho_\theta^n(D) \cap D = \emptyset$ if $n > 0$ and it follows that $\rho_\theta^m(D) \cap \rho_\theta^n(D) = \emptyset$ for $0 \leq m < n$. Thus the rotations of ρ_θ are pairwise disjoint as required and the equidecomposability of $\mathbb{S}^2$ and $\mathbb{S}^2 \setminus D$ follows immediately. □

It then immediately follows that the unit sphere is paradoxical.

Corollary 7.11.1 (The Banach-Tarski Paradox): The sphere $\mathbb{S}^2$ is $SO_3(\mathbb{R})$-paradoxical, as is any sphere centered at the origin. Moreover, any solid ball in $\mathbb{R}^3$ is paradoxical and $\mathbb{R}^3$ is thus paradoxical.

Proof. By combining a few of the preceding results, we have the following:

1. For some countable set D, $\mathbb{S}^2 \setminus D$ is $SO_3(\mathbb{R})$-paradoxical (thm. 7.10)
2. $\mathbb{S}^2 \sim \mathbb{S}^2 \setminus D$ (thm. 7.11)
3. For $A, B \subset X$, if $A \sim B$ and A is paradoxical, so is B (thm. 7.1)

Combining the above immediately yields that $\mathbb{S}^2$ is $SO_3(\mathbb{R})$-paradoxical. Since none of the above is dependent on the radius of a given sphere, it also follows that any sphere is $SO_3(\mathbb{R})$-paradoxical. To explore the paradoxic decompositions of the sphere, it is sufficient to consider spheres centered at the origin as G_3 contains all translations. For clarity, we here consider the unit sphere, but the same proof for spheres of any size. The first decomposition demonstrates that a single point can be absorbed into the sphere: that is, $\mathbb{S}^2 \sim \mathbb{S}^2 \setminus \{\mathbf{0}\}$. Consider the radial correspondence $P \to \{\alpha P : 0 < \alpha \leq 1\}$. Let $P = (0, 0, \frac{1}{2})$

and ρ be a rotation of infinite order about the axis that is the horizontal line in the xz-plane. Then the set $D = \{\rho^n(0) : n \geq 0\}$ can be used to absorb $\mathbf{0}$ as $\rho(D) = D \setminus \mathbf{0}$ and thus $\mathbb{S}^2 \sim \mathbb{S}^2 \setminus \{\mathbf{0}\}$. The same notion of radial correspondence can be used on $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ to get a paradoxical decomposition of $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ using rotations. In the exact same manner as the sphere, then, $\mathbb{R}^3 \setminus \{\mathbf{0}\} \sim \mathbb{R}^3$. □

Given that the proofs used to establish the equidecomposability of certain pieces of the Banach-Tarski Paradox appeal to the Axiom of Choice, it might seem as though an uncountable number of independent rotations is necessary for the actual decomposition. In actuality, only two independent rotations are necessary – indeed, the Satô rotations can be used to generate the decomposition of the Banach-Tarski paradox.

The consequences of the Banach-Tarski paradox are remarkable: from one ball, we can dissect it into two, three, or thousands of balls all identical to the original. This bizarre result led many mathematicians of the day to question – if not outright reject – the Axiom of Choice. However, as we have have seen, paradoxical results do not require the Axiom of Choice. Particularly, Corollary 7.9.1 demonstrates an equally strange result. Hence, the Axiom of Choice need not be discarded merely because paradoxicality can result from it.

While duplication of spheres is an incredible result on its own, that was not all that Banach and Tarski proved. Starting with duplicating balls, Banach and Tarski also showed that a ball is equidecomposable with any solid ball of any size. They further abstracted their work into the following result.

Theorem 7.12 (Banach-Tarski Paradox, Strong Form): If A and B are any two bounded subsets of $\mathbb{R}^3$, each having nonempty interior, then A and B are equidecomposable.

Proof. It is sufficient to show that $A \preceq B$ as the argument for the reverse is identical. Choose K and L to be solid balls such that $A \subseteq K$ and $L \subseteq B$. Let $n \in \mathbb{N}$ be sufficiently large such that K may be covered by n overlapping copies of L. Further, let S be a set of n non-overlapping copies of L. Then, using corollary 7.11.1 and translations, L can be duplicated and the copies moved so that $S \preceq L$. It immediately follows that $A \subseteq K \preceq S \preceq L \subseteq B$ and therefore $A \preceq B$ as required. □

The Banach-Tarski paradox bends our geometric sensibilities of what is possible with a dissection. Less immediately obvious are the consequences on analysis and measure. Several of the above theorems forcefully demonstrate that some measures do not exist. Indeed, many of the above sets used are non-measurable. This is not to say that any and every set-theoretic approach to dissection involves non-measurable sets. In [4], one of the later results is the following corollary.

Corollary 7.12.1: In $\mathbb{R}^3$, any tetrahedron is equidecomposable with a cube using isometries and Lebesgue-measurable pieces.

Despite the consequences of using sets in dissections, the approach remains useful. There are a variety of results, including the above corollary, that would be otherwise unobtainable. As we have seen, using an arbitrary class of sets for dissections inevitably leads to a paradox. Thus, to make use of sets in dissection, the paradoxical must be taken along with the useful.

8. CONCLUSION

As we have seen, geometric dissection puzzles are several thousand years old and yet have changed remarkably little. While not as popular today, the stomachion remains relevant to the study of geometry and geometric puzzles. By contrast, the tangram is still a staple among introductory geometric puzzles, though it is not as popular as it once was. Even so, both puzzles are suitable introductions to the concepts of isometries and dissections.

The discovery of the Wallace-Bolyai-Gerwien theorem greatly improved our understanding of geometry. While recreational mathematics is more concerned with finding particularly clever dissections, such as Dudeney's haberdasher, the WBG theorem allows us to confidently guarantee the existence of a dissection in the first place. The WBG has even been abstracted into spherical and hyperbolic geometric equivalents. While not as elegant as the WBG, Dehn's advancements opened the way to understanding the behavior of geometric figures in higher dimensions. H. Hadwiger ultimately abstracted the Dehn invariant into higher dimensions where it is still used today. While J.P. Sydler demonstrated the sufficiency of the Dehn invariant in Euclidean three-space, it remains an open question as to whether or not the Dehn invariant is even necessary in spherical or hyperbolic three-space.

The Banach-Tarski paradox remains one of the most astonishing results in mathematics. Both seemingly absurd and divorced from physical reality, the B-T paradox caused many to doubt and even reject the Axiom of Choice. Despite the absurdity, the B-T paradox is, as we have seen, a perfectly consistent result of set theory and isometry groups. We have also seen that the Axiom of Choice is not necessary for an actual paradox to arise among isometry groups. Even so, the Axiom of Choice remains one of the most questioned axioms in all of mathematics.

We have seen the trajectory of work in Euclidean two-space and three-space. While there are several results in spherical and hyperbolic two-space, there remain open questions regarding dissections in the three-dimensional versions of each. Similarly, higher Euclidean dimensions hold open questions of their own. With the paradoxes we have seen in three dimensions, future work in dissections will doubtless yield puzzling conclusions.

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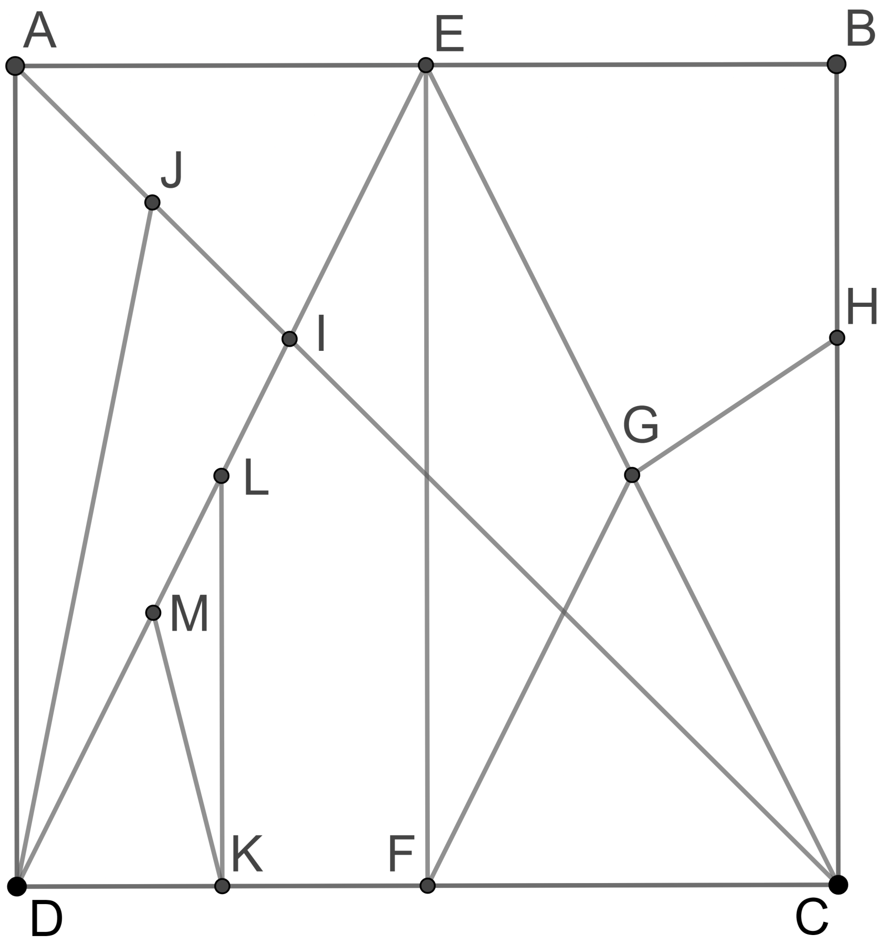


Figure 1: Stomachion Dissection

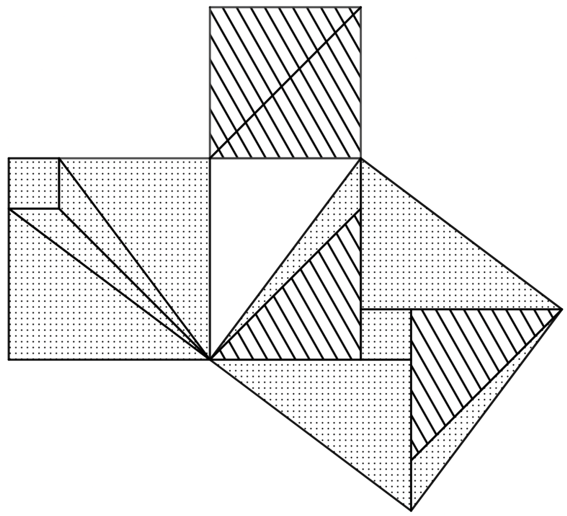


Figure 2: First Possible Liu Hui Dissection

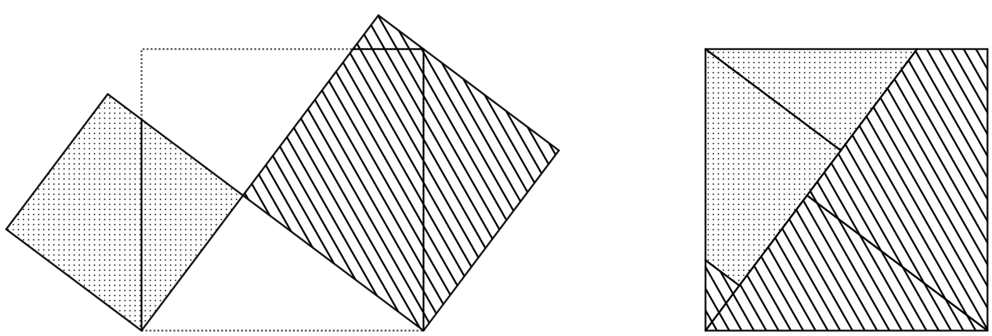


Figure 3: Second Possible Liu Hui Dissection

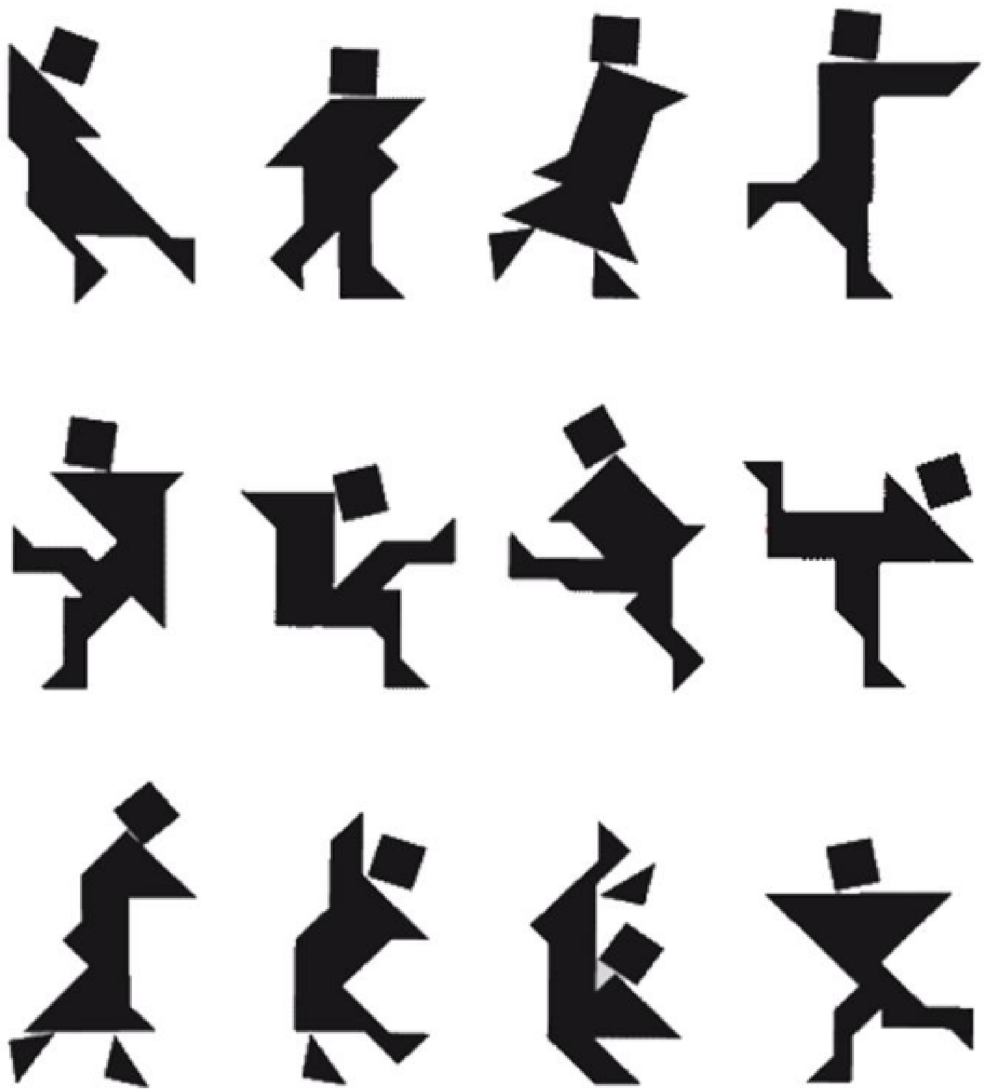
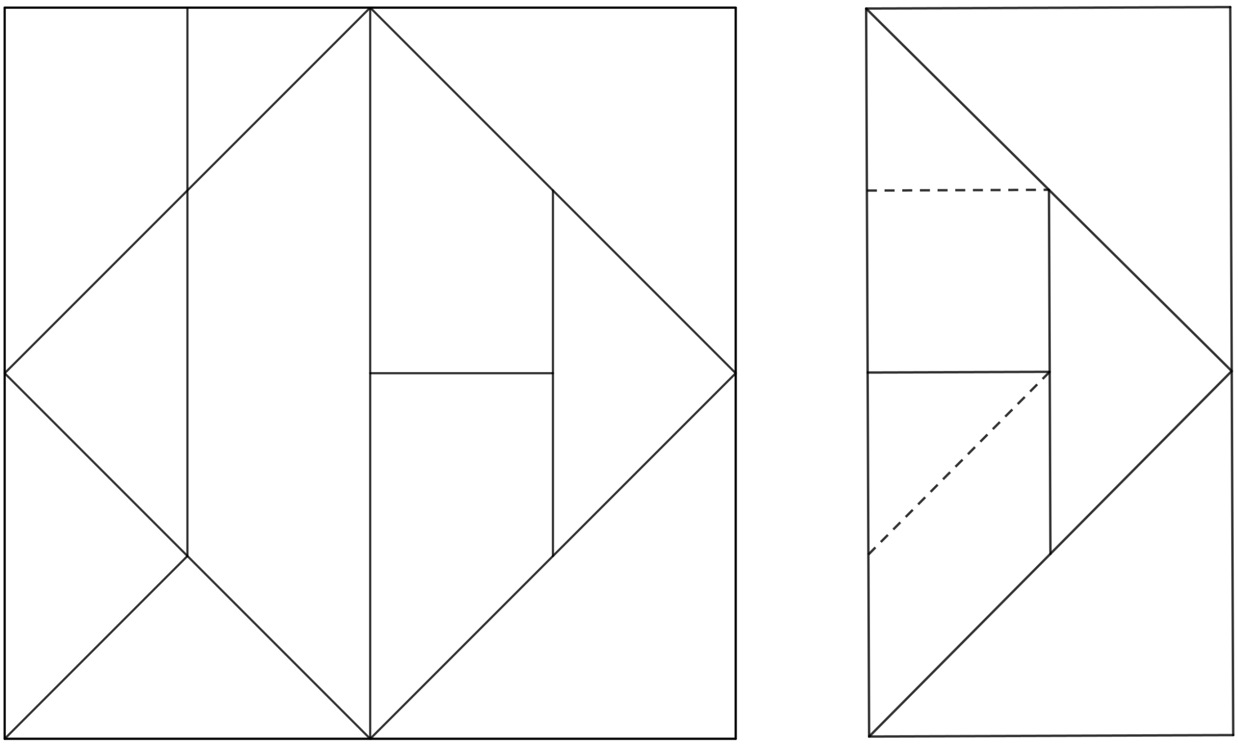


Figure 4: Tangram People



(a) Butterfly Wing Tables (b) Tangram Pieces

Figure 5: Butterfly Wing Tables and Tangram Pieces

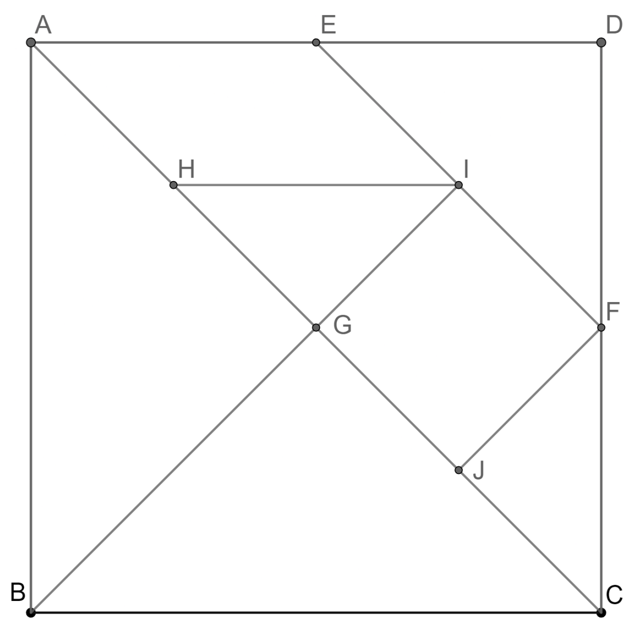


Figure 6: Tangram Dissection

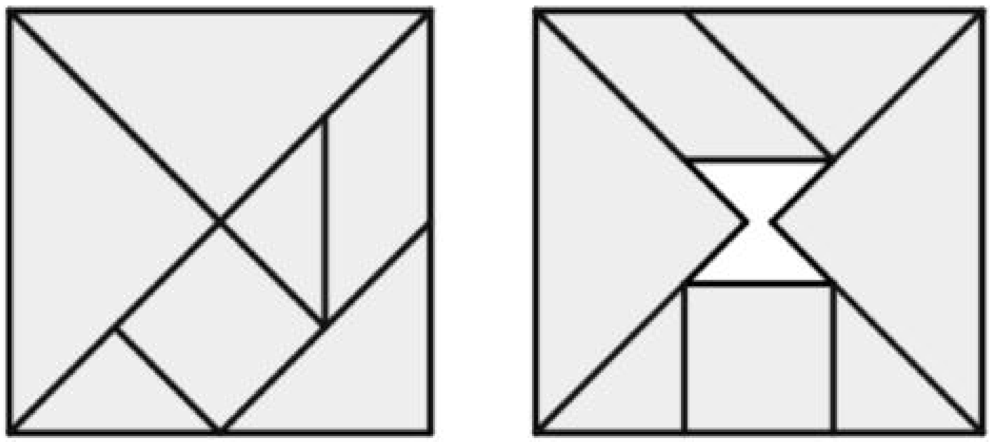


Figure 7: Paradoxical Tangram Squares

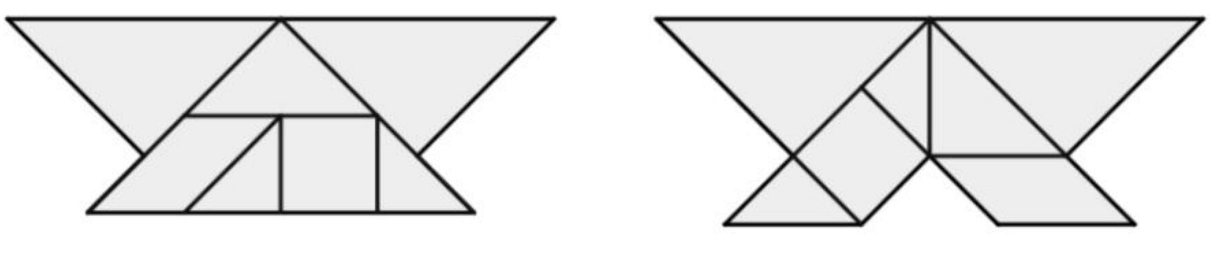


Figure 8: Paradoxical Tangram Dissections

Figure 9: The Haberdasher's Puzzle

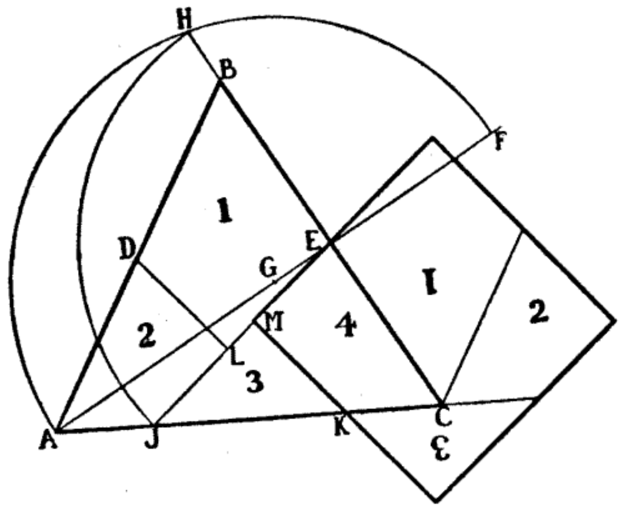


Figure 10: Haberdasher Solution

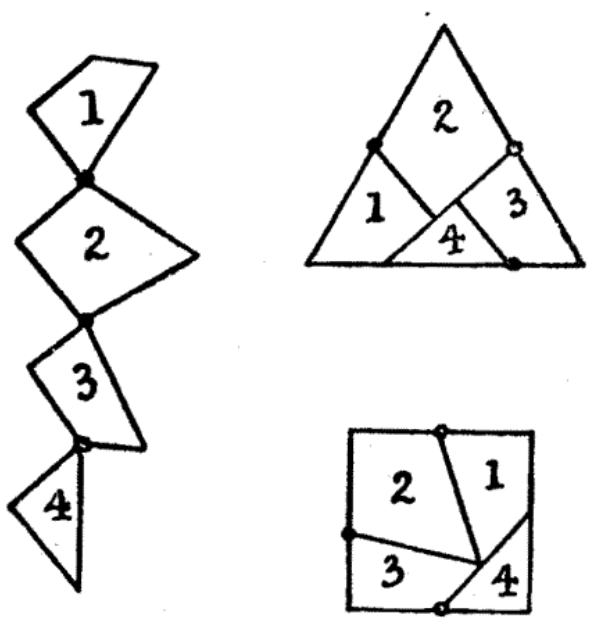


Figure 11: Haberdasher Hinged Dissection

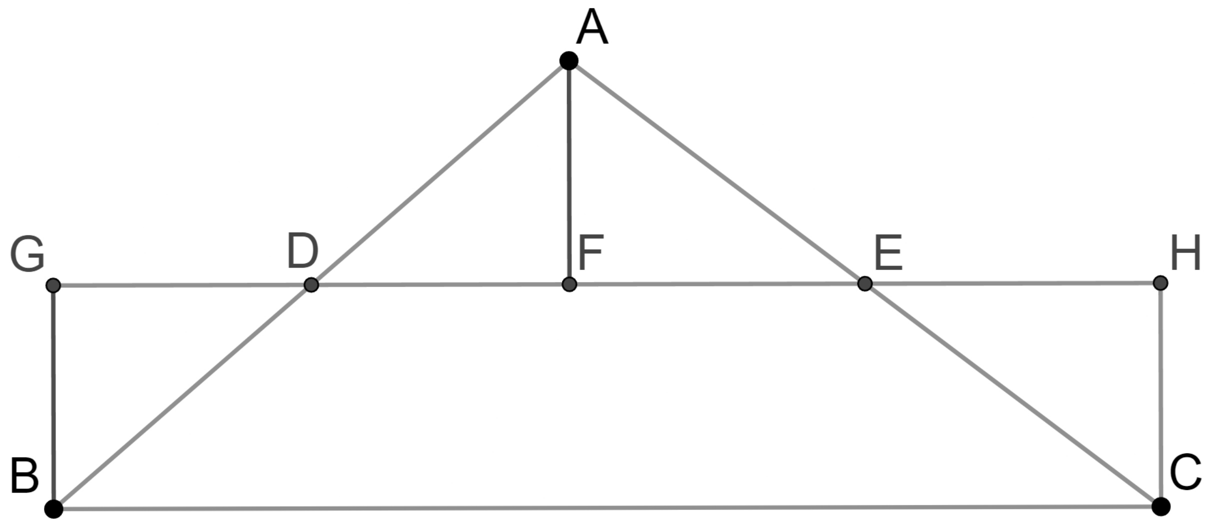


Figure 12: Triangle to Rectangle Dissection

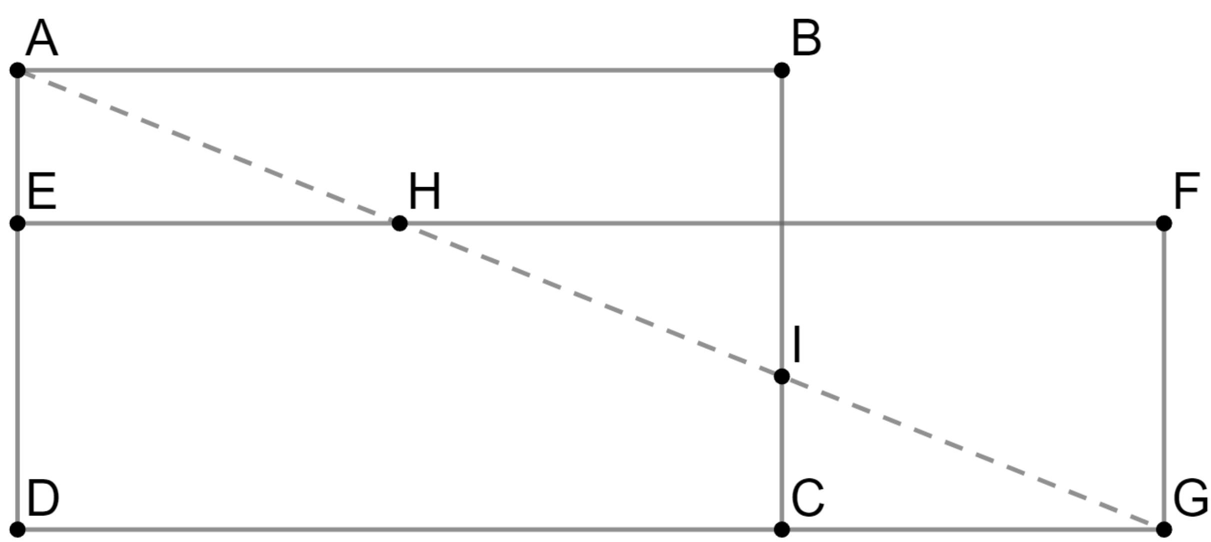


Figure 13: Rectangle Dissection Case 1

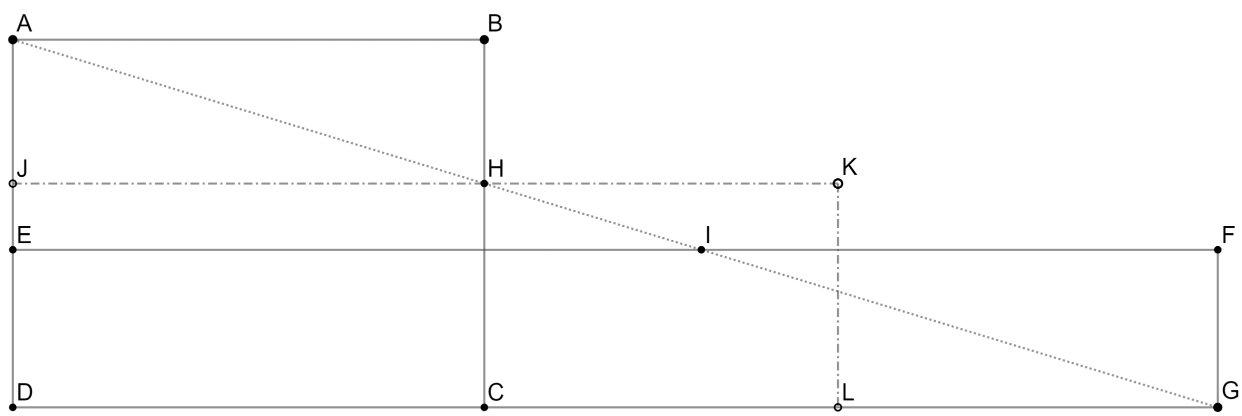


Figure 14: Rectangle Dissection Case 2